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Problems on Low-dimensional Topology, 2016

Edited by T. Ohtsuki[1]

This is a list of open problems on low-dimensional topology with expositions of their history, background, significance, or importance. This list was made by editing manuscripts written by contributors of open problems to the problem session of the conference "Intelligence of Low-dimensional Topology" held at Research Institute for Mathematical Sciences, Kyoto University in May 18–20, 2016.

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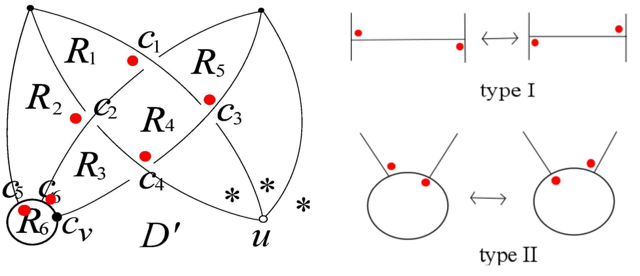
1 Heegaard Floer homology for embedded bipartite graphs

(Yuanyuan Bao)

For a link L in S^3, its link diagram in S^2 with a basepoint determines a Heegaard diagram. Ozsváth and Szabó showed in [34] that (1) the generators of the Heegaard Floer chain complex are given by the Kauffman states of the diagram; (2) the Maslov grading and Alexander grading can be calculated in an explicit and combinatorial way; and as a corollary (3) the Heegaard Floer homology (hat version) is explicitly determined by the signature and the Alexander polynomial of the given link. However in general, a combinatorial description of the differential of the complex is unknown.



For a balanced bipartite graph G_{V_1,V_2} in S^3, we similarly provide a Heegaard diagram for it from its diagram in S^2 with a basepoint. We also prove that (1) the generators of the Heegaard Floer chain complex for the graph are given by the states of the diagram (Figure above); (2) the Alexander polynomial of the graph can be expressed as a state sum. We may ask the following questions, which we suggest to answer in order.

Question 1.1 (Y. Bao)**.** *Can any two states be connected by transpositions of type I and II? Is it possible to calculate the (relative) Maslov grading and Alexander grading combinatorially? For "alternating" (there is no standard definition) bipartite graphs, is the Heegaard Floer complex completely determined by the Alexander polynomial (up to overall shifts of the gradings)?*

In a recent paper [36], from a knot diagram, Ozsváth and Szabó constructed a bigraded chain complex over $\mathbb{F}[U]$, the homology of which is shown to be isomorphic to the knot Floer homology (minus version). It is freely generated by the Kauffman states, and its differential is defined algebraically, built on bordered Floer homology. We may ask the following question.

Question 1.2 (Y. Bao)**.** *For a bipartite graph, or a general graph, is it possible to construct such a complex?*

2 Heegaard Floer homology of knots and 3-manifolds

(Motoo Tange)

The hat version of Heegaard Floer homology is algorithmically computable using the Heegaard diagram. This result is due to [37]. Another development of this topic is the result [33] by Ozsváth, Stipsicz and Szabó. Also, all types of Heegaard Floer homology with coefficients $\mathbb{Z}/2\mathbb{Z}$ is computable combinatorially [27].

Problem 2.1 (M. Tange). *Find an algorithm to compute $HF^+(Y,\mathfrak{s})$, $HF^-(Y,\mathfrak{s})$ with $\mathbb{Z}$ coefficients.*

The Ozsváth-Szabó's d-invariant is difficult to compute for general 3-manifold. Indeed, for suitable graph manifolds (Nemethi's algorithm [29]) or S^1-bundle over a surface.

Problem 2.2 (M. Tange). *Find d-invariant formula for hyperbolic 3-manifold.*

Let K be a knot with annulus presentation. For the definition of annulus presentation, see [1]. Let $A^n(K)$ be the n-fold annulus twist of K along the embedding annulus. The annulus twist is useful method to construct a pair of the diffeomorphic 0-surgery or diffeomorphic 0-framed 2-handlebody.

