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# A family of the Seiberg-Witten equations and configurations of embedded surfaces in 4-manifolds

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**Abstract**

In this paper we consider constraints on configurations consisting of finitely many surfaces embedded in an oriented closed 4-manifold and its genera. A study of a family of the Seiberg-Witten equations, namely, the "high-dimensional" wall crossing phenomenon plays a prominent role in our method. The results in § 2 are based on [3].

## 1 The adjunction inequalities and configurations of embedded surfaces with positive intersection numbers

It is a fundamental problem in 4-dimensional topology to find a lower bound for the genus of an embedded surface which represents a given second homology class of a 4-dimensional manifold. This problem is often called the minimal genus problem. For example, the minimal genus problem for $\mathbb{CP}^2$ is called the Thom conjecture and this is one of the most classical problem in 4-dimensional topology.

Gauge theory provides strong tools to answer the minimal genus problem and a certain type of inequality for genus obtained by gauge theory is often called the *adjunction inequality.* Here we explain this terminology. Let $X$ be a complex surface and $C$ a smooth algebraic curve in $X$. Then it is easy to see that the Euler characteristic $\chi(C) = 2-2g(C)$ satisfies the equality

$$-\chi(C) = c_1(X) \cdot C + C^2,$$

where $\cdot$ means the intersection number and $C^2 = C \cdot C$. This equality is called the *adjunction formula.* When $X$ is a $C^\infty$ 4-manifold and a surface $\Sigma$ is embedded to $X$ in $C^\infty$ sense, then, in general, we cannot determine $g(\Sigma)$ by the homology class $[\Sigma]$. However, a surprising observation in Kronheimer-Mrowka [4] is that, for a suitable characteristic $c \in H^2(X;\mathbb{Z})$, one can expect the inequality

$$-\chi(\Sigma) \geq |c \cdot [\Sigma]| + [\Sigma]^2.$$

This type of inequality is called the adjunction inequality.

After Seiberg-Witten theory appeared, it is successfully used to study the minimal genus problem. Kronheimer-Mrowka [5] proved the Thom conjecture by using the Seiberg-Witten equations. They gave the wall crossing formula for the Seiberg-Witten invariants for 4-manifolds with $b^+ = 1$ and use this formula for the proof of the Thom conjecture. Here $b^+(X)$ is the maximal dimension of positive definite subspaces of $H^2(X;\mathbb{R})$ with respect to the intersection form of $X$.

The direct consequence of arguments in Kronheimer-Mrowka [5] is that the strong relation between the Seiberg-Witten invariants and the adjunction inequalities. For an oriented, closed smooth 4-manifold $X$ with $b^+(X) \geq 2$ and a spin c structure $\mathfrak{s}$ on $X$, let $\mathrm{SW}_X(\mathfrak{s}) \in \mathbb{Z}$ denote the Seiberg-Witten invariant of $X$ with respect to $\mathfrak{s}$. (More precisely, we have to fix a homology orientation of $X$ to determine the sign of $\mathrm{SW}_X(\mathfrak{s})$. Here a homology orientation of $X$ means an orientation of $H^0(X;\mathbb{R}) \oplus H^1(X;\mathbb{R}) \oplus H^+(X;\mathbb{R})$, where $H^+(X;\mathbb{R})$ is a $b^+$-dimensional positive definite subspace of $H^2(X;\mathbb{R})$.) In this paper, we consider only surfaces which are oriented, closed and connected. Put

$$\chi^-(\Sigma) := \max\{-\chi(\Sigma), 0\}$$

for a surface $\Sigma$.

