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Scale invariant structure of a critical Hardy inequality and its sharp remainder

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1 Remainder terms of a critical Hardy inequality

The purpose of this note is to announce the recent results in [20] and [19] which studies scale invariant structure of critical Hardy inequalities and its remainder term.

Let $n\geq 2$ and $B_{1}\subset \mathbb{R}^{n}$ be the n-dimensional unit ball centered at the origin. The critical Hardy inequality with the sharp constant is the following:

$$\left(\frac{n-1}{n}\right)^{n}\int_{B_{1}}\frac{|u(x)|^{n}}{|x|^{n}\left(\log\frac{e}{|x|}\right)^{n}}dx\leq\int_{B_{1}}|\nabla u(x)|^{n}dx, \quad u\in W_{0}^{1,n}(B_{1}). \tag{1.1}$$

The particular interest for (1.1) comes from the best-possible embedding of Sobolev spaces in the framework of rearrangement invariant spaces, where a Banach space X is said to be a rearrangement invariant space if $\|u\|_{X}=\|v\|_{X}$ whenever $u^{\sharp}=v^{\sharp}$. Typical examples of such spaces are Lebesgue spaces, Lorentz spaces, and Orlicz spaces. See [5, Chapter 2] for more details on rearrangement invariant spaces. In view of the best-possible embedding of Sobolev spaces into rearrangement invariant spaces, the important feature of (1.1) is the monotonicity of the potential function $\frac{1}{|x|^{n}\left(\log\frac{e}{|x|}\right)^{n}}$ for $0<|x|<1$, since (1.1) is usually proved by the use of the symmetrization argument which requires this monotonicity. We remark that there exist a large amount of literature on applications, generalizations, and improvements of (1.1) (see e.g. [1, 4, 6, 9, 10, 11, 12, 13, 14, 15, 16, 22] and references therein).

There is another version of the Hardy inequality slightly different from (1.1):

$$\left(\frac{n-1}{n}\right)^{n}\int_{B_{1}}\frac{|u(x)|^{n}}{|x|^{n}\left(\log\frac{1}{|x|}\right)^{n}}dx\leq\int_{B_{1}}|\nabla u(x)|^{n}dx, \quad u\in W_{0}^{1,n}(B_{1}), \tag{1.2}$$

which is essentially proved by Leray [21]. The main difference between (1.2) and (1.1) is that the potential function $\frac{1}{|x|^{n}\left(\log\frac{1}{|x|}\right)^{n}}$ in (1.2) is non-monotone while that in (1.1) is monotone. Moreover, there is a significant difference between (1.1) and (1.2) on the scaling property. Indeed, it is shown in [18] that the inequality (1.2) is invariant under the following power-type scaling:

$$u_\lambda(x) := \lambda^{-\frac{n-1}{n}} u(|x|^{\lambda-1}x), \quad \lambda > 0, \tag{1.3}$$

while (1.1) is not invariant. Moreover, non-attainability of the sharp constant of (1.2) in $W_0^{1,n}(B_1)$ is proved in [18] by applying the invariance of (1.2) under (1.3). Namely, if there is an extremal function $U \in W_0^{1,n}(B_1)$ which attain the sharp constant, then

$$U(x) = U(|x|) = \left(\log \frac{1}{|x|}\right)^{\frac{n-1}{n}} \tag{1.4}$$

should follow by the simplicity of the first eigenvalue (if exists) and the scale invariance of (1.2) under the power-type scaling (1.3). Note that U cannot be a real minimizer since $\|\nabla U\|_{L^n(B_1)} = \infty$ and we will call U as a virtual minimizer.

As in the generalizations of (1.1) like [1, 4, 6, 9, 10, 11, 12, 16], one can expect the existence of the remainder terms in (1.2) because of the absence of the extremal functions. Here we summarize some results on the remainder terms of (1.2) proved in [20] and [19]. First we show the equality proved in [20].

