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GROUPS WHOSE ALL (MINIMAL) CAYLEY GRAPHS HAVE A GIVEN FORBIDDEN STRUCTURE

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ABSTRACT. We give the classification of all (minimal) Cayley bipartite or perfect finite groups as well as finite graphs Γ for which there are only finitely many (minimal) Cayley Γ-free groups.

1. INTRODUCTION

Let G be a non-trivial group and S be an inversed-closed subset of G, that is, $S \subseteq G \setminus \{1\}$ and $S^{-1} = \{s^{-1} : s \in S\} \subseteq S$. The *Cayley graph* of G corresponding to S, denoted by $\mathrm{Cay}(G, S)$, is a graph with G as the vertex set such that two vertices x and y are adjacent if $yx^{-1} \in S$. The Cayley graph $\mathrm{Cay}(G, S)$ is called *minimal* if $S = X \cup X^{-1}$ for some minimal generating subset X of G. Cayley graphs was introduced by Arthur Cayley in 1978 as a geometric description of groups and play a central role in geometric group theory. Being a source and a simple way of constructing symmetric graphs, Cayley graphs has became the subject of extensive research in algebraic graph theory as well as computer science from various points of views.

A group in which all its associated (minimal) Cayley graphs admit a given property $\mathcal{P}$ is called a (minimal) Cayley $\mathcal{P}$-group. Accordingly, a Cayley integral group is that whose all Cayley graphs are integral, that is, they all have integral spectrum. Studying integral graphs was initiated by Harary and Schwenk [8]. As an attempt to describe integral Cayley graphs, among other works, Abdollahi and Jazaeri [1] and simultaneously Ahmady, Bell and Mohar [2] in 2014, complete the classification of all Cayley integral finite groups. Motivated by these works, we are interested in studying the existence of particular subgraphs (mainly odd cycles) in Cayley graphs associated with a finite group. More precisely, we shall give a classification of those finite groups whose all (minimal) Cayley graphs are bipartite or perfect. Since our main results relies on particular forbidden structures in (minimal) Cayley graphs, we review the results on the problem that which graphs are isomorphic to an induced subgraph of a (minimal) Cayley graphs and determine which graphs can be embedded as induced subgraph into infinitely many (minimal) Cayley graphs.

The paper is organized as follows: In section 2, we show that there are only finitely many finite Cayley Γ-free groups for any finite graph Γ while the same result for minimal Cayley Γ-free groups holds if and only if Γ is a union of some paths. We note that a Γ-free graph is one having no induced subgraph isomorphic to Γ. Section 3 gives a description of all (minimal) Cayley bipartite groups, that is, finite groups whose all (minimal) Cayley graphs have no odd cycles as subgraphs.

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Finally, in section 3, we shall restrict our attention to induced odd cycles and determine all (minimal) Cayley perfect finite groups by using the knowledge of their forbidden induced odd cycles. Recall that a graph is *perfect* if the chromatic and clique number of its induced subgraphs coincides. A celebrated theorem of Chudnovsky, Robertson, Seymour and Thomas [5], known as the strong perfect graph theorem, states that a graph Γ is perfect if and only if neither Γ nor its complement has induced odd cycles other that triangles.

Throughout this paper, we adopt the following notations: Given a group G, the minimum size of generating set of G is denoted by $d(G)$. An arbitrary Sylow p-subgroup of G will be denoted by $S_p(G)$. Also, E_p stands for the extra-special p-group of order p^3 and exponent p. The unexplained notions are standard and can be found in any standard book. Recall that the Frattini subgroup $\Phi(G)$ of G is the intersection of all maximal subgroups of G. It is known that $\Phi(G)$ is the set of all non-generators of G, the fact that will be used without further references.

2. (Minimal) Cayley Γ-free groups

Every graph can be simply embedded as induced subgraph into some Cayley graph of sufficiently large order, namely using an elementary abelian 2-group generated freely by the vertices of the graph. As an attempt to decrease the exponential order of the corresponding Cayley graphs, Babai and Sós in [4], using the analysis of Sidon sets, gave a cubic lower bound $9.5|\Gamma|^3$ for the order of a group G, which assures the existence of a Cayley graph on G having Γ as an induced subgraph. This lower bound is further improved to $(2+\sqrt{3})|\Gamma|^3$ by Godsil and Imrich in [7]. Hence, we have the following.