If two knots K and K' are concordant, then 0-surgeries of K and K' are homology cobordant. In general, we do not know whether K and $A^n(K)$ are concordant or not. Hedden [12] gave a formula of $\widehat{CFK}$ of Whitehead double $D_+(K,n)$ from that of K. As an analogy, the following question is considered:

Problem 2.3 (M. Tange). *Compute $CFK^\infty(A^n(K))$ from $CFK^\infty(K)$.*

In the case of $K = 6_3$, the annulus twist of 6_3 has the same τ-invariant, *i.e.*, $\tau(6_3) = \tau(A^n(6_3))$ holds for any n.

Question 2.4 (M. Tange). *For any n does the equality $\tau(K) = \tau(A^n(K))$ hold, in general?*

Question 2.5 (M. Tange). *Are the d-invariants of the 0-surgeries of K and K' equal? Or are the d-invariants of the branched covers of K and K' equal?*

For a knot K in S^3 and an integer n, let $S^3_n(K)$ denote the 3-manifold obtained from S^3 by n-surgery along K. A. Levine in [40] asks the following question:

> *If K_1 is concordant to K_2, then for all n, $S^3_n(K_1)$ is homology cobordant to $S^3_n(K)$. Is the converse true?*

Here we put some stronger problem:

Problem 2.6 (M. Tange). *Find non-concordant knots K_1 and K_2 with same d-invariant for $S^3_n(K_1)$ and $S^3_n(K_2)$ for any integer n.*

Due to [35] knot Floer homology detects the Seifert genus:

$$g(K) = \max\left\{s \mid \widehat{HFK}(S^3,K,s) \neq 0\right\}.$$

The τ-invariant gives the bound for 4-ball genus as follows:

$$|\tau(K)| \leq g_4(K).$$

The following question seems fundamental, but the answer is not known.

Question 2.7 (M. Tange). *Does knot Floer homology detect the 4-ball genus?*

The knot Floer homology also detects fiberedness of a knot due to [30].

$$K \text{ is a fibered knot } \Leftrightarrow \widehat{HFK}(K, g(K)) \cong \mathbb{Z}.$$

What is a geometric property characterizing the knot Floer homology $\widehat{HFK}(K, i)$ for $i < g(K)$?

Suppose that K is a lens space (or L-space) knot $\left(\overset{\text{def}}{\Leftrightarrow} \exists p \in \mathbb{Z} \text{ such that } S^3_p(K) \text{ is}\right.$ $\left.\text{a lens space (or L-space)}\right)$. Then any $\widehat{HFK}(K, i)$ is isomorphic to $\mathbb{Z}$ or $\{0\}$.

Question 2.8 (M. Tange). *What is a geometric property characterizing the isomorphism $\widehat{HFK}(K, i) \cong \mathbb{Z}$?*

Manolescu in [26] defined Pin(2)-equivariant Seiberg-Witten Floer homology. Here we raised the following problem.

Problem 2.9 (M. Tange). *Define Pin(2)-equivariant Heegaard Floer homology which is isomorphic to Manolescu's Pin(2)-equivariant Seiberg-Witten Floer homology.*

Let $f : X \to S^1$ be a circle valued Morse function. Let τ_{top} denote the torsion of cell complex of the infinite cyclic cover $\tilde{X}$ with respect to f. Let τ_{Morse} denote the torsion of Morse complex of the infinite cyclic cover $\tilde{X}$ with respect to f. Morse complex is generated by the critical points of the Morse function and the differentials are defined by counting of trajectories between critical points. Then Hutchings and Lee proved the following formula:

$$\tau_{\text{Morse}}(t) \cdot \zeta(t) = \tau_{\text{top}}.$$

The difference $\zeta(t)$ between two Reidemeister torsions is a zeta function of the dynamical system on a level set. The right-hand side is equivalent to a summation of Seiberg-Witten invariants on X due to the result [28] by Mark.