**Theorem 1.1.** *(Kronheimer-Mrowka [5]) Let $X$ be an oriented, closed smooth 4-manifold with $b^+(X) \geq 2$ and $\Sigma$ be a surface embedded in $X$ with $[\Sigma]^2 \geq 0$. Let $\mathfrak{s}$ be a spin c structure with $\mathrm{SW}_X(\mathfrak{s}) \neq 0$. Then, the inequality*

$$\chi^-(\Sigma) \geq |c_1(\mathfrak{s}) \cdot [\Sigma]| + [\Sigma]^2$$

*holds.*

However, there are many 4-manifolds whose Seiberg-Witten invariants vanish. For example, the Seiberg-Witten invariants for 4-manifolds obtained by connected sum vanish under mild assumptions on $b^+$: let $X_i$ $(i = 1, 2)$ be oriented, closed 4-manifolds with $b^+(X_i) \geq 1$, then $\mathrm{SW}_{X_1 \# X_2}(\mathfrak{s}) = 0$ for any spin c structure $\mathfrak{s}$ on $X_1 \# X_2$. Therefore we cannot use the Seiberg-Witten invariant to show the adjunction inequalities for such 4-manifolds. In fact, Nouh [7] proved that the adjunction inequality for a surface in $\mathbb{CP}^2 \# \mathbb{CP}^2$ does not hold in general. Nouh's result shows that, for such 4-manifolds, not only does one cannot use the Seiberg-Witten invariants, but also one may find examples of surfaces which violate the adjunction inequalities.

Thus a natural question is when one can show the adjunction inequality for 4-manifolds whose Seiberg-Witten invariants vanish. In Strle's paper [12], he showed the following adjunction inequalities for disjoint embedded surfaces with positive self-intersection numbers. Let $\mathrm{sign}(X)$ denote the signature of $X$.

**Theorem 1.2.** *(Strle [12]) Let $X$ be an oriented closed smooth 4-manifold with $b_1(X) = 0$ and $c \in H^2(X;\mathbb{Z})$ be a characteristic with $c^2 > \operatorname{sign}(X)$.*

**(A)** *In the case of $b^+(X) = 1$, let $\alpha \in H_2(X;\mathbb{Z})$ be a homology class with $\alpha^2 > 0$ and $\Sigma \subset X$ be an embedded surface with $[\Sigma] = \alpha$. Then the inequality*

$$-\chi(\Sigma) \geq -|c \cdot \alpha| + \alpha^2 \tag{1}$$

*holds.*

**(B)** *In the case of $b^+(X) > 1$, let $\alpha_1, \dots, \alpha_{b^+} \in H_2(X;\mathbb{Z})$ be homology classes with $\alpha_i^2 > 0$ $(1 \leq i \leq b^+)$ and $\Sigma_1, \dots, \Sigma_{b^+} \subset X$ be embedded surfaces with $[\Sigma_i] = \alpha_i$. Assume that $\Sigma_1, \dots, \Sigma_{b^+}$ are disjoint. Then the inequality*

$$-\chi(\Sigma_i) \geq -|c \cdot \alpha_i| + \alpha_i^2 \tag{2}$$

*holds at least one $i \in \{1, \dots, b^+\}$.*

Note that Strle's theorem can be applied to 4-manifolds whose Seiberg-Witten invariants vanish. His result suggests that one can expect some constraints on *configurations* of embedded surfaces in a 4-manifold even when its Seiberg-Witten invariant vanishes.

In the rest of this paper, we will explain two constraints on configurations of embedded surfaces with self-intersection number zero. Our constraints can be also applied to 4-manifolds whose Seiberg-Witten invariants vanish. While Strle's proof stands on a study of the moduli space of the Seiberg-Witten equations on a 4-manifold with cylindrical ends, our method is to study only compact 4-manifolds and use the "high-dimensional" wall crossing phenomena. In Seiberg-Witten theory, the wall crossing phenomena are usually studied in the case when $b^+ = 1$. Li-Liu [6] gave its generalizations for any $b^+$. While in the usual wall crossing phenomena a 1-parameter family of the Seiberg-Witten equations is the main object, in Li-Liu [6]'s situation a $b^+$-parameter family is it. We call Li-Liu [6]'s generalizations the *high-dimensional wall crossing phenomena.* To use the high-dimensional wall crossing phenomena for constraints on configurations, in [3] the author gave a sufficient condition on a certain $b^+$-parameter family to catch the high-dimensional wall crossing phenomenon in terms of embedded surfaces. This is the foundation of the proof of the results in this paper.

