数理解析研究所講究録
第2004巻 2016年 47-56

HOLOMORPHIC CURVE TECHNIQUE IN SYMPLECTIC GEOMETRY

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1. Introduction

In the middle of 1980's, Gromov introduced pseudo-holomorphic curves in symplectic manifolds and derived many significant results [7]. For example, (1) (non-) squeezing theorem[1], (2) non-existence of embedded exact Lagrangian submanifolds in a symplectic vector space, (3) description of closed symplectic four-manifolds containing a symplectically embedded sphere of non-negative self intersection number (McDuff developed the theory after Gromov's seminal work), (4) homotopy type of the symplectic diffeomorphism group of $(S^2, \omega) \times (S^2, \omega)$ and the fact that the fundamental group of that of $(S^2, \omega) \times (S^2, c\omega)$, $c > 1$, contains an element of infinite order. Around the same time, Conley and Zehnder proved Arnold's conjecture for fixed points of Hamiltonian diffeomoprhisms on tori [2]. Fixed points of a Hamiltonian diffeomorphism correspond to 1-periodic solutions of the corresponding time-dependent Hamiltonian system, which can be captured as critical points of a certain functional on the loop space (the least action principle). Conley and Zehnder used finite dimensional approximation of the functional, hence reduced the problem in a finite dimensional setting. A formal computation leads that the gradient flow lines of the functional can be thought of solutions of Cauchy-Riemann equation perturbed by Hamiltonian term. Shortly after these works, combining the holomorphic curve technique and the variational approach, Floer initiated "$\frac{\infty}{2}$-dimensional" analog of Morse-Novikov theory, which is nowadays called Floer theory [3]. The original motivation is Arnold's conjecture for fixed points of Hamiltonian diffeomorphisms. It related to a question on Lagrangian intersection closely. Since then, Floer theory has been developed in various direction including Heegaard Floer theory presented in Tange's article in this proceedings. In this note, we would like to present a glimpse of the method of holomorphic curves without going into details following the lecture in the workshop.

[1]Gromov called this theorem as squeezing theorem. However, it states. somehow, the ball cannot be "squeezed" to a symplectic cyclinder of smaller width. So it is now often called non-squeezing theorem.

2. A little preliminary

In this section, we collect several notion, definitions in symplectic geometry. A symplectic structure on a manifold X is to equip X with a closed non-degenerate 2-form ω (symplectic form). Namely, $d\omega = 0$ and ω induces the following isomorphism of vector bundles:

$$v \in TX \mapsto i(v)\omega \in T^*X.$$

Thus ω also induces the one-to-one correspondence between vector fields and 1-forms. The most basic example of symplectic manifolds is the symplectic vector space. Let $(x_1, y_1, \dots, x_n, y_n)$ be linear coordiinates on $\mathbb{R}^{2n}$. Write $\omega_0 = \sum_{i=1}^n dx_i \wedge dy_i$. Clearly ω_0 is a symplectic form on $\mathbb{R}^{2n}$. Darboux's theorem guarantees that any symplectic manifold is locally diffeomorphic to a symplectic vector space. In other words, at any point on (X, ω), we can take local coordinates $(x_1, y_1, \dots, x_n, y_n)$, in which ω is written as ω_0.

Other examples include oriented surfaces equipped with area form, cotangent bundle (with a standard symplectic form "$dp \wedge dq$"), Kähler manifolds, in particular, complex projective spaces.

For a smooth function h on X, we define the Hamiltonian vector field X_h associated with h by $i(X_h)\omega = dh$. On the symplectic vector space, the Hamiltonian vector field associated with h is given by

$$X_h = \sum_{i=1}^{n} \left(\frac{\partial h}{\partial y_i}\frac{\partial}{\partial x_i} - \frac{\partial h}{\partial x_i}\frac{\partial}{\partial y_i}\right).$$

By Cartan's formula, X_h satisfies $\mathcal{L}_{X_h}\omega = 0$.