Theorem 2.1. *For every finite graph Γ, the order of a Cayley Γ-free group is bounded above by $(2+\sqrt{3})|\Gamma|^3$.*

While every graph is an induced subgraph of a Cayley graph, it is still unknown which graphs can be embedded as (induced) subgraph into some minimal Cayley graphs. The only known results are due to Babai and Spencer. Indeed, Babai [3] shows that there is no minimal Cayley graphs having $K_4 \setminus e$ or $K_{3,5}$ as subgraph, in which $K_4 \setminus e$ is the diamond graph. Spencer [11], using the ideas of Babai and utilizing probabilistic arguments, proves the existence of graphs of bounded degree and arbitrary girth which cannot be embedded into minimal Cayley graphs as induced subgraphs. In contrast to the above theorem, the situation for minimal Cayley Γ-free groups is completely different as follows.

Theorem 2.2. *Let Γ be a finite graph. Then there are only finitely many minimal Cayley Γ-free groups if and only if Γ is a union of paths. Moreover, $|G| < |\Gamma|^{|\Gamma|}$ for any minimal Cayley Γ-free group G when Γ is a union of paths.*

Proof. Suppose Γ is a graph for which there are just finitely many minimal Cayley Γ-free groups. Since all minimal Cayley graphs of C_{2^n} are isomorphic to the 2^n-cyclic graph, it follows that Γ is an induced subgraph of 2^n-cycles for sufficiently large n. Hence, Γ is a union of paths. Conversely, suppose Γ is a union of paths. Let G be a minimal Cayley Γ-free group and $\Gamma' = \mathrm{Cay}(G, S)$ be a minimal Cayley graph of G in which $S = X \cup X^{-1}$ and $X = \{x_1, \ldots, x_n\}$ is a minimal generating set of G. Also, let N_i denote the ith neighbor of the identity element in Γ', that is, $N_i = \{g \in G : d_{\Gamma'}(1, g) = i\}$ for all $i \geq 0$. Clearly, $|N_0| = 1$ and $|N_i| \leq r(r-1)^{i-1}$

for all $i \geq 1$ in which $r = |S|$ is the degree of Γ'. If $d = \mathrm{diam}(\Gamma')$, then $N_d \neq \emptyset$ and $N_{d+1} = \emptyset$, which imply that

$$\begin{aligned} |G| &= |N_0| + |N_1| + \cdots + |N_d| \\ &\leq 1 + r + r(r-1) + \cdots + r(r-1)^{d-1} = 1 + r \cdot \frac{(r-1)^d - 1}{r-2}. \end{aligned}$$

Since, every path connecting 1 to any element of N_d is an induced path of length d in Γ', it follows that $d < |\Gamma| - 1$. On the other hand,

$$x_1 \sim x_2 x_1 \sim \cdots \sim x_n \cdots x_1 \sim x_1 x_n \cdots x_1 \sim \cdots \sim x_{n-1} \cdots x_1 x_n \cdots x_1$$

is an induced path in Γ', which implies that $2n - 1 \leq d$. Since $r \leq 2n$, we observe that $|G|$ is bounded above by $|\Gamma|^{|\Gamma|}$, as required. □

3. (Minimal) Cayley bipartite groups

It is well-known that bipartite graphs are perfect. Hence, in order to classify (minimal) Cayley perfect groups, we need to know the structure of (minimal) Cayley bipartite groups. As we shall see in the next section, almost all (minimal) Cayley perfect groups are (minimal) Cayley bipartite groups.

Since the only bipartite complete graphs are those with at most 2 vertices, the only Cayley bipartite groups are simply groups with at most 2 elements. Hence, in what follows, we just consider minimal Cayley bipartite groups. To end this, we use the following characterization of finite bipartite Cayley graphs.

Lemma 3.1. *Let G be a finite group. A Cayley graph* $\mathrm{Cay}(G, S)$ *is bipartite if and only if* $[G : \langle S^2 \rangle] = 2$ *and* $S \subseteq G \setminus \langle S^2 \rangle$.

Proof. Suppose $\mathrm{Cay}(G, S)$ is bipartite with a bipartition (X, Y). Let $H = \langle S^2 \rangle$. Since, $sX \subseteq Y$ and $sY \subseteq X$ for all $s \in S$, it follows that $|X| = |Y| = |G|/2$. In addition, $s_1 s_2 X = X$ and $s_1 s_2 Y = Y$ for all $s_1, s_2 \in S$, which imply that $HX = X$ and $HY = Y$. Since H contains all products of even number of elements of S, we must have $s_1 H = s_2 H$ for all $s_1, s_2 \in S$ whence $[G : H] = 2$ and $S \subseteq G \setminus H$. Moreover, X and Y are right cosets of H. The converse is obvious. □

Theorem 3.2. *A finite group G is a minimal Cayley bipartite group if and only if it is a 2-group.*