As a Hutchings-Lee type formula, Goda, Matsuda, and Pajitnov in [9] proved the formula below. Let K be a knot in S^3 and $f : S^3 - K \to S^1$ a circle valued Morse function. Let v be a half transversal flow for f and τ_v denote the Novikov torsion for v. Let R be the regular surface of f and $h : R \to R$ 'a monodromy map' generalized to even any non-fibered knot. Let and $\zeta_h(t)$ the zeta function of dynamical system h. Then the following equality holds:

$$\tau_v(t)\,\zeta_h(t) = \frac{\Delta_K(t)}{t-1}.$$

Problem 2.10 (M. Tange). *Interpret this formula by using some Floer theory.*

Heegaard Floer counterpart of the right-hand side is the Heegaard Floer homology $HFK^{-}(K)$. This problem implies a decomposition of $HFK^{-}(K)$ into two some homology theories for Novikov torsion and zeta function. The ζ-part would be symplectic Floer homology for mapping classes.

Casson invariant is the first term of LMO invariant $Z^{LMO}(Y)$ of a homology 3-sphere Y. To the higher terms of this invariant we have not found geometric meanings. Here we propose the following problem.

Problem 2.11 (M. Tange). *By deforming Heegaard (or instanton) Floer theory in some sense, find the higher terms in it again.*

3 Modifying constructions of Lagrangian and Heegaard Floer theory

(Kaoru Ono)

For a pair of Lagrangian submanifolds L_1, L_2 in a closed symplectic manifold, Lagrangian Floer homology $HF_*(L_1, L_2)$ is defined from a chain complex generated by intersection points of L_1 and L_2 whose differential counts pseudo-holomorphic disks; for details see [6, 7]. Further, in [7], it is extended to $HF_*\big((L_1, b_1), (L_2, b_2)\big)$ for "bounding cochains" b_i of L_i ($i = 1, 2$). For a 3-manifold M, the instanton Floer homology of M is defined by "(infinite dimensional) Morse theory" for the Chern-Simons functional on the space of SU(2) connections on M; see *e.g.* [5]. Motivated by the instanton Floer homology, for a 3-manifold M, Heegaard Floer homology of M is defined as Lagrangian Floer homology associated to a Heegaard diagram of M; for details see *e.g.* [3].

Question 3.1 (K. Ono). *There are some constructions in Lagrangian Floer theory. Is it possible to consider such constructions in Heegaard Floer theory and apply them to low dimensional topology?*

Construction 1 (bounding cochain). For a topological space, (co)homology theory can be twisted by a local system. In particular, we have Morse (co)homology with coefficients in a local system. Under certain conditions, we can also construct Lagrangian Floer complex twisted by local systems on Lagrangian submanifolds.

This construction is a part of the following story. Let (L_1, L_2) be a (transversal) pair of Lagrangian submanifolds in a closed (or 'tame') symplectic manifold (X, ω). In general, Floer chain complex for (L_1, L_2) is not defined. The obstruction is formulated in terms of filtered A_∞-algebras associated with L_i, $i = 1, 2$. If Maurer-Cartan equations in these filtered A_∞-algebras have solutions (we call them bounding cochains or Maurer-Cartan elements), we can modify the definition of the boundary operator to obtain a chain complex. (More generally, we can work with a pair of weak bounding cochains (weak Maurer-Cartan elements) with the same potential value.)

We can introduce an equivalence relation among (weak) bounding cochains so that the cohomology groups associated to equivalent (weak) bounding cochains are isomorphic. The notion of augmentation in the setting of contact homology is an analog of bounding cochains.

There is also a construction, called bulk deformations [8]. In fact, a special kind of bulk deformations had already used in Heegaard Floer theory from the early stage.

Construction 2 (filtration by the action functional). Floer complex is a kind of Morse-Novikov complex associated to so-called action functional (corresponding to the Chern-Simons functional for the instanton Floer homology), which naturally induces a filtration on the complex. Using this filtration, one can obtain "essential critical values" of the action functional. (In the case of Morse theory, essential critical values mean critical values corresponding to non-zero homology classes.) In symplectic Floer theories, it provides useful information such as spectral invariants [32] for Hamiltonian diffeomorphisms, torsion exponents in Lagrangian Floer (co)homology, etc.

4 Braid groups and C^p-groups

(Yuta Nozaki)

Let G be a group, and let p be a positive integer. As in [31], we define $\mathrm{C}^p(G)$ to be the subgroup of G generated by the set $\{g^p \mid g \in G\} \cup \{[g,h] \mid g,h \in G\}$, where $[g,h] = ghg^{-1}h^{-1}$. In other words, $\mathrm{C}^p(G)$ is the kernel of the natural projection $G \to G_{\mathrm{ab}}/pG_{\mathrm{ab}}$, where G_{ab} denotes the abelianization of G.