A diffeomorphism ψ of X is called a symplectic diffeomorphism (symplectomorphism), if $\psi^*\omega = \omega$. There is a specific class of symplectic diffeomorphism called Hamiltonian diffeomorphisms. Let H be a smooth function on $[0,1] \times X$, (with compact support). We set $h_t = H(t, \bullet)$. Integrating $\{X_{h_t}\}$, we obtain an isotopy φ_H^t with $\varphi_H^0 = id_X$. A diffeomorphism ψ of X is called a Hamiltonian diffeomorphism, if there exists H such that $\psi = \varphi_H^1$. A diffeomorphism on an m-dimensional manifold is locally expressed by m functions of m-variables. A Hamiltonian diffeomorphism ψ on a $2n$-dimensional symplectic manifold is, in a sense, described by a function on $2n$-variables. (For example, if ψ is close to the identity, it is locally described by a so-called *generating function*. Then the fixed points are critical points of the generating function.)

A submanifold S in (X, ω) is called symplectic, if the restriction of ω to S is a symplectic form on S. S is called isotropic, if the restriction of ω to S vanishes everywhere on S, in other words, $TS \subset (TS)^{\perp_\omega}$. Here

$$(T_pS)^{\perp_\omega} = \{v \in T_pX \mid \omega(v, w) = 0, \ \forall w \in T_pS\}.$$

S is called coisotropic, if $(TS)^{\perp_\omega} \subset TS$. For an isotropic (resp. coisotropic) submanifold, we have $\dim S \leq$ (resp. $\geq$) $\frac{1}{2}\dim X$. A particularly important class of submanifolds is that of Lagrangian submanifolds, which are isotropic and coisotropic. Typical examples of Lagrangian submanifolds are the graph of closed 1-forms in the cotangent bundle of a smooth manifold M, the conormal bundle of a submanifold of M in the cotangent bundle of M, the real part of non-singular algebraic variety definied over $\mathbb{R}$, the graph of a symplectic diffeomorphism of (X, ω) considered as a submanifold in $(X \times X, -\omega \oplus \omega)$.

Since the group of linear symplectic transformations (linear isomorphism on a symplectic vector space preserving the symplectic structure) contains the unitary group as a maximal compact subgroup, the structure group of the tangent bundle of a symplectic manifold can be reduced to the unitary group. In other words, there exists an almost complex strcutre J (an endomrophism of the tangent bundle with $J^2 = -id$) such that $g_J(\bullet, \bullet) := \omega(\bullet, J\bullet)$ is a Riemannian metric on X (almost complex structure compatible with ω). Moreover, the space of almost complex structures compatible with ω is contractible. A map f between almost complex manifolds (Σ, j) and (X, J) is called J-holomorphic (or simply holomorphic), if the differential df is complex linear with respect to j and J, i.e., $J \circ df = df \circ j$. In contrast to the fact that J-holomorphic functions are rare, there are at least locally plenty of holomorphic curves. Non-integrability of J, measured by Nijenhuis tensor, gives restrictions for J-holomorphic submanifolds. In the case of real two-dimension (complex one-dimension), Nijenhuis tensor vanishes, hence no restriction. In particular, we call (the image of) a holomorphic map from the Riemann sphere a holomorphic sphere. Moreover, the deformation theory of holomorphic maps from closed Riemann surfaces (resp. compact Riemann surface with Lagrangian boundary condition) is controlled by two-step elliptic complex (elliptic operator). The compactness of the moduli space is also estabilished in works starting with [7].

3. HOLOMORPHIC CURVES

In this section, we discuss a couple of results in [7] and try to give rough ideas of proofs.

