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A remark on generic structures with the full amalgamation property

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Abstract

We prove that any generic structure with the full amalgamation property is stable.

1 Preliminaries

The reader is assumed to be familiar with the basics of generic structures. This paper was influenced by papers of Baldwin-Shi [1] and Wagner [5].

Let L be a finite relational language, where each relation $R \in L$ has arity $n \geq 2$ and satisfies the following:

- If $\models R(\bar{a})$ then the elements of $\bar{a}$ are without repetition and,
- $\models R(\sigma(\bar{a}))$ for any permutation σ.

Thus, for any L-structure A and $R \in L$ with arity n, R^A can be thought of as a set of n-element subsets of A. For a finite L-structure A, a predimension of A is defined by

$$\delta_\alpha(A) = |A| - \sum_{R \in L} \alpha_R |R^A|,$$

where $0 < \alpha_R \leq 1$ and $\alpha = (\alpha_R)_{R \in L}$. $\delta_\alpha(A)$ is usually abbreviated to $\delta(A)$. Let $\delta(B/A)$ denote $\delta(BA) - \delta(A)$. For $A \subset B$ and $n \in \omega$, A is said to be n-closed in B, denoted by $A \leq_n B$, if

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$\delta(X/A \cap X) \geq 0$ for any finite $X \subset B$ with $|X \cap (B - A)| \leq n$.

In addition, A is said to be closed in B, denoted by $A \leq B$, if

$$A \leq_n B \text{ for any } n \in \omega.$$

The closure $\mathrm{cl}_B(A)$ of A in B is defined by $\bigcap\{C : A \subset C \leq B\}$.

Let $\mathbf{K}_\alpha$ be the class of the finite L-structures A with $\delta(B) \geq 0$ for any $B \subset A$.

Definition 1.1 Let $\mathbf{K} \subset \mathbf{K}_\alpha$. Then a countable L-structure M is said to be $(\mathbf{K}, \leq)$-generic, if

1. any finite $A \subset M$ belongs to $\mathbf{K}$;
2. whenever $A \leq B \in \mathbf{K}$ and $A \leq M$, then there is a $B \cong_A B'$ with $B \leq M$;
3. for any finite $A \subset M$, $|\mathrm{cl}_M(A)|$ is finite.

2 The full amalgamation property

In what follows, M is a $(\mathbf{K}, \leq)$-generic structure for some $\mathbf{K} \subset \mathbf{K}_\alpha$, and $\mathcal{M}$ is a big model of $\mathrm{Th}(M)$.

$\mathrm{cl}_{\mathcal{M}}(A)$ is abbreviated to $\mathrm{cl}(A)$. For $A, B, C \subset \mathcal{M}$ with $B \cap C \subset A$, B and C are said to be free over A, denoted by $B \bot_A C$, if

$$R^{ABC} = R^{AB} \cup R^{AC}$$

for any $R \in L$. Moreover, $B \oplus_A C$ denotes an L-structure $(BCA, R^{AB} \cup R^{AC})_{R \in L}$.

Definition 2.1 Let A, B be finite with $A \leq B \subset \mathcal{M}$. Then B is said to be closed over A, if $\mathrm{cl}(B) = B \cup \mathrm{cl}(A)$ and $B \bot_A \mathrm{cl}(A)$.

Lemma 2.2 Let A, B be finite with $A \leq B \subset \mathcal{M}$. Then the following are equivalent.

1. B is closed over A;
2. For any finite $D \subset \mathcal{M} - B$ with $\mathrm{cl}_{BD}(B) = BD$, $B \bot_A D$.

Proof. (1→2) If 2 does not hold, then there is a finite $D \subset \mathcal{M} - B$ with

$$\mathrm{cl}_{BD}(B) = BD \text{ and } B \not\perp_A D.$$

Clearly $D \subset \mathrm{cl}(B)$. Since B is closed over A, we have $B \perp_A \mathrm{cl}(A)$. So $D \not\subset \mathrm{cl}(A)$. Hence $\mathrm{cl}(B) \neq B \cup \mathrm{cl}(A)$. A contradiction.
(2→1) By 2, $B \perp_A \mathrm{cl}(A)$. So it is enough to show that $\mathrm{cl}(B) = B \cup \mathrm{cl}(A)$. If not, then there is a $D \subset \mathrm{cl}(B) - B \cup \mathrm{cl}(A)$. We can assume that

$$\mathrm{cl}_{BD}(B) = BD \text{ and } B \not\perp_A D.$$

On the other hand, by 2 again, we have $B \perp_A D$. A contradiction.