When there is a p-fold cyclic covering $(S^3, K) \to (L(p,q), K')$ for a knot K in S^3 and a knot K' in the lens space $L(p,q)$, it is shown in [31] that the knot group $G(K)$ is isomorphic to $\mathrm{C}^p(\pi_1(L(p,q) \setminus K'))$, since the following sequence is exact,

$$G(K) \longrightarrow \pi_1(L(p,q) \setminus K') \longrightarrow \mathbb{Z}/p\mathbb{Z}.$$

Therefore, for an arbitrary knot K, the following question naturally arises, which was discussed in [31]. Here, as in [31], we call a group G a C^p-*group* if there exists G' such that G is isomorphic to $\mathrm{C}^p(G')$.

Question 4.1 (Y. Nozaki). *Let K be a knot. Is $G(K)$ a C^p-group?*

Remark. For a given knot K in S^3, it is a non-trivial problem to determine whether there is a knot K' in $L(p,q)$ such that K is isotopic to the preimage of K' by the projection $S^3 \to L(p,q)$. We note that, if such a K' exists, $G(K)$ is a C^p-group. Hence, if we can show that $G(K)$ is not a C^p-group, it follows that such a K' does not exist.

Remark. Hartley [10] gave a list of possible free period of prime knots K with up to 10 crossings. Such a K can be obtained as the preimage of a knot in $L(p,q)$.

Further, we consider the corresponding problem for braid groups.

Problem 4.2 (Y. Nozaki). *Let p be odd and $n \geq 3$. Is the nth braid group B_n a C^p-group?*

It is known in [31] that, if G is a C^p-group and there is an epimorphism $G \to G'$ whose kernel is a characteristic subgroup of G, then G' is also an C^p-group. Here, a *characteristic subgroup* of G is a subgroup which is invariant under all automorphisms of G. Let $\mathfrak{S}_n$ be the nth symmetric group. The kernel of the natural homomorphism $B_n \to \mathfrak{S}_n$ is the pure braid group, which is a characteristic subgroup of B_n. Since $\mathfrak{S}_n$ is not a C^p-group for even p (as shown in [31, Example 2.8]), it follows that B_n is *not* a C^p-group for even p. On the other hand, B_3 is a C^p-group for p with $\gcd(p, 6) = 1$.

Remark. Let X_n be the configuration space of n distinct points in $\mathbb{R}^2$. The braid group B_n is isomorphic to $\pi_1(X_n/\mathfrak{S}_n)$. Problem 4.2 is related to a problem to find an appropriate space whose p-fold cyclic cover is homeomorphic to $X_n/\mathfrak{S}_n$.

5 Complex of surfaces of a 4-manifold and the adjunction inequalities

(Hokuto Konno)

The notion of the "complex of curves" of a surface was introduced by Harvey [11] in the 1980s, and has been studied from the viewpoint of the Teichmüller space and the action of the mapping class group. The *complex of curves* (also called *curve complex*) of a surface S is defined to be the simplicial complex whose vertices are the isotopy classes of essential simple closed curves on S and whose simplices are spanned by collections of such curves which can be realized disjointly.

A 4-dimensional analog of this notion, namely, "complex of surfaces" was introduced by Mikio Furuta.[2]

Definition (M. Furuta) Let X be an oriented, closed smooth 4-manifold. The *complex of surfaces* $\mathcal{K} = \mathcal{K}(X)$ of X is the abstract simplicial complex defined as follows:

- The set of vertices $V(\mathcal{K})$ is given as the set of smooth embeddings of surfaces with self-intersection number zero:

$$V(\mathcal{K}) := \{\Sigma \hookrightarrow X \mid [\Sigma]^2 = 0\}.$$

  Here we consider only oriented, closed, connected surfaces. We denote each vertex $(\Sigma \hookrightarrow X) \in V(\mathcal{K})$ briefly by Σ.

