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Endo-trivial modules for finite groups with dihedral Sylow 2-subgruops

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1. Endo-trivial modules

This is joint work with Caroline Lassueur (see [8], [7]). We assume that k is a field with characteristic $p > 0$, and G is a finite group with $p \Big| |G|$. A finitely generated right kG-module M is called *endo-trivial* if

$$\mathrm{End}_k(M) \cong k_G \oplus (\text{projective})$$

as right kG-modules where k_G is the trivial kG-module. Then in the stable module category $\mathsf{stmod}(kG)$ of finitely generated kG-modules, the set

$$T(G) := \{ [M] \in \mathsf{stmod}(kG) \mid M \text{ is endo-trivial} \}$$

has an abelian group structure by making use of the tensor product over k. Namely, we define an addition $+$ in $T(G)$ by

$$[M] + [N] := [M \otimes_k N]$$

(note that $M \otimes_k N$ is again an endo-trivial kG-module if so are M and N, and recall also that $\mathrm{End}_k(M) \cong M^* \otimes_k M$ as kG-modules where $M^* := \mathrm{Hom}_k(M, k)$. Remind also that for right kG-modules M and N, $M \otimes_k N$ is considered as left kG-module by the diagonal action, namely $(m \otimes n)g := mg \otimes ng$ for $g \in G$, $m \in M$ and $n \in N$, and that M^* is again a left kG-modlue via $(\phi g)(m) := \phi(mg^{-1})$ for $g \in G$, $m \in M$ and $\phi \in M^*$. Then it is easy to know that the zero element in $T(G)$ is $[k_G]$ and the inverse element $-[M]$ of $[M]$ in $T(G)$ is $[M^*]$.

The endo-trivial modules go back to Dade in 1978 [6]. Since then endo-trivial modules show up in many places in the modular representation theory of finite groups and they do have played very important role in this area. Actually, Dade classifies all the endo-trivial kG-modules for the case where G are finite abelian p-groups (see [6]). Since then the determination of the structure of $T(G)$ has been done for the case where G are finite p-groups by Carlson and Thévenaz (see [13], [2]). Then, we might be interested in $T(G)$ for artibrary finite groups G. This problem is still open as far as the author knows. It is known that $T(G)$ is finitely generated by a result of [3] (depending on initiated work of Puig). From now on, let $TT(G)$ and $TF(G)$ be the torsion part and the torsion free part of $T(G)$ respectively, and let P be a Sylow p-subgroup of G.

First, if P is cyclic, then $T(G)$ is considered in [9]. This is of course exactly the case where the group algebra kG is of finite-representation type as is well-known since many years ago. Further, if $p = 2$ and P is generalized quaternion or semidihedral, then $T(G)$ is treated by [4]. Recall that these cases cover *almost* of all the cases where the group algebras kG are of finite-representation type and of tame-representation type. Then, what's missing? Yes, the case where P is dihedral (possibly the Klein-four group of order 4) is missing. This is actually our motivaion for the work (see [7] and [8]).

Our main result is the folllowing:

Theorem 1.1. *Suppose that G is a finite group with a dihedral Sylow 2-subgroup P of order at least 8, and assume that $T(G)$ is the abelian group of endo-trivial kG-modules over an algebraically closed field k of characteristic 2. Set $\overline{G} := G/O_{2'}(G)$. Then we have the following:*

(i) *If $\overline{G} \not\cong \mathfrak{A}_6$ (here $\mathfrak{A}_6$ is the alternating group on 6 letters), then $TT(G) = X(G)$, where $X(G) := \{[M] \in T(G) \mid \dim_k(M) = 1\}$.*

(ii) *If $\overline{G} \cong \mathfrak{A}_6$, then either*

  (a) *$TT(G) = X(G)$, or*

  (b) *$TT(G)/X(G)$ is an elementary abelian 3-group and each indecomposable endo-trivial kG-module M with $[M] \in TT(G) \backslash X(G)$ is a 9-dimensional simple module.*

Proof. See [8, Theorem 1.2]. □

Remark 1.2. For the case where the Sylow 2-subgroup P of G is the Klein-four group $C_2 \times C_2$, see [8, Theorem 1.5] and also [7].