Definition 2.3 $(\mathbf{K}, \leq)$ is said to have the full amalgamation property, if whenever $A \leq B \in \mathbf{K}, A \subset C \in \mathbf{K}$ and $B \perp_A C$ then $B \oplus_A C \in \mathbf{K}$.

Lemma 2.4 Suppose that $(\mathbf{K}, \leq)$ has the full amalgamation property. Then, whenever $A \subset \mathcal{M}$ and $A \leq B \in \mathbf{K}$, then there is a $B' \subset \mathcal{M}$ such that B' is closed over A and $B' \cong_A B$.

Proof. Let $D_0, D_1, ...$ be an enumeration of the elements of $\mathbf{K}$ with

$$B \cap D_i = \emptyset,\ \mathrm{cl}_{BD_i}(B) = BD_i \text{ and } B \not\perp_A D_i$$

for each $i \in \omega$.

Claim: For any $n \in \omega$ there is a $B' \subset \mathcal{M}$ such that

1. $B' \cong_A B$;
2. for each $i \leq n$ there is no $D_i' \subset \mathcal{M}$ with $B'D_i' \cong_A BD_i$.

Proof of Claim: It is enough to show that for each $n \in \omega$,

$$M \models \forall X(X \cong A \rightarrow \exists Y(XY \cong AB \wedge \bigwedge_{i \leq n} \neg \exists Z_i(XYZ_i \cong ABD_i)).$$

Take any $A^* \subset M$ with $A^* \cong A$. Then $C = \mathrm{cl}_M(A^*)$ is finite. Take B^* with

$$B^*A^* \cong BA \text{ and } B^* \perp_{A^*} C.$$

By the full amalgamation property,

$$E^* = B^* \oplus_{A^*} C \in \mathbf{K}.$$

By genericity, we can assume that $E^* \leq M$. Then B^* is closed over A^*. By Lemma 2.2, we have $M \models \bigwedge_{i \leq n} \neg\exists Z_i(A^*B^*Z_i \cong ABD_i))$. (End of Proof of Claim)

By the above claim,

$$\Sigma(Y) = \{Y \cong_A B\} \cup \{\neg\exists Z_i(YZ_i \cong_A BD_i) : i \in \omega\}$$

is consistent. Take a realization B' of $\Sigma(Y)$. By Lemma 2.2 again, B' is closed over A.

Definition 2.5 Th(M) is said to be ultra-homogeneous over closed sets, if whenever $A, A' \subset \mathcal{M}$ are isomorphic then tp(A) = tp(A').

Note 2.6 It can be seen that Th(M) is ultra-homogeneous over closed sets if and only if whenever $A, A' \subset \mathcal{M}$ are isomorphic and finitely generated then tp(A) = tp(A').

Proposition 2.7 Let M be $(\mathbf{K}, \leq)$-generic. Suppose that $(\mathbf{K}, \leq)$ has the full amalgamation property. Then Th(M) is ultra-homogeneous over closed sets.

Proof. Let $\mathcal{M}$ be a big model. Take any $A, A' \leq \mathcal{M}$ with $A \cong A'$. We want to prove that

$$\text{tp}(A) = \text{tp}(A').$$

By Note 2.6, we can assume that A, A' are finitely generated. So take a finite $A_0 \subset A$ with $\text{cl}(A_0) = A$, and let A'_0 be such that $A'_0A' \cong A_0A$. Take any $b \in \mathcal{M} - A$ and let $B = \text{cl}(bA)$. To show that tp(A) = tp(A'), it is enough to prove that

$$\text{there is a } B' \leq \mathcal{M} \text{ with } B'A' \cong BA.$$

Note that B is countable since B is also finitely generated. Let $B_1, B_2, \ldots$ be a tower of finite subsets of B such that

- each B_i is i-closed:
- $\bigcup_i B_i = B$;
- $A_0 \subset B_1$.

For each $i \in \omega$ let $A_i = B_i \cap A$ and take A'_i with $A'_i A'_0 A' \cong A_i A_0 A$. Fix any $i \in \omega$. Since $B_i \leq_i \mathcal{M}$ and $A \leq \mathcal{M}$, we have $A_i \leq_i \mathcal{M}$, and hence $A'_i \leq_i \mathcal{M}$. On the other hand, by Lemma 2.4, there is a $B'_i \subset \mathcal{M}$ such that

$$B'_i A'_i \cong B_i A_i \text{ and } B'_i \text{ is closed over } A'_i.$$

Claim: $B'_i \leq_i \mathcal{M}$.