- For $k \geq 1$, a collection of $(k+1)$ vertices $\Sigma_0, \dots, \Sigma_k \in V(\mathcal{K})$ spans a k-simplex if and only if $\Sigma_0, \dots, \Sigma_k$ are disjoint.

In the above definition of the "complex of surfaces", we do not consider the isotopy classes of embeddings of surfaces. On the other hand, in the same way as the definition of the complex of curves, one can define an abstract simplicial complex whose vertices are the isotopy classes of embeddings of surfaces and whose

[2]The author heard this definition from Mikio Furuta in private communication in April 2015.

simplices are spanned by collections of such isotopy classes which can be realized disjointly. However, to give the following application to the adjunction inequalities using Seiberg-Witten theory, the first definition of the "complex of surfaces" might be appropriate.

One can easily see that $\mathcal{K}(X)$ is contractible for any X. Thus it is natural to seek a significant subcomplex of $\mathcal{K}(X)$ having non-trivial homotopy type.

Definition (H. Konno) Let $\mathfrak{s}$ be a spin c structure on X. Then, the *complex of surfaces violating the adjunction inequality* $\mathcal{K}_V = \mathcal{K}_V(X, \mathfrak{s})$ is the subcomplex of $\mathcal{K}(X)$ defined as the set of vertices is given by

$$V(\mathcal{K}_V) := \{\Sigma \in V(\mathcal{K}) \mid \max\{-\chi(\Sigma), 0\} < |c_1(\mathfrak{s}) \cdot [\Sigma]|\}$$

and having the induced structure of an abstract simplicial complex from $\mathcal{K}$.

We showed that, for any $k \geq 0$, there exists infinitely many pairs $(X, \mathfrak{s})$ satisfying

$$\tilde{H}_k(\mathcal{K}_V(X, \mathfrak{s}); \mathbb{Z}) \neq 0.$$

This result gives a classical application; we can drive certain adjunction inequalities for surfaces embedded to 4-manifolds whose Seiberg-Witten invariants vanish.

On the other hand, we do not know any example of $(X, \mathfrak{s})$ and k with $\tilde{H}_k(\mathcal{K}_V(X, \mathfrak{s}); \mathbb{Z}) = 0$ except for the case of $V(\mathcal{K}_V(X, \mathfrak{s})) = \emptyset$.

Problem 5.1 (H. Konno). *Find an example of* $(X, \mathfrak{s})$ *and* k *with* $\tilde{H}_k(\mathcal{K}_V(X, \mathfrak{s}); \mathbb{Z}) = 0$ *and* $V(\mathcal{K}_V(X, \mathfrak{s})) \neq \emptyset$.

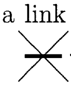
The complex of curves is used to describe the end of the moduli space of complex structures on the base surface. On the other hand, the complex $\mathcal{K}_V$ is used to describe "stretching" of neighborhoods of embedded surfaces in 4-manifold. The Seiberg-Witten equations on the stretched neighborhoods play a key role to consider the adjunction inequalities. One natural "limit" of this stretching is a non-compact 4-manifold whose ends are given by the cylinders, *i.e.*, product of the embedded surfaces, S^1, and the half line.

Problem 5.2 (H. Konno). *For a closed* 4*-manifold, construct the "moduli space" of a certain structure whose end is given by* 4*-manifolds with cylindrical ends. Describe the end of the moduli space in terms of the complex of surfaces.*

6 Surface-links and marked graph diagrams

(Sang Youl Lee)

A *surface-link* is a closed 2-manifold smoothly (or piecewise linearly and locally flatly) embedded in $\mathbb{R}^4$. Two surface-links are said to be *equivalent* if they are ambient isotopic.

A *marked graph diagram* (or *ch-diagram*) is a link diagram in $\mathbb{R}^2$ possibly with some 4-valent vertices equipped with markers: . An *oriented marked graph* *diagram* is a marked graph diagram in which every edge has an orientation such that each marked vertex looks like or (see Figure 1).

For a given (oriented) marked graph diagram D, let $L_-(D)$ and $L_+(D)$ be classical (oriented) link diagrams obtained from D by replacing each marked vertex with and , respectively (see Figure 1). An (oriented) marked graph diagram D is said to be *admissible* if both resolutions $L_-(D)$ and $L_+(D)$ are diagrams of (oriented) trivial links.