3.1. Non-squeezing theorem.

Non-squeezing theorem is a manifestation of symplectic rigidity. We briefly present its statement and implication followed by a flavor of the proof. Define the ball of radius R and a cylinder of width R by

$$B^{2n}(R) = \{(x_i, y_i) \in \mathbb{R}^{2n} | \sum_{i=1}^{n}(x_i^2 + y_i^2) < R\},$$

$$Z^{2n}(R) = \{(x_i, y_i) \in \mathbb{R}^{2n} | x_1^2 + y_1^2 < R\} = B^2(R) \times \mathbb{R}^{2n-2}.$$

Theorem.(non-squeezing theorem) Let $\psi : B^{2n}(R) \to Z^{2n}(R')$ be a symplectic embedding, i.e., an embedding such that $\psi^*\omega_0|_{Z^{2n}(R')} = \omega_0|_{B^{2n}(R)}$. Then $R \leq R'$.

Remark. If we replace the condition that ψ preserves the symplectic structure by that ψ preserves the volume form, or if we replace $Z^{2n}(R')$ by another kind of cylinder defined by $x_1^2 + x_2^2 < R'$, the conclusion does not hold. As a corollary of non-squeezing theorem, the following result holds.

Theorem. (C^0-rigidity of symplectic diffeomorphisms) Let ψ_n be a sequence of symplectic diffeomorphisms of $(\mathbb{R}^{2n}, \omega_0)$. If ψ_n converges to a diffeomorphism ψ of $\mathbb{R}^{2n}$ in C^0-topology, ψ is a symplectic diffeomorphism.

A very rough sketch of the proof of non-squeezing theorem goes as follows. Firstly, we take a sufficiently large L such that, after translation in 3-rd, $\dots, 2n$-th coordinates, $\psi(B^{2n}(R))$ is contained in $B^2(R') \times ((L/4, 3L/4))^{2n-2}$. Embed $B^2(R')$ to $S^2(A)$, the sphere of area $A = \pi R'^2$. Then $B^{2n}(R)$ is symplectically embedded in $S^2(A) \times \mathbb{R}^{2n-2}/L\mathbb{Z}^{2n-2}$. Denote by ω the product symplectic structure. Let J_0 be the standard complex structure on $\mathbb{R}^{2n}$ such that $J_0\frac{\partial}{\partial x_i} = \frac{\partial}{\partial y_i}$, $J_0\frac{\partial}{\partial y_i} = -\frac{\partial}{\partial x_i}$. Pick an almost complex structure J compatible with ω which is an extension of $\psi_*(J_0)$ and coincides with J_0 outside $S^2(A) \times [L/4, 3L/4]^{2n-2}$. (Here we regard $[L/4, 3L/4]^{2n-2}$ as a subset of $\mathbb{R}^{2n-2}/L\mathbb{Z}^{2n-2}$. Then we can find J-holomorphic sphere $S^2(A) \times \{p\}$ with $p \notin [L/4, 3L/4]^{2n-2}$. Using the deformation theory and compactness theorem for holomorphic curves, we can show that there is a family of holomorphic spheres in the same homology class sweep the whole space $S^2(A) \times \mathbb{R}^{2n-2}/L\mathbb{Z}^{2n-2}$. In particular, there is a holomorphic sphere S passing through $\psi(O)$, where O is the origin of $B^{2n}(R)$. The monotoniciy formula for the area of holomorphic curves, the symplectic area of S, which is A, is at least πR^2. Hence $\pi R^2 \leq A$. Since $\epsilon > 0$ can be arbitrary small, we obtain $R \leq R'$.

3.2. **Non-existence of exact Lagrangian submanifold in** $(\mathbb{R}^{2n}, \omega_0)$**.** A symplectic manifold (X, ω) is called exact, if ω is an exact 2-form, i.e., $\omega = d\lambda$ for some 1-form λ. A typical example is $(\mathbb{R}^{2n}, \omega_0)$, where $\omega_0 = d(\sum_{i=1}^n x_i dy_i)$, for example. Cotangent bundles with the standard symplectic structure, Liouville domains, which are generalization of convex domains in $(\mathbb{R}^{2n}, \omega_0)$ are also exact symplectic manifolds. A Lagrangian submanifold L in $(X, \omega = d\lambda)$ is called exact, if the restriction of λ is an exact 1-form on L. Note that the Lagrangian condition implies that the restriction of λ to L is a closed 1-form. The following result is also due to Gromov [7].