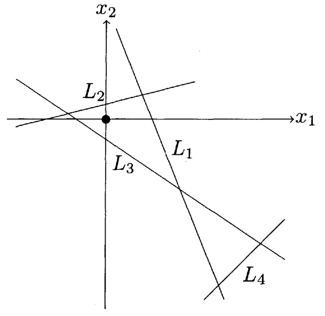


Figure 1: An example of a quadrilateral including the origin of $\mathbb{R}^2$

## 2 Constraints on configurations obtained by the high-dimensional wall crossing phenomena

In this section, we explain a special case of the result obtained by the high-dimensional wall crossing phenomena, namely, the adjunction inequalities for configurations of surfaces in $2\mathbb{CP}^2\#n(-\mathbb{CP}^2)$. For a generalization of this result, see [3]. (From the general form of our results, we can give a simple alternative proof of Strle's results: Thoerem 1.2.) We will often use the identification $H^2(\cdot) \simeq H_2(\cdot)$ obtained by Poincaré duality.

Let consider the 4-manifold

$$X = 2\mathbb{CP}^2\#n(-\mathbb{CP}^2) = (\#_{p=1}^2\mathbb{CP}_p^2)\#(\#_{q=1}^n(-\mathbb{CP}_q^2)) \quad (n > 0).$$

Let $H_p$ denote a generator of $H_2(\mathbb{CP}_p^2;\mathbb{Z})$ and $E_q$ a generator of $H_2(-\mathbb{CP}_q^2;\mathbb{Z})$. For a cohomology class $c \in H^2(X;\mathbb{Z})$ and homology classes $\alpha_1,\dots,\alpha_4 \in H_2(X;\mathbb{Z})$, we define a line $L_i$ ($i = 1,\dots,4$) in $\mathbb{R}^2$ by

$$L_i := \left\{\, (x_1,x_2) \in \mathbb{R}^2 \;\middle|\; (x_1H_1 + x_2H_2)\cdot\alpha_i = c\cdot\alpha_i \,\right\}. \tag{3}$$

For these lines, we will consider the condition that (parts of) lines $L_1,\dots,L_4$ form sides of a "quadrilateral" by this order. Here we use the word "quadrilateral" in the following sense. Let $L_1',\dots,L_4'$ be four line segments in $\mathbb{R}^2$. If an orientation of $L_i'$ is given, we can define the initial point $I(L_i')$ and the terminal point $T(L_i')$ of $L_i'$. We call the ordered set $(L_1',\dots,L_4')$ a *quadrilateral* when there exists an orientation for each $L_i'$ such that $T(L_i') = I(L_{i+1}')$ holds for each $i \in \mathbb{Z}/4$. (We admit a point as a line segment. Thus a "triangle" also a quadrilateral in our definition.)

**Theorem 2.1.** *For the* 4*-manifold*

$$X = 2\mathbb{CP}^2\#n(-\mathbb{CP}^2) \quad (n > 0),$$

*let $c \in H^2(X;\mathbb{Z})$ be a characteristic with $c^2 > \operatorname{sign}(X)$ and $\alpha_1, \ldots, \alpha_4 \in H_2(X;\mathbb{Z})$ be homology classes with $\alpha_i^2 = 0$ $(i = 1, \ldots, 4)$. Let $\Sigma_1, \ldots, \Sigma_4 \subset X$ be embedded surfaces with $[\Sigma_i] = \alpha_i$. Assume that $\alpha_i$ and $c$ satisfy the following (A) and $\Sigma_i$ satisfy (B) :*