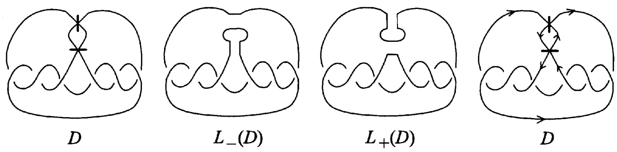


Figure 1: A marked graph diagram and its resolutions

S. J. Lomonaco, Jr. [25] and K. Yoshikawa [39] introduced a method of describing surface-links using marked graph diagrams. Indeed, every surface-link $\mathcal{L}$ is represented by an admissible marked graph diagram D. Moreover, if D is an admissible marked graph diagram representing a surface-link $\mathcal{L}$, then one can construct a surface-link $\mathcal{L}_D$ from D in a canonical way such that $\mathcal{L}_D$ is equivalent to $\mathcal{L}$.

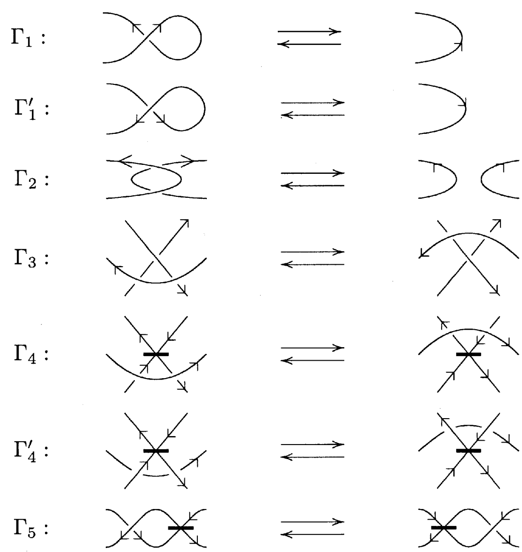


Figure 2: Yoshikawa moves of type I

The *Yoshikawa moves* for oriented marked graph diagrams are the local moves

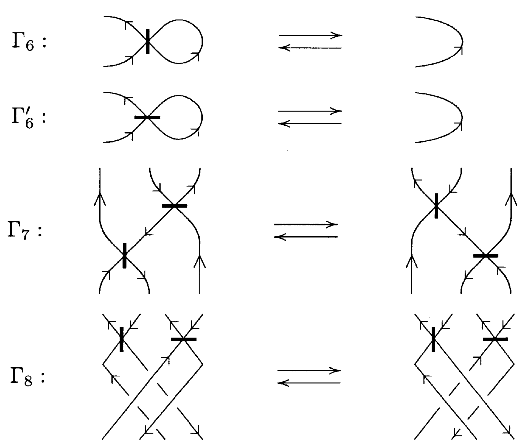


Figure 3: Yoshikawa moves of type II

$\Gamma_1, \dots, \Gamma_5$ (Type I) and $\Gamma_6, \dots, \Gamma_8$ (Type II) illustrated in Figures 2 and 3. Let

$$\mathfrak{S} = \{\Gamma_1, \Gamma_1', \Gamma_2, \Gamma_3, \Gamma_4, \Gamma_4', \Gamma_5, \Gamma_6, \Gamma_6', \Gamma_7, \Gamma_8\}.$$

It is known that two admissible marked graph diagrams represent equivalent surface-links if and only if they are transformed into each other by a finite sequence of 11 Yoshikawa moves in $\mathfrak{S}$ [22, 20, 38]. Therefore any oriented surface-link can be represented by an oriented marked graph diagram [25, 39], and such a representation diagram is unique up to the Yoshikawa moves in $\mathfrak{S}$. For unoriented surface-links (*i.e.*, non-orientable surface-links or orientable surface-links without orientations), the Yoshikawa moves in $\mathfrak{S}$ forgetting the orientations are enough to describe their marked graph representations [19, 21, 22, 38].

On the other hand, it is proved that if $\Gamma \in \mathfrak{S} - \{\Gamma_5, \Gamma_8\}$, then Γ is independent from the other moves in $\mathfrak{S}$ [21]. If the answers of the following two questions are all affirmative, then $\mathfrak{S}$ is a minimal generating set for oriented Yoshikawa moves.