Theorem. (non-existence of exact Lagrangian submanifolds) Let L be a closed embedded Lagrangian submanifold in $(\mathbb{R}^{2n}, \omega_0)$. Then L is not exact.

In particular, this theorem implies that a closed embedded Lagrangian submanifold in $(\mathbb{R}^{2n}, \omega_0)$ has non-zero first Betti number. For example, the sphere S^n cannot be embedded in $(\mathbb{R}^{2n}, \omega_0)$ as a Lagrangian submanifold for $n > 1$. It contrast to the fact that S^3 can be embedded in $\mathbb{C}^3$ as a totally real submanifold [1], [8] page 193.

Here is a rough sketch of the proof. In order to show that $\lambda|_L$ is not exact, we will find a loop γ in L such that $\int_\gamma \lambda \neq 0$. It suffices to find a non-constant holomorphic disc u in $(\mathbb{R}^{2n}, J_0)$ with boundary on L, since $\int_{\partial D^2} \lambda = \int_{D^2} u^*\omega_0 > 0$, where the first equality is due to Stokes' formula and the second inequality follows from compatibility of J_0 and ω_0 as well as non-constancy of u. The main body of the proof is to find such a holomorphic disc.

Pick and fix $\mathbf{a} \in \mathbb{C}^n$ with $\| \mathbf{a} \| = 1$ and $p_0 \in L$. Consider the pair of $u : (D^2, \partial D^2) \to (\mathbb{C}^n, L)$ and $s \in \mathbb{R}$ satisfying $u(1) = p_0$, $u_*[D^2, \partial D^2] = 0$ in $H_2(\mathbb{C}^n, L; \mathbb{Z})$ and

$$\frac{\partial u}{\partial \overline{z}} := \frac{1}{2}\left(\frac{\partial u}{\partial x} + J_0 \frac{\partial u}{\partial y}\right) = s \cdot \mathbf{a}. \tag{1}$$

When $s = 0$, u must be the constant map to p_0. This is because a holomorphic map u with $\int_{D^2} \omega_0 = 0$ must be a constant map. When s is sufficiently large, there are no solutions u for (1). The reason is the following. If u is a solution of (1), u is harmonic, i.e., $\Delta u = 0$. Hence $\frac{\partial u}{\partial \overline{z}}$ is also harmonic. By the mean value theorem for harmonic functions and Stokes' formula, we find that

$$\frac{\partial u}{\partial \overline{z}}(0) = \frac{1}{\int_{D^2} dxdy} \int_{D^2} \frac{\partial u}{\partial \overline{z}} dxdy = \frac{-\sqrt{-1}}{\int_{D^2} dxdy} \int_{\partial D^2} u dz.$$

Let $D = \max\{\| p \| \, | p \in L\}$. Then the norm of the right hand side is bounded by $2D$, while the left hand side is $s \cdot \mathbf{a}$. Hence, if $s > 2D$, the equation (1) has no solution.

The energy of a map u is defined by $E(u) = \frac{1}{2} \int_{D^2} \| du \|^2 \, dxdy$. By a simple computation, we find that

$$E(u) = \int_{D^2} u^*\omega_0 + \int_{D^2} \| \frac{\partial u}{\partial \overline{z}} \|^2 \, dxdy.$$