**(A)** *The lines $L_1, \ldots, L_4$ form sides of a quadrilateral including the origin of $\mathbb{R}^2$ by this order.*

**(B)** $\Sigma_i \cap \Sigma_{i+1} = \emptyset$ $(i \in \mathbb{Z}/4)$.

*Then, the inequality*

$$-\chi(\Sigma_i) \geq |c \cdot \alpha_i|$$

*holds for at least one $i \in \{1, \ldots, 4\}$.*

*Example* 2.2. Let $X = 2\mathbb{CP}^2\#19(-\mathbb{CP}^2)$, $c = H_1 - 3H_2 - \sum_{q=1}^{19} E_q$. The homology classes

$$\begin{aligned}
\alpha_1 &:= 3H_1 + 3H_2 - \textstyle\sum_{q=1}^{3} E_q + \sum_{q=4}^{10} E_q + 2(E_{11} + E_{12}),\\
\alpha_2 &:= -3H_1 + 2H_2 + \textstyle\sum_{q=1}^{3} E_q + \sum_{q=13}^{18} E_q + 2E_{19}\\
\alpha_3 &:= H_1 + H_2 + E_{12} - E_{13},\\
\alpha_4 &:= 2H_1 - H_2 - \textstyle\sum_{q=1}^{3} E_q - E_{13} + E_{14}
\end{aligned}$$

satisfy that $\alpha_i^2 = 0$ and $\alpha_i \cdot \alpha_{i+1} = 0$ $(i \in \mathbb{Z}/4)$. It is easy to check that these $\alpha_i$ and $c$ satisfy (A) in Theorem 2.1. (See Figure 2.)

Thus, by Theorem 2.1, for embedded surfaces $\Sigma_i$ satisfying $[\Sigma_i] = \alpha_i$, if they also satisfy that $\Sigma_i \cap \Sigma_{i+1} = \emptyset$ $(i \in \mathbb{Z}/4)$,

$$-\chi(\Sigma_i) \geq |c \cdot \alpha_i|$$

holds for at least one $i \in \{1, \ldots, 4\}$. This means that the genus bound

$$g(\Sigma_i) \geq 2$$

holds for at least one $i \in \{1, \ldots, 4\}$.

Under certain assumptions on geometric intersections with embedded surfaces violating the adjunction inequalities, we can derive the adjunction inequality for a single surface. Before giving an example, we mention an easy method to make surfaces with small genera. For a homology class $\beta = aH_2 + \sum_{q=1}^{n} b_q E_q \in H_2(\mathbb{CP}_2^2\#n(-\mathbb{CP}^2);\mathbb{Z})$, considering algebraic curves $C \subset \mathbb{CP}_2^2$ and $C_q \subset \mathbb{CP}_q^2$ and reversing orientations of them if we need, we can easily construct the surface $S \subset \mathbb{CP}_2^2\#n(-\mathbb{CP}^2)$ by

$$S := C\#(\#_{q=1}^{n} C_q) \tag{4}$$

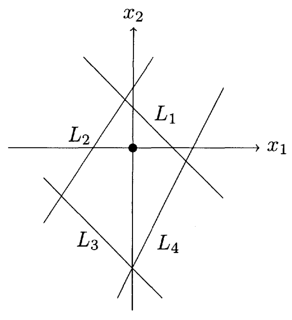


Figure 2: $L_1, \ldots, L_4$ in Example 2.2

satisfing

$$[S] = \beta, \quad g(S) = \frac{(|a|-1)(|a|-2)}{2} + \sum_{q=1}^{n} \frac{(|b_q|-1)(|b_q|-2)}{2}.$$

For example, on the characteristic $H_2 - \sum_{q=1}^{n} E_q$, such naive construction is sufficient to give many examples of surfaces which violate the adjunction inequality on this characteristic.