Question 6.1 (J. Kim, Y. Joung, S. Y. Lee [21]). *Is the Yoshikawa move Γ_5 independent from the other moves in $\mathfrak{S}$?*

Question 6.2 (J. Kim, Y. Joung, S. Y. Lee [21]). *Is the Yoshikawa move Γ_8 independent from the other moves in $\mathfrak{S}$?*

Let $\mathcal{L}$ be a surface-link and let D be a marked graph diagram of $\mathcal{L}$. Let $|V(D)|$ and $|C(D)|$ denote the number of all marked vertices and classical crossings in D, respectively. In [39], Yoshikawa introduced the *ch-index*, denoted by $\mathrm{ch}(\mathcal{L})$, of a surface-link $\mathcal{L}$, which is defined to be the minimum number $\mathrm{ch}(\mathcal{L}) = \min_{D \in \mathcal{D}}(|V(D)| + |C(D)|)$, where $\mathcal{D}$ denotes the set of all marked graph diagrams representing $\mathcal{L}$. Clearly, $\mathrm{ch}(\mathcal{L})$ is an ambient isotopy invariant of $\mathcal{L}$. Using the terminology, he gave a table of 23 surface-links with ch-index ≤ 10 [39]. So it is natural to raise the following problem.

Problem 6.3 (S. Y. Lee). *Create a complete table of admissible marked graph diagrams representing surface-links with ch-index ≥ 11.*

Up to now, many invariants for surface-links have been defined by using various representations of surface-links, for example, broken surface diagrams, 2-dimensional braids, charts, etc. So the following problem can be considered.

Problem 6.4 (S. Y. Lee). *How to compute known invariants for surface-links using marked graph diagrams?*

In [25], S. J. Lomonaco, Jr. used marked graph diagrams to calculate the surface-link groups. In [2], S. Ashihara gave a method of calculating the fundamental biquandles of surface links from their marked graph diagrams and Y. Joung, J. Kim and S. Y. Lee compute the Alexander biquandles of oriented surface-links via marked graph diagrams in [20]. Recently, it is also shown that the quandle cocycle invariants for surface-links can be computed by using marked graph diagrams [17].

The answers of the following problems would enrich the theory of surface-links.

Problem 6.5 (S. Y. Lee). *Construct new invariants for surface-links with marked graph diagrams.*

Especially, the following problem is important.

Problem 6.6 (S. Y. Lee). *Construct polynomial invariants for surface-links with marked graph diagrams which can be computed by recursive rules (skein relation) and categorifications.*

So far, there have been several attempts to construct new invariants with marked graph diagrams [13, 14, 15, 23, 24]. Finally, one may ask the following questions.

Question 6.7 (S. Y. Lee). *Is it possible to construct quantum invariants for surface-links with marked graph diagrams?*

Question 6.8 (S. Y. Lee). *Is it possible to construct a surface-link (co)homology with marked graph diagrams.*

7 Surface-links which bound immersed handlebodies

(Kengo Kawamura)

An *immersed surface-link* or simply a *surface-link* means a closed oriented surface generically immersed in $\mathbb{R}^4$. When it is embedded, we also call it an embedded surface-link. Two surface-links are *equivalent* if there is an orientation-preserving diffeomorphism $f : \mathbb{R}^4 \to \mathbb{R}^4$ sending one to the other preserving their orientations. A surface-link is said to be *ribbon* if it is equivalent to a surface-link which bounds immersed handlebodies in $\mathbb{R}^4$ whose multiple point set consists of ribbon singularities. (Note that ribbon surface-links are embedded surface-links.) A surface-link is said to be *ribbon-clasp* if it is equivalent to a surface-link which bounds immersed handlebodies in $\mathbb{R}^4$ whose multiple point set consists of ribbon singularities and clasp singularities; for details, see [16].