Since the first term on the right hand side vanishes and the second term is bounded by $4D^2$ from the above, $E(u)$ is uniformly bounded. Now the compactness argument (Gromov's compactness, removal of singularities, etc.) yields the following. When the energy is uniformly bounded and s_n converges to s_∞, the sequence u_n of solutions for (1) with $s = s_n$ converges to a solution u_∞ of (1) with $s = s_\infty$ away from a finite number of points. Rescaling u_n suitably around these finitely many points, the new sequence converges to a non-constant holomorphic map from either a Riemann sphere or a disc with Lagrangian boundary condition. In our setting, there are no compact holomorphic curves in $\mathbb{C}^n$, hence the only possibility is a holomorphic disc. By the deformation theory and bubbling-off argument of holomorphic curves, if no bubble appear during $0 \leq s \leq 2D$, the space of solutions (s, u) of (1) is a one-dimensional manifold and the constant solution to p_0 at $s = 0$ is deformed to a solution at $s = 2D$, which is a contradiction. Therefore a non-constant holomorphic disc $v : (D^2, \partial D^2) \to (\mathbb{C}^n, L)$ must appear as a bubble between $s = 0$ and $s = 4D^2$.

Remark. Combining the argument above with "figure-eight trick", Gromov derived the following result on Lagrangian intersection. The condition for tameness of a symplectic manifold gurantees good control of holomorphic curves at the end of the symplectic manifold. For example, $(\mathbb{R}^{2n}, \omega_0)$, cotangent bundles equipped with the standard symplectic structure, Liouville domains are tame symplectic manifolds.

Theorem. (persistence of Lagrangian intersection) Let (X, ω) be an exact symplectic manifold. Let L be a closed embedded exact Lagrangian submanifold and ψ a Hamiltonian diffeomorphism of (X, ω). Then we have $L \cap \psi(L) \neq \emptyset$.

4. Naive idea of Lagrangian Floer theory

In this section, we explain Floer's idea very briefly. Let L, L' be closed embedded Lagrangian submanifolds in a closed symplectic manifold (X, ω) such that L and L' intersects transversally. In good situations, we can define a complex $(CF^\bullet(L', L), \delta)$, where $CF^\bullet(L', L)$ is generated by intersection points of L and L', such that the resulting cohomology is invariant under Hamiltonian deformations of L'.

Let us consider the space of paths from L to L':

$$\mathcal{P}(L, L') = \{\gamma : [0,1] \to X | \gamma(0) \in L, \gamma(1) \in L'\}.$$

Define a "1-form" α on $\mathcal{P}(L, L')$ by

$$\alpha_\gamma(\xi) := \int_0^1 \omega(\xi(t), \dot{\gamma}(t)) dt,$$

where $\dot{\gamma}$ is the velocity vector of γ and ξ is a tangent vector of $\mathcal{P}(L, L')$ at γ, in other words, ξ is a section of $\gamma^* TX$ such that $\xi(0)$ (resp. $\xi(1)$) tangents to L (resp. L'). We can see that α is an "exact 1-form" as follows. Fix $\gamma_0 \in \mathcal{P}(L, L')$ and consider γ close to γ_0. Then there is a map $w : [0,1] \times [0,1] \to X$ such that $w(0,t) = \gamma_0(t)$, $w(1,t) = \gamma(t)$, $w(s,0) \in L$ and $w(s,1) \in L'$. For γ sufficiently close to γ_0, we can find such w with the image close to γ_0. Such w is unique up to homotopy respecting the boundary condition above. Then we set

