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An exposition of a conjecture on Non-dividing formulas in NIP theories

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Abstract

We give a survey of a question of Chernikov and Simon: given a formula $\phi(x,b)$ in an NIP theory, that does not fork over M, can we find a $\psi(y)\in tp(b/M)$ such that the set $\{\phi(x,b'):b'\models\psi(y)\}$ is consistent.

1 Introduction

In this paper we survey recent progress on a question first asked in [3]. It is part of on–going investigations into the behaviour of forking outside stable theories. The question subsequently became a conjecture (e.g. [7]), and is principally concerned with NIP theories, where the behaviour forking formulas is very different from stable theories (for example forking independence is no longer an independence relation). The conjecture is the following:

Conjecture 1.1. *Let $M\prec N$ be NIP L-structures with N sufficiently saturated. Let $\varphi(x,y)$ be an L_M-formula and let $b\in N^{|y|}$. Assume $\phi(x,b)$ does not fork over M. Then there is an L_M-formula $\psi(y)\in tp(b/M)$ such that $\{\phi(x,b'):b'\in\psi(N)\}$ is consistent.*

In Section 2 of this survey we go over the basic facts and definitions needed to understand the statement of Conjecture 1.1. We also go over why the conjecture is true in stable theories. In Section 3 we summarise the progress made so far on this conjecture. In Section 4 we highlight some open questions. None of the material contained in this survey paper is original.

Throughout we assume some familiarity with classification theory, there are many books that cover the subject in detail, see for example [12], or [10] for detail related to NIP theories.

2 Background

In this section we go through all the basic definitions and facts, including showing that the conjecture is true for stable theories.

In the following: T will be a complete theory; M will be a small model; often with $M\prec N$ and N sufficiently saturated; A, B, C will all be small subsets of N; and all variables and parameters could be tuples. We use standard model theoretic notation, for example we write $b_1\equiv_M b_2$ to mean that $tp(b_1/M)=tp(b_2/M)$, and $p\vdash\phi(x,b)$ to mean the type p contains the formula $\phi(x,b)$ (i.e.

in a saturated model the solution set of the type p is contained in the set defined by $\phi(x, b)$.

Definition 2.1.
- *A sequence $\{b_i\}_{i \in I}$ is A-indiscernible if whenever $i_1 < ... < i_n$ and $j_1 < .. < j_n$ then $b_{i_1}...b_{i_n} \equiv_A b_{j_1}...b_{j_n}$.*
- *A formula $\phi(x, b)$ divides over a set A if there is an A-indiscernible sequence such that $b_i \equiv_A b \ \forall i$ and $\{\phi(x, b_i) : i \in \omega\}$ is inconsistent.*
- *A formula $\phi(x, b)$ forks over a set A if it implies a finite disjunction of formulas each of which divide over A.*
- *A type $tp(c/B)$ forks over A if it contains a formula which forks over A.*
- *$C \perp_A B$ if $tp(C/AB)$ does not fork over A.If $C \perp_A B$ we say " C is independent from B over A".*

Conjecture 1.1 only considers formulas which do not fork over a model. We know from [2] that if the theory is NTP_2 then a formula forks over a model if and only if it divides over a model. Hence, as all NIP theories are NTP_2, we can assume that the formula $\phi(x, b)$ does not divides over M. This conjecture then says that if a family of formulas is consistent whenever the parameter set is an indiscernible sequence, then we can in fact find a *definable* set of parameters over which the family is consistent.

We will say that the conjecture holds for a particular non-forking (over M) formula $\phi(x, b)$ if we can find a $\psi(y) \in tp(b/M)$ such that $\{\phi(x, b') : b' \in \psi(M)\}$ is consistent. Similarly, we will say Conjecture 1.1 holds for a particular theory if it holds for all non-forking formulas in the theory.

For the rest of this section we will assume that T is an NIP theory. In this context all formulas have finite VC-dimension, this means that the (p, q)–theorem can be applied. In fact Conjecture 1.1 is often referred to as the "definable (p, q)–theorem." This is because, if it were true, Conjecture 1.1 would imply that a particularly nice consequence of the (p, q)–theorem would be true in a definable form. See section 2 of [9] for an explanation of this.