A *chord graph* $(O;\alpha)$ is a spatial trivalent graph which consists of a trivial link O and disjoint simple arcs α spanning O. A *chord diagram* $C(O;\alpha)$ is a diagram of a chord graph $(O;\alpha)$. It is known [18] that every ribbon surface-link can be obtained from a chord graph $(O;\alpha)$ up to equivalence. The resulting ribbon surface-link, denoted by $F(O;\alpha)$, is obtained from a trivial 2-link whose equator is O by 1-handle surgeries along 1-handles $h(\alpha)$ whose cores are α. A ribbon surface-link $F(O;\alpha)$ is *faithfully equivalent* to a ribbon surface-link $F(O';\alpha')$ if there is an equivalence $f:\mathbb{R}^4\to\mathbb{R}^4$ sending $F(O;\alpha)$ to $F(O';\alpha')$ and meridian curves of $h(\alpha)$ to null-homotopic curves in $F(O';\alpha')\cup h(\alpha')$. It is proved in [18] that two ribbon surface-links $F(O;\alpha)$ and $F(O';\alpha')$ are faithfully equivalent if and only if the chord diagrams $C(O;\alpha)$ and $C(O';\alpha')$ are related by a finite sequence of certain moves.

We generalize above arguments as follows. A *chord graph* $(O\cup H;\alpha)$ is a spatial trivalent graph which consists of a split union of a trivial link O and Hopf links H, and disjoint simple arcs α spanning $O\cup H$. A *chord diagram* $C(O\cup H;\alpha)$ is a diagram of a chord graph $(O\cup H;\alpha)$. It can be seen that every ribbon-clasp surface-link can be obtained from a chord graph $(O\cup H;\alpha)$ up to equivalence. The resulting ribbon-clasp surface-link, denoted by $F(O\cup H;\alpha)$, is obtained from an M-trivial 2-link (for details; see [16]) whose equator is $O\cup H$ by 1-handle surgeries along 1-handles $h(\alpha)$ whose cores are α. We similarly define a faithful equivalence for ribbon-clasp surface-links. Then, we ask whether an analogous result holds.

Problem 7.1 (K. Kawamura). *Find certain moves for chord diagrams $C(O\cup H;\alpha)$ which generate the faithful equivalence on ribbon-clasp surface-links $F(O\cup H;\alpha)$.*

This problem is a specialized version of the following problem.

Problem 7.2 (K. Kawamura). *Find certain moves for chord diagrams $C(O\cup H;\alpha)$ which generate the equivalence on ribbon-clasp surface-links $F(O\cup H;\alpha)$.*

8 Morse-Novikov numbers of surface-links

(Hisaaki Endo and Andrei Pajitnov)

A *2-knot* is a smoothly embedded 2-sphere in S^4. A Morse function $f:S^4\setminus K\to S^1$ on the complement to a 2-knot K is called *strongly minimal* if its number of critical points $m_p(f)$ of index p is minimal possible for every p. The *Morse-Novikov number* $\mathcal{MN}(K)$ is the minimal possible number of critical points of a Morse function $S^4\setminus K\to S^1$ belonging to the canonical class in $H^1(S^4\setminus K)$.

Question 8.1 (H. Endo, A. Pajitnov [4]). *Is it true that for any 2-knot K there exists a strongly minimal Morse function $S^4\setminus K\to S^1$?*

This is true for spun knots $K=S(k)$ where k is a classical knot with $\mathcal{MN}(k)=2$.

Question 8.2 (H. Endo, A. Pajitnov). *Is it true that for any classical knot k we have $\mathcal{MN}(S(k))=2\,\mathcal{MN}(k)$?*

This is true for any classical knot k of tunnel number 1.

It is known [4] that $\mathcal{MN}(K_1 \# K_2) \leq \mathcal{MN}(K_1) + \mathcal{MN}(K_2)$ for knots K_1, K_2 of any dimension.

Question 8.3 (H. Endo, A. Pajitnov). *Is it true that*

$$\mathcal{MN}(K_1 \# K_2) = \mathcal{MN}(K_1) + \mathcal{MN}(K_2)$$

for 2-knots?

Problem 8.4 (H. Endo, A. Pajitnov). *Compute Morse-Novikov numbers for the surface-links*

$$9_1, \quad 9_1^{0,1}, \quad 10_2^{0,1}, \quad 10_1^{1,1}$$

of the Yoshikawa's table [39].

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