Proof. First assume that G is a minimal Cayley bipartite group. Let K be the intersection of all subgroups of G of index 2. If $\mathrm{Cay}(G, S)$ is a minimal Cayley graph of G, then $S \subseteq G \setminus H$ for some subgroup H of G of index 2 by Lemma 3.1. Since $K \subseteq H$, we have $K \cap S = \emptyset$, from which it follows that $K \subseteq \Phi(G)$, the Frattini subgroup of G. Thus, $G/\Phi(G)$ is a 2-group so that G is a 2-group too. Conversely, assume G is a 2-group. If $\mathrm{Cay}(G, S)$ is a minimal Cayley graph of G, then $S = X \cup X^{-1}$, where $X = \{x_1, \ldots, x_n\}$ is a minimal generating set of G. Let $H = \langle \Phi(G), x_1 x_2, \ldots, x_1 x_n \rangle$. Then H is a maximal subgroup of G and $S \subseteq G \setminus H$. Hence, by Lemma 3.1, $\mathrm{Cay}(G, S)$ is bipartite. Therefore, G is a minimal Cayley bipartite group. □

4. (Minimal) Cayley perfect groups

In this section, we shall give a classification of those finite groups all of whose minimal Cayley graphs are perfect. As a result we show that there are only few Cayley perfect groups. The following simple lemma will be used frequently.

Lemma 4.1. *Let $G = \langle g \rangle$ be a cyclic group. Then*

(1) $\mathrm{Cay}(G, \{g^{\pm 2}, g^{\pm 3}\})$ *has an induced* 5*-cycle* $1 \sim g^2 \sim g^4 \sim g^6 \sim g^3 \sim 1$ *for* $|g| \geq 10$*; and*

(2) $\mathrm{Cay}(G, \{g^{\pm 1}, g^{\pm 4}\})$ *has an induced* 5*-cycle* $1 \sim g \sim g^2 \sim g^3 \sim g^4 \sim 1$ *for* $|g| \geq 8$.

The proof of our theorems rely also on the following result of the first author. In what follows, $^-: G \longrightarrow G/\Phi(G)$ denotes the natural epimorphism, in which G is a given fixed group.

Theorem 4.2 ([6]). *Let G be a finite solvable group and P be a Sylow p-subgroup of G. If either $\overline{P} \trianglelefteq G$ or $\overline{P}$ is cyclic, then $d(P) = d(\overline{P})$.*

Now, we can state and prove our main results.

Theorem 4.3. *A finite group G is a minimal Cayley perfect group if and only if either G is a 2-group, or it is isomorphic to one of the groups C_3, C_6, S_3, $C_3 \times C_3$, A_4 or E_3.*

Proof. From Theorem 3.2, we know that every 2-group is a minimal Cayley perfect group. Also, a simple verification shows that the other six groups are also minimal Cayley perfect groups.

To prove the converse assume that G is a minimal Cayley perfect group and that G is not a 2-group. Hence $G \setminus \Phi(G)$ contains an element g of odd order. Let X be a minimal generating set of G containing g and $S = X \cup X^{-1}$. Then the subgraph induced by $\langle g \rangle$ in $\mathrm{Cay}(G, S)$ is an odd cycle, which implies that $|g| = 3$. Hence, G is a $\{2,3\}$-group. Let Q be a Sylow 3-subgroup of G. If $\exp(Q) \neq 3$, then $S_3(\Phi(G)) = H_3(Q)$ is a maximal subgroup of Q by [12], in which $H_3(Q) = \langle x \in Q : x^3 \neq 1 \rangle$. From the solvability of G in conjunction with Theorem 4.2, we observe that Q is cyclic and hence $Q \cong C_3$, a contradiction. Thus $\exp(Q) = 3$. First assume that $G = Q$ is a 3-group. If $d(G) \geq 3$ and a, b, c are elements of a minimal generating set X of G, then we observe that

$$1 \sim c \sim bc \sim abc \sim cabc \sim bcabc \sim abcabc \sim cabcabc \sim bcabcabc \sim 1$$

is an induced 9-cycle in $\mathrm{Cay}(G, X \cup X^{-1})$ arose from the relation $(abc)^3 = 1$, which is a contradiction. Thus $d(G) \leq 2$. By [10, 12.3.5], G is a group of nilpotent class 2 so that $|G| \leq 27$. As a group of exponent 3, $G \cong C_3$, $C_3 \times C_3$ or E_3.

Finally, assume that G is neither a 2-group nor a 3-group. Let $\mathcal{C}$ be the class of all groups isomorphic to C_6, S_3 or A_4. A simple computation shows that, in a group of order 6 or 12, a minimal generating set involving an element of order 3 gives rise to a perfect Cayley graph only if the group belongs to $\mathcal{C}$. We show that $G \in \mathcal{C}$ too. Assume G is a minimal counter example. Let l_f be the number of non-Frattini factors in a chief series of G. First assume that $l_f = 2$. One can easily see that $\overline{G} \cong C_6$, S_3 or A_4. We have three cases:

Case 1. $\overline{G} \cong C_6$. Then G is cyclic. If $|G| > 6$, then G has a minimal Cayley graph with an induced 5-cycle as illustrated in Lemma 4.1(1), a contradiction. Thus $G \cong C_6$, a contradiction.