If T is NIP then a non-forking formula $\phi(x, b)$ extends to (i.e. is contained in) a global invariant type, by work in [5]. Below we give the definition of an invariant type. If, in addition, $\phi(x, b)$ extend to a global type which is either finitely satisfiable or definable, then the conjecture holds for $\phi(x, b)$. This is explained the Examples 2.4 and 2.5 below.

Definition 2.2.
1. *Let $A \subset N$, and $p \in S_x(N)$ (the set of x types over N), we say that p is A-invariant if $\sigma p = p$ for any $\sigma \in Aut(N/A)$, i.e. if $b \equiv_A b'$ then $p \vdash \phi(x, b)$ if and only if $p \vdash \phi(x, b')$.*
2. *We say that p is invariant if p is A-invariant for some small $A \subset N$*

Proposition 2.3. *Let $M \prec \mathcal{N}$ be NIP L-structures with N sufficiently saturated. Let $\varphi(x, y)$ be an L_M-formula and let $b \in N^{|y|}$. Assume $\varphi(x, b)$ does not divide over M. Then $\{\varphi(x, b') : N \models q(b')\}$ is consistent.*

Proof By [5] $\phi(x, b)$ extends to an M-invariant global type p. Therefore, for every $b' \models tp(b/M)$ we have that $p \vdash \phi(x, b')$, hence $p \vdash \{\varphi(x, b') : N \models q(b')\}$. As p is consistent, $\{\varphi(x, b') : N \models q(b')\}$ is consistent.

□

Example 2.4. *Let $\phi(x,b)$ be non-forking over M, and suppose it extends to a definable type $p \in S_x(\mathcal{U})$. That is to say for every formula $\theta(x,y)$, we have a defining schema, $\psi_\theta(y)$ such that for every $c \in \mathcal{U}$ we have $\theta(x,c) \in p$ if and only if $c \models \psi_\theta(y)$.*

To see that the conjecture holds for any such $\phi(x,b)$, take $\psi(y)$ to be $\theta_\phi(y)$. Now $p \vdash \{\phi(x,b') : b' \models \theta_\phi(y)\}$, so as p is consistent, $\{\phi(x,b') : b' \models \theta_\phi(y)\}$ must be consistent.

It follows from this example that Conjecture 1.1 is true if T is a stable theory. In stable theories every type is definable. In particular, as non-forking formulas extend to global invariant types (T stable implies T NIP) they will extend to to definable types. Thus, by the above, Conjecture 1.1 will hold for T a stable theory.

Example 2.5. *Let $\phi(x,b)$ be non-forking over M, and suppose it extends to a type finitely satisfiable in M. Then in particular $\phi(x,b)$ is satisfiable, so there is an $a \in M$ such that $M \models \phi(a,b)$. We can then take $\psi(y)$ to be $\phi(a,y)$, which is an L_M-formula. Clearly $a \in \{\phi(x,b') : b' \models \phi(a,y)\}$, so it is consistent.*

3 Progress in NIP theories

There has been substantial progress on this conjecture in non-stable NIP theories. The following is discussed in [7], and follows from work in [3].

Proposition 3.1. *Let $M \prec N$ be NIP L-structures with N sufficiently saturated. Let $\varphi(x,y)$ be an L_M-formula and let $b \in N^{|y|}$. Assume $\varphi(x,b)$ does not divide over M. Then there exist an L_M-formula $\psi(y) \in tp(b/M)$ and a finite $A_\psi \subseteq N^{|x|}$ such that, for each $b' \in \psi(M)$, there is some $a \in A_\psi$ such that $N \models \varphi(a,b')$.*

Proposition 3.1 gives an approach to solving the conjecture, i.e. by reducing A_ψ to a singleton, this is used in the proof of Theorem 3.8. In [7] Simon reduces the problem further, remarking that one can assume that both the language L and the model M are countable, and showing that the conjecture holds if $tp(b/M)$ has only countably many coheirs.

There has also been substantial progress made through considering dp-minimal structures. This notion was introduced by Shelah, and includes all o-minimal and C-minimal theories.