$$\mathcal{A}^{loc}(\gamma) := \int_{[0,1]\times[0,1]} w^* \omega.$$

A direct computation shows that $d\mathcal{A}^{loc} = \alpha$ on a neighborhood of γ_0. Thus α has a local primitive function around any γ_0, hence α is a "closed 1-form". The function $\mathcal{A}^{loc}$ may not be extended to a globally well-defined function on $\mathcal{P}(L, L')$. However, we have a well-defined function $\mathcal{A}$ on a suitalble covering space $\widetilde{\mathcal{P}}(L, L')$ of $\mathcal{P}(L, L')$. (For any closed 1-form η, there exists a covering space on which the pull-back of η becomes exact.) We mimick Morse complex (or Novikov complex of a closed 1-form), although we cannot follow the construction in finite dimension by the following reasons. (1) the gradient flow lines (see below) may not exist passing through a given point (no gradient flow), (2) the Hessian of the function has infinitely many positive and negative subspaces (Morse index must be replaced by something else). For each intersection point $p \in L \cap L'$, we take a path $\Lambda(t)$ in the space of Lagrangian subspaces joining T_pL to T_pL' and can assign Maslov-Viterbo index. (It depends on the path $\Lambda(t)$. Sometimes, we can take these paths at each intersection points in a coherent way.) We use Maslov-Viterbo index in place of Morse index in finite dimensional case.

Using the Riemannian metric g_J, we can define an inner product on the tangent space $T_\gamma\mathcal{P}(L, L')$ by

$$\langle \xi_1, \xi_2 \rangle := \int_0^1 g_J(\xi_1(t), \xi_2(t))dt.$$

With respect to this inner product, a formal computation yields that the "gradient vector field" of $\mathcal{A}$ is given by

$$\mathrm{grad}\mathcal{A}(\gamma) = -J\dot{\gamma}.$$

We can interpret gradient flow trajectories as solutions of the Cauchy-Riemann equation for $u : \mathbb{R} \times [0,1] \to X$ with $u(\mathbb{R} \times \{0\}) \subset L$ and $u(\mathbb{R} \times \{1\}) \subset L'$

$$\frac{\partial u}{\partial \tau}(\tau, t) + J(u)\frac{\partial u}{\partial t}(\tau, t) = 0. \tag{2}$$

Note that the equation (2) may not have a solution for a given γ such that $u(0,t) = \gamma(t)$. In other words, there may not exist a "gradient flow trajectory" passing through γ. A compactness argument for holomorphic curves implies that for a solution u of (2), the energy $E(u)$ is finite if and only if $\lim_{\tau\to\pm infty} u(\tau, t) = p^{\pm}$ for some $p^{\pm} \in L \cap L'$. We call such u a Floer trajectory. We denote by $\widetilde{M}(p^-, p^+)$ the space of solutions of (2) such that $\lim_{\tau\to\pm\infty} u(\tau, t) = p^{\pm}$. Since (2) is invariant under the shift in τ-direction, $\mathbb{R}$ acts on $\widetilde{\mathcal{M}}(p^-, p^+)$. We write its quotient space by $\mathcal{M}(p^-, p^+) = \widetilde{\mathcal{M}}(p^-, p^+)/\mathbb{R}$. Floer coboundary operator $\delta : CF^\bullet(L', L) \to CF^{\bullet+1}(L', L)$ is defined[2]

[2]In order to define δ, it is necessary to make sense of the cardinality of $\mathcal{M}^{\dim=0}$ using perturbation of J (some cases) and/or abstract pertrubation. In general situation, $p \in L \cap L'$ should be replaced by the inverse images in the covering space, just as in Novikov theory for closed 1-forms on a finite dimensional case. We must also take Novikov completion in order to define $CF^\bullet(L', L)$, in general.