*Example* 2.3. Let give natural numbers $d_1 \geq 4,\ d_2 \geq 1,\ d_3 \geq 2$ and $n \geq d_1^2 + \max\{d_2^2, d_3^2\}$. For $X = 2\mathbb{CP}^2 \# n(-\mathbb{CP}^2)$, let consider the homology classes

$$\alpha := d_1 H_1 - \sum_{q=1}^{d_1^2} E_q,$$

$$\beta_1 := d_2 H_2 + \sum_{q=d_1^2+1}^{d_1^2+d_2^2} E_q,$$

$$\beta_2 := d_3 H_2 - \sum_{q=d_1^2+1}^{d_1^2+d_3^2} E_q.$$

Let $S_i \subset \mathbb{CP}_2^2 \# n(-\mathbb{CP}^2) \setminus (\text{disk}) \subset X$ be surfaces with $[S_i] = \beta_i$ obtained as (4). For an embedded surface $\Sigma \subset X$ satisfying $[\Sigma] = \alpha$ and $\Sigma \cap S_i = \emptyset$ $(i = 1, 2)$, we can show that

$$g(\Sigma) \geq \frac{(d_1-1)(d_1-2)}{2} \tag{5}$$

from Theorem 2.1.

By the adjunction formula for $\mathbb{CP}_1^2 \# n(-\mathbb{CP}^2)$, the homology class $\alpha$ can be represented by a surface $\Sigma$ of genus $(d_1-1)(d_1-2)/2$ satisfying $\Sigma \cap S_i = \emptyset$. Thus the inequality (5) is the optimal bound under the condition $\Sigma \cap S_i = \emptyset$.

## 3 Constraints on configurations obtained by the high-dimensional wall crossing phenomena and the gluing technique

To obtain the results in § 2, the author used the high-dimensional wall crossing phenomena in [3]. On the other hand, Ruberman ([8], [9] and [10]) studied the combination of the usual (i.e. $b^+ = 1$) wall crossing phenomena and the *gluing* technique. The gluing technique is a deep analytical tool in gauge theory. A typical application of the gluing technique is the proof the blowup formula, which describes the behavior of the Seiberg-Witten invariants (or Donaldson invariants) under blowups of 4-manifolds. Ruberman's arguments can be regarded as a 1-parameter version of the proof of the blowup formula. Namely, Ruberman considered the gluing argument as the proof of the blowup formula for the 1-parameter family to use the wall crossing argument. This argument can be generalized to higher-dimensional families to use the high-dimensional wall crossing argument. This generalization gives new constraints on configurations. In this section, we give the formulation of these results.

To describe the results, for a spin c 4-manifold, we introduce an abstract simplicial complex which consists of surfaces violating the adjunction inequalities. Before the definition of this simplicial complex, we need an "ambient" simplicial complex. This ambient simplicial complex was introduced to the author by Mikio Furuta.

**Definition 3.1.** (Furuta) For an oriented, closed 4-manifold $X$, we define the abstract simplicial complex $\mathcal{K} = \mathcal{K}(X)$ as follows:

- The set of vertices $V(\mathcal{K})$ is given as the set of smooth embeddings of surfaces with self-intersection number zero:

  $$V(\mathcal{K}) := \left\{ \Sigma \hookrightarrow X \mid [\Sigma]^2 = 0 \right\}.$$

  Here we consider only oriented, closed, connected surfaces. We denote each vertex $(\Sigma \hookrightarrow X) \in V(\mathcal{K})$ briefly by $\Sigma$.

- For $k \geq 1$, a collection of $(k+1)$ vertices $\Sigma_0, \dots, \Sigma_k \in V(\mathcal{K})$ spans a $k$-simplex if and only if $\Sigma_0, \dots, \Sigma_k$ are disjoint.

We call $\mathcal{K}$ the *complex of surfaces* of $X$.

Of course, any abstract simplicial complex is a CW complex, thus $\mathcal{K}$ is a CW complex although $\mathcal{K}$ is a huge space. We topologize $\mathcal{K}$ as a CW complex, i.e., by the weak topology.