Definition 3.2. *Let Let $M \prec N$ be NIP L-structures with N sufficiently saturated, both models of T. We say T is dp-minimal if for every $A \subset N$, every singleton a and any two infinite sequences I_0, I_1 of tuples, if I_0 and I_1 are indiscernible over AI_1, AI_0 respectively then one of I_0 or I_1 is indiscernible*

In [9] Simon proves that in dp-minimal theories all invariant one-types are either definable or finitely satisfiable, thus showing the conjecture holds for formulas with a single variable. It is also shown in [9] that the conjecture holds for formulas with two variables. However, to show conjecture holds for all formulas Simon needs the additional assumption of either low or medium directionality. The following definition is taken from [9], the concepts were originally investigated in [4].

Definition 3.3. *An NIP theory T is of*

- *small directionality, if given a model M and $p \in S(M)$, then for any finite set Δ of formulas, the global coheirs of p determine only finitely many Δ-types (and thus p has at most $2^{|T|}$ coheirs).*
- *medium directionality, if it is not of small directionality and if the global coheirs of every such p determine at most $|M|$ Δ-types (and thus p has at most $|M|^{|T|}$ coheirs).*
- *large directionality, otherwise.*

Theorem 3.4. *(From [9]) Suppose T dp-minimal with either low or medium directionality, then Conjecture 1.1 holds for T.*

As the theory of real closed fields (RCF) has large directionality, this does not cover all dp-minimal theories. In [11] the authors show that the conjecture holds for dp-minimal theories with what they call property D – essentially formulas in one variable extend to definable types.

Definition 3.5. *Let $M \prec N$ be NIP L-structures with N sufficiently saturated, both models of T. We say T has property D if for every $A \subset N$, every consistent L_A-formula $\phi(x)$ in one variable extends to an A-definable complete type $p \in S_x(A)$.*

Theorem 3.6. *(From [11]) Suppose T dp-minimal with property D, then Conjecture 1.1 holds for T.*

The proof in this case is the main content of [11]. They prove a stronger fact here: that every non-forking (over M) formula extends to a complete M-definable type. This case in particular shows that the conjecture is true for any theory with definable Skolem functions, so for example $T = Th(\mathbb{Q}_p)$.

More recently progress has been made outside the context of dp-minimal theories. In particular, [1] the conjecture is shown to hold for distal NIP theories. The notion of distality was introduced by Simon in [8]. The idea is that distal theories are those NIP theories which are, in some sense, completely unstable. There are various equivalent definitions. The following is the one used in [3].

Definition 3.7. *Let T be NIP, $M \prec N$ models, N sufficiently saturated. We say T is distal if for any $\phi(x, y)$ there is $\theta(x, z)$ such that: for any A, a and finite $C \subseteq A$ (with $|C| \geq 2$), there is a $b \in A$ such that $N \models \phi(a, b)$ and $\theta(x, b) \vdash tp_\phi(a/C)$ (where $tp_\phi(C)$ is the ϕ-type of a over C).*

Theorem 3.8. *(From [1]) Conjecture 1.1 holds for T a distal NIP theory.*

The proof uses strict Morley sequences, which exist in NIP theories by [2], combined with a combinatorial lemma to force the $|A_\psi|$ from Definition 3.1 to have size one.

4 Further Work

The main question for further work is clearly to prove the full conjecture, i.e. for all NIP theories. The idea behind distality was that every type in an NIP theory would break down into a "distal" part and a "stable" part, see [6] for more details. The hope is that this decomposition would lead to the full conjecture as it is known to hold in both parts. However, thus far attempts have been unsuccessful. To find a counterexample one would have to look at NIP theories which are non–distal, non–stable, are of large directionality and do not have property D. The author is unaware of any natural examples of such a theory.

One could also ask if the main conjecture holds in fuller generality, that is to say, outside the context of NIP structures. The main reason NIP structures are considered is that the (p, q)-theorem holds here, whereas in the wider context we have formulas with infinite VC-dimension, for which the (p, q)–theorem does not apply. However, there is no a priori reason why Conjecture 1.1 could not be true in other theories. The first step might be to look at non–forking formulas which have finite VC-dimension in a theory which is NTP_2. It would also be interesting to look at particular examples, such as the random graph.

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