by counting number of Floer trajectories joining p^- and p^+

$$\delta p^- = \sum \#_2\mathcal{M}^{\dim=0}(p^-,p^+)p^+.$$

Here $\#_2\mathcal{M}^{\dim=0}(p^-,p^+)$ is the cardinality modulo 2 of 0-dimensional components of $\mathcal{M}(p^-,p^+)$. In general, $\mathcal{M}(p^-,p^+)$ is not canonically oriented. In the case that L and L' are equipped with spin structures (or more generally (L,L') is a relatively spin pair), then $\mathcal{M}(p^-,p^+)$ are oriented in a consistent way [5]. In order to show that $\delta \circ \delta = 0$, we study the ends of 1-dimensional components of $\mathcal{M}(p^-,p^+)$. There are the following possibilities of ends. The first type is splitting into several Floer trajectories. This is similar to limit behavor of gradient trajectories in finite dimensional Morse theory. The second type is bubbling-off of holomorphic discs at the boundary of $\mathbb{R} \times [0,1]$. There may also happen bubbling-off of holomorphic spheres. However it occurs in real codimension 2, while the second type occurs in real codimension 1. Hence the last type can be excluded by pertubation of J (under certain assumption of X and L, L') or abstract perturbation technique. We can see the bubbling-off of holomorphic discs in the following simple local example. Let $L = \mathbb{R}$ and L' the unit circle around the origin in $\mathbb{C}$. Then consider Floer trajectories from -1 to itself. In this case, $\mathcal{M}(-1,-1)$ is an interval $(-1,1)$. The boundary point 1 corresponds to splitting into two Floer trajectories (upper hemidisc as Floer trajectory from -1 to 1 and lower hemidisc as Floer trajectory from 1 to -1). The boundary point -1 corresponds to bubbling-off of a holomorphic disc, i.e., a constant Floer trajectory at -1 with the unit disc as a holomorphic disc bubble. In fact, we can see that $\delta \circ \delta \neq 0$ in this case.

If we can exclude the bubbling-off of holomorphic discs, we can see $\delta \circ \delta = 0$ and obtain Floer cochain complex $(CF^\bullet(L',L),\delta)$. The resulting cohomology denoted by $HF^\bullet(L',L)$ is called Floer cohomology of L, L'. We can also show that $HF^\bullet(L',L) \cong HF^\bullet(\psi(L'),L)$ for a Hamiltonian diffeomorphism ψ such that L and $\psi(L')$ intersects transversally. If we can compute Floer cohomology, we can give a lower bound of the number of intersection points provided they intersect transversally.

Floer [3] realized these lines of ideas under the condition that $\pi_2(X,L) = 0$ and $L' = \psi(L)$ for some Hamiltonian diffeomorphism ψ. We can also study Floer theory for Hamiltonian diffeomorphisms. (By taking the graph of a Hamiltonian diffeomorphism, it can be also put in Lagrangian intersection setting.) In [4], Floer succeeded the construction for a Hamiltonian diffeomorphism on monotone symplectic manifolds. Here monotonicity is that the first Chern class is positively proportional to the de Rham cohomology class represented by the symplectic form. Yong-Geun Oh considered an analogous situation in Lagrangian intersection setting and performed

Since the right hand side of the definition of δ may be an infinite sum. So we allow the infinite sum as long as the value of the function $\mathcal{A}$ grows to $+\infty$.

the construction of Floer cohomology of (L, L') under the condition that L, L' are monotone Lagrangian submanifolds[3] with minimal Maslov number > 2. Later, he obtained the construction for monotoen L and $L' = \psi(L)$ with minimal Maslov number 2.

As we mentioned before, we may not have $\delta \circ \delta = 0$. Such an obstruction is caused by bubbling-off of holomorphic discs. In order to understand obstructions, we have to study all holomorphic discs in a systematic way. This is done by Fukaya, Oh, Ohta and the author [5]. We formulate it in terms of *filtered A_∞-algebra* associated with Lagrangian submanifolds. (We cannot explain terminology here and would like to invite interested readers to [5].) If the Maurer-Cartan equation admits weak solutions for L and L' with the same *potential value*, then we can rectify δ to a differential of $CF^\bullet(L', L)$ and define Floer cohomology depends on (weak) solutions of Maurer-Cartan equations. The resulting cohomology is also invariant under Hamiltonian deformations of L' in a suitable sense.

Another way to deform Floer coboundary operator is *bulk deformations.* Namely, we can deform all constructions such as filtered A_∞-algebras, filtered A_∞-bimodules, etc. using an cycle in X. These machinery may be considered as too abstract. However, happily enough, we can see the efficiency of all these machinery in application to concrete examples such as Lagrangian torus fibers in compact Kähler toric manifolds, see, e.g., a survey article [6].

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