*Remark* 3.2. The complex of surfaces is a 4-dimensional analog of the complex of curves due to Harvey [2] in 2-dimensional topology. In the above definition of the complex of surfaces, we do not consider the isotopy classes of embeddings of surfaces. On the other hand, in the same way of the definition of the complex of curves, one can define an abstract simplicial complex whose vertices are the isotopy classes of embeddings of surfaces and whose simplices are spanned by collections of such isotopy classes which can be realized disjointly. However, to give certain applications to the adjunction inequalities using Seiberg-Witten theory, the first definition of the complex of surfaces might be appropriate.

Here we consider the special phenomena in 4-dimensional topology, namely, the adjunction inequalities.

**Definition 3.3.** Let $\mathfrak{s}$ be a spin c structure on $X$. Then, the *complex of surfaces violating the adjunction inequality* $\mathcal{K}_V = \mathcal{K}_V(X, \mathfrak{s})$ is the subcomplex of $\mathcal{K}(X)$ defined as the set of vertices is given by

$$V(\mathcal{K}_V) := \{\, \Sigma \in V(\mathcal{K}) \mid \chi^-(\Sigma) < |c_1(\mathfrak{s}) \cdot [\Sigma]| \,\}$$

and having the induced structure of an abstract simplicial complex from $\mathcal{K}$. Namely, $\Sigma_0, \ldots, \Sigma_k \in V(\mathcal{K}_V)$ spans a $k$-simplex if and only if $\Sigma_0, \ldots, \Sigma_k$ span a $k$-simplex in $\mathcal{K}$.

Let $\mathfrak{s}^+$ be a spin c structure on $\mathbb{CP}^2$ such that $c_1(\mathfrak{s}^+)$ is a generator in $H_2(\mathbb{CP}^2; \mathbb{Z})$ and $\mathfrak{s}^-$ be a spin c structure on $-\mathbb{CP}^2$ such that $c_1(\mathfrak{s}^-)$ is a generator in $H_2(-\mathbb{CP}^2; \mathbb{Z})$. (We have two choices of each $\mathfrak{s}^+$ and $\mathfrak{s}^-$.)

The main result in this section is the following statement.

**Theorem 3.4.** *Let $(X, \mathfrak{s}_X)$ be an oriented, closed spin c 4-manifold with $b^+(X) \geq 2$. Suppose that $\mathrm{SW}_X(\mathfrak{s}) \neq 0$ and $d(\mathfrak{s}_X) = 0$, where $d(\mathfrak{s}) := (c_1(\mathfrak{s})^2 - 2\chi(X) - 3\mathrm{sign}(X))/4$. Put*

$$\begin{aligned} Z &:= X \# m\mathbb{CP}^2 \# n(-\mathbb{CP}^2) \quad (m \geq 1,\ n \geq 4m), \\ \mathfrak{s}_Z &:= \mathfrak{s}_X \# (\#_{p=1}^m \mathfrak{s}_p^+) \# (\#_{q=1}^n \mathfrak{s}_q^-). \end{aligned}$$

*Then*

$$\tilde{H}_{m-1}(\mathcal{K}_V(Z, \mathfrak{s}_Z); \mathbb{Z}) \neq 0$$

*holds.*

*Remark* 3.5. More precisely, we can give a concrete non-trivial element of $\tilde{H}_{m-1}(\mathcal{K}_V(Z, \mathfrak{s}_Z); \mathbb{Z})$.