Case 2. $\overline{G} \cong S_3$. From [6] we know that $G = \langle x, y : x^3 = y^{2^k} = 1, x^y = x^{-1} \rangle$ for some $k \geq 1$. For $k \geq 2$, the relation $y^{2^k - 2}xyx^{-1}yx = 1$ defines an induced odd cycle of length $2^k + 3$ in $\mathrm{Cay}(G, \{x^{\pm 1}, y^{\pm 1}\})$. Hence, we must have $k = 1$ so that $G \cong S_3$, a contradiction.

Case 3. $\overline{G} \cong A_4$. Then $\overline{G} = \langle \overline{x}, \overline{y} : \overline{x}^2 = \overline{y}^3 = (\overline{xy})^3 = \overline{1} \rangle$. Let P and Q be the Sylow 2-subgroup and a Sylow 3-subgroup of G, respectively. By Theorem 4.2, $P = \langle x, x^y \rangle$ is a maximal subgroup of G and $Q \cong C_3$. We may assume $Q = \langle y \rangle$. If $Q^x \subseteq \Phi(P)Q$, then $x^{-y}x = [y, x] \in \Phi(P)$, which is impossible as $x^{-y}x$ is a generator of P. Thus $y^x = x'y$ for some $x' \in P \setminus \Phi(P)$. Hence, replacing x by x' if necessary, we may assume that $(xy)^3 = 1$, from which it follows that $x^{y^{-1}}x^y x = 1$. Clearly, $|x| = 2^m > 2$ for $G \ncong A_4$. Since the group $\langle a, b : a^{2^m} = b^{2^m} = (ab)^{2^m} = 1 \rangle$ is infinite, there must exists a relation $w = 1$ in x, x^y independent of the relations $x^{2^m} = 1$, $(x^y)^{2^m} = 1$ and $(x^y x)^{2^m} = 1$. Assume w has minimum length among all such relations. Clearly, $|w| \geq 7$ in which $|w|$ denotes the length of $|w|$ as a word in x, y. After a suitable cyclic shift and inverse if required, we may assume that $w = x^{a_1 y}x^{b_1} \cdots x^{a_k y}x^{b_k}$ in which $a_i, b_i \neq 0$ for $i = 1, \ldots, k$, $a_1 > 0$ and $(a_1, b_1) \neq (1, 1)$. Let us call a word in x, x^y is good if it has even length as a word in x, x^y. Since $\mathrm{Cay}(P, \{x^{\pm 1}, x^{\pm y}\})$ is bipartite by Theorem 3.2 and $y \notin P$, one can easily see that a subword u of a good word u^* equals an element $g \in \{1, x^{\pm 1}, y^{\pm 1}\}$ only if either $g^{-1}u$ or ug^{-1} is a good subword of u^*. Having this in mind, $w = 1$ gives rise to an induced odd cycle in $\mathrm{Cay}(G, \{x^{\pm 1}, y^{\pm 1}\})$ when $|w|$ is odd, which is a contradiction. Thus $|w|$ is even. From $w = 1$ we may construct a new relation $w' = 1$, where w' is defined as

$$w' = x^{-y^{-1}}x^{-1}x^{(a_1-1)y}x^{b_1}x^{a_2 y}x^{b_2} \cdots x^{a_k y}x^{b_k}.$$

Suppose w' has a proper subword w'' which is equal to an element $g \in \{1, x^{\pm 1}, y^{\pm 1}\}$ and that neither $g^{-1}w''$ nor $w''g^{-1}$ is a subword of w' otherwise we may replace w'' by $g^{-1}w''$ or $w''g^{-1}$ and g by 1. If w'' is a subword of $x^{y^{-1}}w'$ then, by the argument above, either $g^{-1}w''$ or $w''g^{-1}$ is a good subword of $x^{y^{-1}}w'$ and we may assume $w'' = 1$. Moreover, $w'' = x^{-1}x^{-y}w'''$ should be an initial subword of $x^{y^{-1}}w'$ in which w''' is a good initial subword of w. But then, it follows that $w''' = x^y x$ contradicting the assumption on a_1, b_1 and the length of w. Hence w'' should contain some letters of the initial term $x^{-y^{-1}}$ of w'. If w'' is not an initial subword of w', then since $g^{-1}w'' \equiv w''g^{-1} \equiv 1 \pmod{P}$, we must have a relation of the form $yw''y = 1$ in which either $yw''y$ is an initial subword of $x^{y^{-1}}w'$ or w' ends at $x^{a'_i y}$ with $|a'_i| < |a_i|$ as a subword of $x^{a_i y} = x^{a'_i y}x^{(a_i - a'_i)y}$. Since the former case was ruled out by the above discussions, $(yw''y)^{-1}w' = 1$ is a relation in which $(yw''y)^{-1}w'$ is a proper subword of w contradicting the assumption on w. Thus $w'' = x^{-y^{-1}}x^{-1}w'''$ is an initial subword of w' possessing the initial term $x^{-y^{-1}}x^{-1}$. But then $x^y w''' = g$ where $x^y w''' g^{-1}$ or $g^{-1}x^y w'''$ is a proper subword of w after possibly a cyclic shift contradicting the assumption on w. Now, the relation $w' = 1$ determines an induced odd cycle of length $|w'| = |w| + 1$ in $\mathrm{Cay}(G, \{x^{\pm 1}, y^{\pm 1}\})$, the final contradiction.