Actually we do have a pretty much general result to compute $T(G)$. It is stated as follows:

Theorem 1.3. *Suppose that k is an algebraically closed field of characteristic $p > 0$, and that G is a finite group of p-rank at least 2, namely G contains a subgroup $C_p \times C_p$ where C_p is the cyclic group of order p. Further assume that G has no strongly p-embedded subgroups. Now, we suppose that $H \lhd G$ with $p \not| |H|$. Set $\overline{G} := G/H$, and assume more over that $\mathrm{H}^2(\overline{G}, k^\times) = 1$. Then, we have*

$$k(G) \cong X(G) + k(\overline{G})$$

where $K(G)$ is the kernel of the restriction map

$$\mathrm{Res}^G_P : T(G) \to T(P) \quad \textit{given by} \quad [M] \mapsto [M{\downarrow}^G_P],$$

and further

$$k(G) := \{[M] \in T(G) \,|\, M = f^{-1}(L)$$
$$\textit{for a 1-dimensional } kN_G(P)\textit{-module } L\}$$

where f is the Green correspondence with respect to $(G, P, N_G(P))$.

Proof. See [8, Theorem 1.1]. □

Remark 1.4. The point in Theorem 1.3 is the following. We actually would like to compute $TT(G)$, and it is most likely the same as $K(G)$. Precisely speaking, $K(G) = TT(G)$ unless P is cyclic or generalized quaternion. So at least for our purpose this is the case. So our final aim is recuded from the computation of $TT(G)$ to that of $K(G)$. Now, let us look at the right hand side. Then, first of all, $X(G)$ is computable (it is nothing but the p'-part of $G/[G, G]$ where $[G, G]$ is the commutator subgroup of G. Then what about the second term $K(\overline{G})$. Since we can assume that $H \neq 1$, we are able to use inductive argument. So, it works!

2. Okuyama's theorem

The author would like to introduce a theorem of Okuyama in 1981 with a proof, which showed up only in Japanese [12, Theorem 1], because the author hopes/believes that Okuyama's theorem may/should be useful even for understanding endo-trivial modules and also endo-permutation modules and so on. Who knows? Okuyama's theorem here makes Brandt's result [1, Theorem B] more precise. For a finite dimentional k-algebra A and $I \subseteq A$ we denote by $\mathrm{Ann}_A(I)$ the (right) annihilator of I in A, namely, $\mathrm{Ann}_A(I) := \{a \in A \mid Ia = \{0\}\}$, and we denote by $Z(A)$ the center of A.

Theorem 2.1 (Okuyama (1981) [12]). *Let B be a block algebra of kG of a finite group G over an algebraically closed field k of characteristic $p > 0$ (actually, this theorem holds even for an arbitrary finite dimensional* symmetric *k-algebra B). Further, let $\ell(B)$ be the number of non-isomorphic simple right B-modules, and let $S_1, \cdots, S_{\ell(B)}$ be the all non-isomorphic simple right B-modules. Then, it holds*

$$\dim_k[\mathrm{Ann}_B(J(B)^2) \cap Z(B)] \;=\; \ell(B) + \sum_{i=1}^{\ell(B)} \dim_k[\mathrm{Ext}_B^1(S_i,\, S_i)].$$

Proof. (Okuyama in [12, Theorem 1]). Set $\ell := \ell(B)$, and let $e_1, \cdots, e_\ell$ be the set of all primitive idempotents of B such that $P_i := P(S_i) := e_iB$ is the projective cover of S_i for $i = 1, \cdots, \ell$. Set $e := e_1 + \cdots + e_\ell$, and $A := eBe$, and hence A is the basic ring (algebra) of B, and A is a finite dimensional symmetric k-algebra (recall that A and B are Morita equivalent). Further, set $J := J(B)$ the Jacobson radical of B.

Step 1. $\dim_k[\mathrm{Ext}_B^1(S_i, S_i)] = \dim_k(e_iJe_i/e_iJ^2e_i)$.

Step 2. If C is a finite dimensional k-algebra such that $C/J(C) \cong k$, then $\mathrm{Ann}_C(J(C)^2) \subseteq Z(C)$.

Proof of Step 2. Follows by [10, Lines 7–8 of the proof of Lemma 2].

Step 3. For any two-sided ideal I of B, we have that $\dim_k[\mathrm{Ann}_B(I) \cap Z(B)] = \dim_k[\mathrm{Ann}_A(eJe) \cap Z(A)]$.

Proof of Step 3. Recall that B and A are Morita equivalent.

Step 4. If $0 \neq f = f^2 \in C$ for a finite-dimensional symmetric k-algebra C, then fCf is again symmetric.

Proof of Step 4. Take a similar way to prove that a Morita equivalence preserves being symmetric, though fCf is not necessarily Morita equivalent to C, of course.

Step 5. Set

$$T := \sum_i e_iJ^2e_i + \sum_{i \neq j} e_iJe_j.$$

Then T is a two-sided ideal of A.

Step 6. $\mathrm{Ann}_A(T) \;=\; \mathrm{Ann}_A[(e{\cdot}J(A)^2{\cdot}e) \cap Z(A)]$. □

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