Proof of Claim: Take any $X \subset \mathcal{M} - B'_i$ with $|X| \leq i$. Let $X_0 = X \cap A'$ and $X_1 = X \cap (\mathcal{M} - A')$. Since B'_i is closed over A'_i, we have

$$B'_i A' \leq \mathcal{M} \text{ and } B'_i \perp_{A'_i} A'.$$

Then

$$\begin{aligned}
\delta(X/B'_i) &= \delta(X_1/B'_i X_0) + \delta(X_0/B'_i) \\
&\geq \delta(X_0/B'_i) && \text{(by } B'_i A' \leq \mathcal{M}) \\
&= \delta(X_0/A'_i) && \text{(by } B'_i \perp_{A'_i} A') \\
&\geq 0 && \text{(by } A'_i \leq_i \mathcal{M})
\end{aligned}$$

Hence $B'_i \leq_i \mathcal{M}$. (End of Proof of Claim)

For each $i \in \omega$ let

$$\Sigma_i(X_i) = \{X_i A'_i \cong B_i A_i\} \cup \{X_i \text{ is } i\text{-closed }\}.$$

By the above claim, each $\Sigma_i(X_i)$ is consistent. Therefore $\bigcup_i \Sigma_i(X_i)$ is also consistent. Hence we can take a realization B' of $\bigcup_i \Sigma_i(X_i)$, and then we have $B' \leq \mathcal{M}$ and $B'A' \cong BA$.

3 Theorem

For a finite $B \subset \mathcal{M}$, a dimension of B is defined by

$$d(B) = \inf\{\delta(C) : B \subset_\omega C \subset \mathcal{M}\}.$$

For a tuple $e \in \mathcal{M}$ and a finite $A \subset \mathcal{M}$, $d(e/A)$ denotes $d(eA) - d(A)$. In case that A is infinite, $d(e/A)$ is defined by $\inf\{d(e/A_0) : A_0 \subset_\omega A\}$. The following fact can be found in [1] and [5].

Fact 3.1 Let $A \leq B \leq \mathcal{M}$ and $e \in \mathcal{M} - B$ with $\mathrm{cl}(eA) \cap B = A$. Then $d(e/B) = d(e/A)$ if and only if $\mathrm{cl}(eA) \perp_A B$ and $\mathrm{cl}(eA) \cup B \leq \mathcal{M}$.

Theorem 3.2 Let M be $(\mathbf{K}, \leq)$-generic. Suppose that $(\mathbf{K}, \leq)$ has the full amalgamation property. Then $\mathrm{Th}(M)$ is stable.

Proof. Let $\mathcal{M}$ be a big model. Take any κ with $\kappa^\omega = \kappa$. Take any $N \prec \mathcal{M}$ with $|N| = \kappa$. Take any $e \in \mathcal{M} - N$. Then there is a countable $A \subset N$ with $d(e/N) = d(e/A)$ and $\mathrm{cl}(eA) \cap N = A$.

Claim: $\mathrm{tp}(e/A)$ determines $\mathrm{tp}(e/N)$.

Proof of Claim: Take any $e' \models \mathrm{tp}(e/A)$ with $d(e'/N) = d(e'/A)$ and $\mathrm{cl}(e'A) \cap N = A$. Let $E = \mathrm{cl}(eA)$ and $E' = \mathrm{cl}(e'A)$. Since $\mathrm{tp}(e/A) = \mathrm{tp}(e'/A)$, we have $E \cong_A E'$. By Fact 3.1, we have

$$E \cong_N E' \text{ and } EN, E'N \leq \mathcal{M}.$$

By Proposition 2.7, $\mathrm{tp}(E/N) = \mathrm{tp}(E'/N)$, and hence $\mathrm{tp}(e/N) = \mathrm{tp}(e'/N)$. (End of Proof of Claim)

By the above claim, $|S(N)| \leq \kappa^\omega \cdot |S(A)| = \kappa^\omega = \kappa$. Hence the theory is stable.

Remark 3.3 Take any irrational α with $0 < \alpha < 1$. Then the $(\mathbf{K}_\alpha, \leq)$-generic structure is called the Shelah-Spencer random graph. (For instance, see [2].) In [1], it was proved that the theory is stable. Since $(\mathbf{K}_\alpha, \leq)$ has the full amalgamation property, by Theorem 3.2, it can be also checked that $\mathrm{Th}(M)$ is stable.

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