In the sequel, we assume that $l_f \geq 3$. Let

$$\Phi(G) = G_0 \trianglelefteq G_1 \trianglelefteq \cdots \trianglelefteq G_{l-1} \trianglelefteq G_l = G,$$

be the inverse image of a chief series of $G/\Phi(G)$ and assume $M = G_{l-1}$. From [9, Theorem 2], we know that G has a minimal generating set $X = \{x_1, \ldots, x_{l_f}\}$ in which $x_i \in G_{n_i} \setminus G_{n_i - 1}$ $(i = 1, \ldots, l_f)$ is an element of prime power order, $n_1 = 1$, $n_f = l$ and G_{n_i}/G_{n_i-1} are the non-Frattini factors of the chief series, for $i = 1, \ldots, l_f$. Replacing the elements of X by suitable conjugates, we can also assume that x_i, x_j belong to the same Sylow p-subgroup whenever x_i, x_j are both

p-elements for some $p = 2, 3$. Let $Y_i = X \setminus \{x_i\}$ for $i = 1, \ldots, l_f$. First observe the every x_i belongs to some Y_j having elements of odd and even orders. Hence, by assumption on G and perfectness of $\mathrm{Cay}(\langle Y_j\rangle, Y_j \cup Y_j^{-1})$, it follows that $\langle Y_i\rangle \in \mathcal{C}$ so that x_i has prime order. Furthermore, $l_f = 3$.

We claim that $l = l_f$, that is, there are no Frattini factors. Clearly, Y_i contains elements of order 2 and 3 for some $i \in \{1, 2\}$. Then $G = G_{n_2}\langle Y_i\rangle$ implies $G/G_{n_2} \cong \langle Y_i\rangle/(G_{n_2} \cap \langle Y_i\rangle) \cong C_2$ or C_3. Hence, $n_2 = n_3 - 1 = l - 1$. Therefore, $\Phi(G/G_1) = G_{l-2}/G_1$. On the other hand, we have $G/G_1 = \langle G_1x_2, G_1x_3\rangle$. In case $\{|x_2|, |x_3|\} = \{2, 3\}$, we have $\langle x_2, x_3\rangle \in \mathcal{C}$ showing that $l = 3$. If $|x_2| = |x_3| = 2$, then G/G_1 is a dihedral 2-group. Clearly, $\{x_1, x_2x_3, x_3\}$ is a minimal generating set of G forcing $(x_2x_3)^2 = 1$. Hence, $G_{l-2} = \langle G_1, (x_2x_3)^2\rangle = G_1$ and again $l = 3$. Finally, assume that $|x_2| = |x_3| = 3$. Being a 2-generated 3-group of exponent 3, G/G_1 is isomorphic to $C_3 \times C_3$ or E_3. Assume G/G_1 is non-abelian. If x_2, x_3 commute with x_1, then $G = \langle x_1\rangle \times \langle x_2, x_3\rangle$ so that $[x_2, x_3] \in \Phi(G)$, a contradiction. Hence, we may assume that $[x_1, x_3] \neq 1$. If $[x_1, x_2x_3] \neq 1$, then since $\{x_1, x_2x_3, x_3\}$ is a minimal generating subset of G, $\langle x_1, x_2x_3\rangle \cong A_4$ by assumption on G. Accordingly, $(x_1x_2x_3)^3 = 1$ giving rise to an induced 9-cycle in $\mathrm{Cay}(G, X \cup X^{-1})$, a contradiction. Thus, by replacing x_2 by x_2x_3 if required, we may assume that x_1 and x_2 commute. Since $[x_1, x_2x_3^{-1}] \neq 1$ and $\{x_1, x_2, x_2x_3^{-1}\}$ is a minimal generating subset of G, we observe that