To prove Theorem 3.4, we define a group homomorphism

$$\mathrm{SW} = \mathrm{SW}_{Z, \mathfrak{s}_Z} : \mathcal{H}_*(Z, \mathfrak{s}_Z) \to \mathbb{Z}$$

and show that this map is non-tivial. Here $\mathcal{H}_*(Z, \mathfrak{s}_Z)$ is a certain subgroup of $\tilde{H}_*(\mathcal{K}_V(Z, \mathfrak{s}_Z); \mathbb{Z})$. The map SW is defined, roughly speaking, by "counting" the parametrized moduli space of the Seiberg-Witten equations for parameter space introduced in [3]. (To justify the

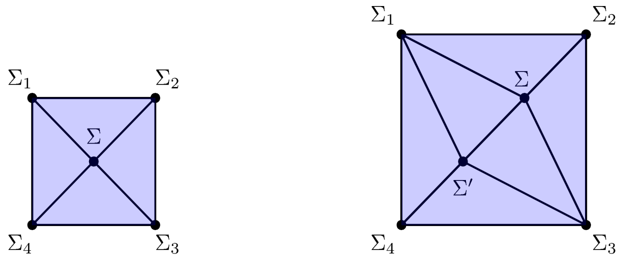


Figure 3: Some boundings for $\gamma$

counting argument, we use Ruan's virtual neighborhood technique and its family version.) This parameter space is obtained by stretching neighborhoods of embedded surfaces which forms the element of $\mathcal{H}_*$. This construction of the parameter space is a slight generalization of one due to Frøyshov [1]. The proof of the non-triviality of this map is given by the combination of the high-dimensional wall crossing phenomena and the gluing technique.

Here we explain why a non-trivial element of $\tilde{H}_*(\mathcal{K}_V(Z, \mathfrak{s}_Z); \mathbb{Z})$ is useful to give constraints on configurations of embedded surfaces and its genera. For example, let $\Sigma_1, \dots, \Sigma_4 \in V(\mathcal{K}_V)$ be embedded surfaces with $\Sigma_i \cap \Sigma_{i+1} = \emptyset$ $(i \in \mathbb{Z}/4)$. Then the collection $\{\langle \Sigma_i, \Sigma_{i+1} \rangle\}_{i \in \mathbb{Z}/4}$ forms a 1-cycle $\gamma = \langle \Sigma_1, \Sigma_2 \rangle + \dots + \langle \Sigma_4, \Sigma_1 \rangle$ in $\mathcal{K}_V$. Assume that $[\gamma] \neq 0$ in $H_1(\mathcal{K}_V; \mathbb{Z})$. For $\Sigma \in V(\mathcal{K})$ with $\Sigma \cap \Sigma_i = \emptyset$ $(i \in \mathbb{Z}/4)$, the collection $\{\langle \Sigma, \Sigma_i, \Sigma_{i+1} \rangle\}_{i \in \mathbb{Z}/4}$ can be regarded as the "cone" of this 1-cycle. (See Figure 3.) If $\Sigma \in V(\mathcal{K}_V)$ holds, this "cone" is contained in $\mathcal{K}_V$, thus $[\gamma] = 0$ in $H_1(\mathcal{K}_V; \mathbb{Z})$. This contradicts our assumption, therefore we have $\Sigma \notin V(\mathcal{K}_V)$. In conclusion, we have the adjunction inequality for an embedded surface $\Sigma$ with $\Sigma \cap \Sigma_i = \emptyset$ $(i \in \mathbb{Z}/4)$.

In the same way, for embedded surface $\Sigma, \Sigma' \in V(\mathcal{K})$, if $\Sigma$ and $\Sigma'$ satisfies

$$\Sigma \cap \Sigma_i = \emptyset \ (i = 1, 2, 3), \ \Sigma' \cap \Sigma_i = \emptyset \ (i = 1, 3, 4), \ \Sigma \cap \Sigma' = \emptyset,$$

then $\Sigma \notin V(\mathcal{K}_V)$ or $\Sigma' \notin V(\mathcal{K}_V)$ holds. Namely, the adjuction inequality holds for $\Sigma$ or $\Sigma'$. (See Figure 3.)

As in these example, if we find a non-trivial element in $H_*(\mathcal{K}_V; \mathbb{Z})$, we obtain constraints on genera for infinitly many configurations of surfaces.

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