$$x_1x_1^{x_3^{-1}x_2^{-1}} = (x_1x_1^{x_3^{-1}})^{x_2^{-1}} = x_1^{x_3x_2^{-1}} = x_1^{(x_2x_3^{-1})^{-1}} = x_1x_1^{x_2x_3^{-1}} = x_1x_1^{x_2^{-1}x_3^{-1}},$$

which implies that $[x_2, x_3]$ commutes with x_1. Hence, $G = \langle x_1, x_1^{x_3}\rangle \rtimes \langle x_2, x_3\rangle$ and one can verify that $[x_2, x_3] \in \Phi(G)$, which is a contradiction. Thus G/G_1 is abelian and consequently $l = 3$, as required. In addition, we have shown that when $|x_2| = |x_3| = p$, either $\langle x_2, x_3\rangle \cong C_p \times C_p$ or $G \cong E_3$ with $[x_2, x_3] \in \Phi(G)$.

Further we show that every involution $x_u \in X$ acts by inversion on $\Phi(G)$ and that $\Phi(G)$ is elementary abelian when $\langle x_u, x_v\rangle \cong A_4$ for some $x_v \in X$. To end this, let $g \in \Phi(G)$. Since x_u can be replaced with gx_u in X, we deduce that $(gx_u)^2 = 1$, that is, $g^{x_u} = g^{-1}$. Replacing x_u by $x_u^{x_v^{\pm 1}}$ in X results in a new minimal generating subset, from which it follows that $g^{x_u^{x_v^{\pm 1}}} = g^{-1}$. Hence, $g^{-1} = g^{x_u^{x_v^{-1}}} = g^{x_ux_u^{x_v}} = g$, as claimed.

Now, put $H = \langle Y_3\rangle$. Since $\mathrm{Cay}(H, Y_3 \cup Y_3^{-1})$ is an induced subgraph of $\mathrm{Cay}(G, X \cup X^{-1})$, it is perfect. We distinguish three cases:

Case $1'$. H is a 2-group. Then $[G : M] = 3$ and M is the Sylow 2-subgroup of G by Theorem 4.2 and the fact that the Sylow 3-subgroups of G have exponent three. Moreover, since $\Phi(G) = \Phi(M)$, it follows that $\overline{M}$ is elementary abelian. As $\mathrm{Cay}(\langle Y_i\rangle, Y_i \cup Y_i^{-1})$ is perfect for $i = 1, 2$, the minimality of G shows that $\langle Y_i\rangle \in \mathcal{C}$ and subsequently $\langle Y_i\rangle \cong C_6$ or A_4. Hence, $|\overline{M}| \leq 16$. First suppose that $\langle x_1, x_3\rangle \cong \langle x_2, x_3\rangle \cong C_6$. Then $M = H$, $G/\Phi(G) \cong C_6 \times C_2$ and G is nilpotent. If $M \setminus \Phi(M)$ contains an element x of order ≥ 4, then Lemma 4.1(1) shows that the Cayley graph corresponding to every minimal generating subset of G containing x and x_3 contains an induced 5-cycle, a contradiction. Thus $M \setminus \Phi(M)$ contains only involutions, which yields $M \cong C_2 \times C_2$. Hence, $G \cong C_6 \times C_2$ contradicting the choice of G. Thus, $\langle x_i, x_3\rangle \cong A_4$ for some $i = 1, 2$. We show that $\langle x_j, x_3\rangle \cong C_6$ for $j \in \{1, 2\} \setminus \{i\}$. Indeed, if $\langle x_2, x_3\rangle \cong A_4$, then $\{x_1, x_2, x_2x_3\}$ is a minimal generating subset of G and $|x_2x_3| = 3$, from which it follows that $\langle x_1, x_2x_3\rangle \cong C_6$. Otherwise $(x_1x_2x_3)^3 = 1$ and hence we obtain an induced 9-cycle in $\mathrm{Cay}(G, X \cup X^{-1})$. Thus, by replacing x_3 by x_2x_3 if necessary, we may assume that x_1 and x_3 commute, as required. Hence, $\Phi(G)$ is elementary abelian as shown before. For $g \in \Phi(G)$, we observe that $(gx_i)^{x_3^{-1}} = (gx_i)(gx_i)^{x_3}$ and $(gx_j)^{x_3} = (gx_j)$ for x_i can be replaced by gx_i in X. Thus $g^{x_3^{-1}} = gg^{x_3} = 1$, which yields $g = 1$. Therefore, $\Phi(G) = 1$. Now, it is obvious that $G \cong A_4 \times C_2$. Putting $a := x_3^{x_i}$ and $b := x_jx_3$, we observe that $G = \langle a, b\rangle$ and $\mathrm{Cay}(G, \{a^{\pm1}, b^{\pm1}\})$ has an induced 7-cycle determined by $b^{-1}abab^2a^{-1} = 1$, which is impossible.

Case 2$'$. H is a 3-group. Then $[G : M] = 2$ and M is the Sylow 3-subgroup of G by Theorem 4.2. As in case 1$'$, for $i = 1, 2$, we have $\langle Y_i\rangle \in \mathcal{C}$ showing that $\langle Y_i\rangle \cong C_6$ or S_3. Hence, $x_i^{x_3} = x_i^{\epsilon_i}$ with $\epsilon_i = \pm1$ for $i = 1, 2$. Moreover, $M = H$ is a group of order 9 or 27. Let $w = x_2x_1^{-1}x_2^{-1}x_1$. Then $\mathrm{Cay}(G, X \cup X^{-1})$ has an induced 7-cycle or 11-cycle determined by the relation $x_3x_2^{\epsilon_2}x_1^{-\epsilon_1}x_3x_2x_1wx_2 = 1$ according as $w = 1$ or not, respectively, which is a contradiction.

Case 3$'$. H is neither a 2-group nor a 3-group. By assumption, $H \in \mathcal{C}$ so that $H \cong C_6$, S_3 or A_4. Assume $|x_i| = 2$ and $|x_j| = 3$ for $\{i, j\} = \{1, 2\}$. Let $k \in \{1, 2\}$ be such that $|x_k| \neq |x_3|$. As $\langle x_k, x_3\rangle \in \mathcal{C}$, we also have $\langle x_k, x_3\rangle \cong C_6$, S_3 or A_4.

First assume that $|x_2| = |x_3| = p$. If $p = 2$ then $[x_2, x_3] = 1$ and $x_1^{x_2}, x_1^{x_3} \in \langle x_1\rangle$, from which it follows that $G \cong C_6 \times C_2$ or $S_3 \times C_2$ contradicting the assumption on G. Thus $p = 3$. As in case 1$'$, we may assume that x_1 commutes with x_3 and consequently x_1 commutes with $[x_2, x_3]$ as shown before. Hence, $G = \langle x_1\rangle \times \langle x_2, x_3\rangle$ if $[x_1, x_2] = 1$ and $G = \langle x_1, x_1^{x_2}\rangle \rtimes \langle x_2, x_3\rangle$ if $[x_1, x_2] \neq 1$. In the former case, $x_1x_2x_3x_2^{-1}x_1x_2^{-1}x_3x_2x_3 = 1$ determines an induced 9-cycle in $\mathrm{Cay}(G, X \cup X^{-1})$, a contradiction. Also, in the latter case, the relation $(x_1x_2x_3)^2wx_1x_3x_2 = 1$ in which $w = 1$ or $x_2^{-1}x_3^{-1}x_2x_3$ according as $[x_2, x_3] = 1$ or not, determines an induced 9-cycle or 13-cycle in $\mathrm{Cay}(G, X \cup X^{-1})$, respectively, which is a contradiction.

Thus, we have left with the case $|x_2| \neq |x_3|$. If x_2 and x_3 commute, then we can interchanging x_2 and x_3 after which H will be a 2-group or a 3-group. Hence, without loss of generality, we assume that $[x_2, x_3] \neq 1$. In the case x_1 and x_2 commute, $\langle x_2, x_2^{x_3}, x_2^{x_3^{-1}}\rangle$ is an elementary abelian normal subgroup of G otherwise $[x_1, x_3]$ does not commutes with x_2 so that $\langle x_2, x_3\rangle \cong A_4$ and $(x_1^{x_3})^{x_2} = (x_1^{x_3})^{-1}$. Then $[x_1, x_3]^{x_2} = x_1[x_3, x_1]$, which implies that $\{[x_1, x_3], x_2, x_3\}$ is a minimal generating subset of G. But then, we must have $\langle [x_1, x_3], x_2\rangle \cong C_6$ or S_3, which is impossible. It means we can also interchange x_1 and x_2 after which we are in the situation that $|x_2| = |x_3|$ as discussed above. Thus, we may further assume that $[x_1, x_2] \neq 1$. As a result, $\langle x_1, x_2\rangle$ and $\langle x_k, x_3\rangle$ are isomorphic to S_3 and A_4 in some order, which implies that $\Phi(G)$ is an elementary abelian subgroup of G. Now, we have only two possibilities. If $(|x_1|, |x_2|, |x_3|) = (2, 3, 2)$, then $\langle x_1, x_2\rangle \cong A_4$ and $\langle x_2, x_3\rangle \cong S_3$. Clearly, $\langle x_1, x_3\rangle$ is a dihedral 2-group. If $|x_1x_3| = 2^m$, then we observe that $\mathrm{Cay}(G, X \cup X^{-1})$ contains an induced $(2^{m+1} + 5)$-cycle determined by $(x_1x_2)^2(x_3x_1)^{2^m-1}x_2x_3x_2^{-1} = 1$, which is a contradiction. Thus, we should have $(|x_1|, |x_2|, |x_3|) = (3, 2, 3)$. Then $\langle x_1, x_2\rangle \cong S_3$ and $\langle x_2, x_3\rangle \cong A_4$, hence $|x_2x_3| = 3$. Let Q be a Sylow 3-subgroup of G containing x_2x_3. Let $y \in \Phi(G)\langle x_2, x_2^{x_3}\rangle$, a Sylow 2-subgroup of G, be such that $x_1^y \in Q$. Replacing x_1 by x_1^y in X, one can assume that $x_1 \in Q$. Being elements of Q, it follows that $(x_1x_2x_3)^3 = 1$ giving rise to an induced 9-cycle in $\mathrm{Cay}(G, X \cup X^{-1})$, the final contradiction. The proof is complete. $\square$

Utilizing the above theorem, it is now easy to obtain the classification of all Cayley perfect finite groups.

Theorem 4.4. *Let G be a nontrivial finite group. Then G is a Cayley perfect group if and only if G is isomorphic to one of the groups C_2, C_3, C_4, $C_2 \times C_2$, S_3, C_6, $C_2 \times C_2 \times C_2$, $C_2 \times C_4$, D_8, Q_8, $C_3 \times C_3$.*

Proof. Assume G is a Cayley perfect group. By Theorem 4.3, either G is a 2-group or G is isomorphic to one of the groups C_3, C_6, S_3, $C_3 \times C_3$, A_4 or E_3. From rows (9) and (10) of Table I, it follows that $G \ncong A_4$ and E_3. Furthermore, if G is a 2-group, by Lemma 4.1(2) and rows (1)–(8) of Table I, we observe that $|G| \leq 8$. Hence, $G \cong C_2$, C_4, $C_2 \times C_2$, $C_4 \times C_2$, $C_2 \times C_2 \times C_2$, D_8 or Q_8 and the result follows. The converse is straightforward. $\square$

Table I

	G	S	5-cycle
1	$\langle a\rangle \times \langle b\rangle,\ \lvert a\rvert = \lvert b\rvert = 4$	$\{a^{\pm 1}, b^{\pm 1}, a^2b^2\}$	$1, a^{-1}, (ab)^{-1}, ab, a, 1$
2	$\langle a, b : a^4 = b^4 = 1, [a, b] = a^2\rangle$	$\{a^{\pm 1}, b^{\pm 1}, a^2b^2\}$	$1, a^{-1}, (ba)^{-1}, ab, a, 1$
3	$\langle a\rangle \times \langle b\rangle \times \langle c\rangle,\ \lvert a\rvert = 4, \lvert b\rvert = \lvert c\rvert = 2$	$\{a^{\pm 1}, b, c, a^2bc\}$	$1, c, ca^{-1}, ba, b, 1$
4	$\langle a, b, c : a^4 = b^2 = c^2 = 1, a^c = a^{-1}b\rangle$	$\{a^{\pm 1}, b, c\}$	$1, c, ca^{-1}, ba, b, 1$
5	$\langle a, b, c : b^2 = c^2 = [a, b] = [a, c] = 1, [b, c] = a^2\rangle$	$\{a^{\pm 1}, b, c, a^2bc\}$	$1, a^{-1}, a^{-1}c, ab, a, 1$
6	$\langle a, b : a^4 = b^2 = 1, a^b = a^{-1}\rangle \times \langle c\rangle,\ \lvert c\rvert = 2$	$\{a^{\pm 1}, b, c, a^2bc\}$	$1, a^{-1}, a^{-1}c, ab, a, 1$
7	$\langle a, b : a^4 = 1, a^2 = b^2, a^b = a^{-1}\rangle \times \langle c\rangle,\ \lvert c\rvert = 2$	$\{a^{\pm 1}, b^{\pm 1}, c, (a^2b^{-1}c)^{\pm 1}\}$	$1, a^{-1}, a^{-1}c, ab, a, 1$
8	$\langle a\rangle \times \langle b\rangle \times \langle c\rangle \times \langle d\rangle,\ \lvert a\rvert = \lvert b\rvert = \lvert c\rvert = \lvert d\rvert = 2$	$\{a, b, c, d, abcd\}$	$1, a, ab, abc, abcd, 1$
9	$\langle a, b, c : a^3 = b^3 = [a, c] = [b, c] = 1, c = [a, b]\rangle$	$\{a^{\pm 1}, b^{\pm 1}, c^{\pm 1}, (a^{-1}bc)^{\pm 1}\}$	$1, a, ac, bc^{-1}, b, 1$
10	$\langle a, b : a^2 = b^3 = (ab)^3 = 1\rangle$	$\{a, b^{\pm 1}, a^b\}$	$1, a^b, b^a, ab, a, 1$

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