Theorem 1.1 ([20]). *For $a, b \in \mathbb{R}$, we define*

$$R_n(a,b) = (n-1)|a|^n + |b|^n - n|a|^{n-1}ab.$$

Then the equality

$$\begin{aligned}\int_{B_1} \left|\frac{x}{|x|}\cdot\nabla u(x)\right|^n dx - \left(\frac{n}{n-1}\right)^n \int_{B_1} \frac{|u(x)|^n}{|x|^n \left(\log\frac{1}{|x|}\right)^n} dx \\ = \int_{B_1} R_n\left(\frac{n-1}{n}\frac{u(x)}{|x|\log\frac{1}{|x|}}, -\frac{x}{|x|}\cdot\nabla u\right) dx\end{aligned} \tag{1.5}$$

holds for every $u \in W_0^{1,n}(B_1)$. Moreover, the remainder term (the right hand side of (1.5)*) vanishes if and only if u takes the form*

$$u(x) = \left|\log\frac{1}{|x|}\right|^{\frac{n-1}{n}} \varphi\left(\frac{x}{|x|}\right) \tag{1.6}$$

for some function $\varphi : \mathbb{S}^{n-1} \to \mathbb{R}$.

We should mention a relation between (1.2) and (1.5). Pointwise Young's inequality implies

$$R_n(a,b) = (n-1)|a|^n + |b|^n - n|a|^{n-1}ab \geq 0$$

for all $a, b \in \mathbb{R}$. Therefore (1.5) yields (1.2) by neglecting R_n. Furthermore, non-attainability of the sharp constant of (1.2) can be obtained from Theorem 1.1. Namely, if the sharp constant attained by some function $u \in W_0^{1,n}(B_1)$, then u satisfies $R_n(u) = 0$, hence

$$u(x) = \left|\log \frac{1}{|x|}\right|^{\frac{n-1}{n}} \varphi\left(\frac{x}{|x|}\right)$$

for some function $\varphi : \mathbb{S}^{n-1} \to \mathbb{R}$ by the second assertion in the theorem. This contradicts to $u \in W_0^{1,n}(B_1)$.

Since (1.5) is an equality, there is no more improvement of the remainder term of (1.2). However, it seems important to characterize the remainder term R_n in (1.5) by a meaningful term. For instance, we give a characterization of the remainder term in (1.5) by a ratio and a distance from the virtual extremal function $U(|x|) = \left(\log \frac{1}{|x|}\right)^{\frac{n-1}{n}}$.

Let us state an expression by a ratio of u to $\left(\log \frac{1}{|x|}\right)^{\frac{n-1}{n}}$.

Theorem 1.2 ([19]). *Let $n \geq 2$. Then there exists a positive constant $C > 0$ such that there holds*

$$\int_{B_1} \left|\frac{x}{|x|} \cdot \nabla u(x)\right|^n dx - \left(\frac{n}{n-1}\right)^n \int_{B_1} \frac{|u(x)|^n}{|x|^n \left(\log \frac{1}{|x|}\right)^n} dx \geq C \int_{B_1} \left(\log \frac{1}{|x|}\right)^{n-1} \left|\frac{x}{|x|} \cdot \nabla \left(\frac{u(x)}{U(|x|)}\right)\right|^n dx \tag{1.7}$$

for every $u \in W_0^{1,n}(B_1)$.

Next Theorem is an expression by a distance of u to $\left(\log \frac{1}{|x|}\right)^{\frac{n-1}{n}}$.

Theorem 1.3 ([19]). *Let $n \geq 2$. Define*

$$d(f,g) := \sup_{0<r<1} \frac{|f(r) - g(r)|}{\left(\log \frac{1}{r}\right)^{\frac{n-1}{n}}}, \qquad \tilde{u}(r) := \int_{\mathbb{S}^{n-1}} u(r,\theta) d\theta.$$

Then there exists $C > 0$ such that the inequality

$$\int_{B_1} \left|\frac{x}{|x|} \cdot \nabla u(x)\right|^n dx - \left(\frac{n}{n-1}\right)^n \int_{B_1} \frac{|u(x)|^n}{|x|^n \left(\log \frac{1}{|x|}\right)^n} dx \geq C \frac{\left[\inf_{a>0} d\left(\tilde{u}(r), aU(r)\right)\right]^{n^2}}{\left(\int_{B_1} \frac{|u|^n}{|x|^n \left(\log \frac{1}{|x|}\right)^n} dx\right)^{n-1}}$$

holds for every $u \in W_0^{1,n}(B_1)$.

Some studies in which a remainder term is estimated by a distance can be found in [3] for the Sobolev inequality and in [9] for Hardy inequalities. Similar characterization by a ratio can be found in [15, 12] for Hardy inequalities. See [20] for the proof of Theorem 1.1. A proof of Theorems 1.2, 1.3 and its generalization can be found in [19].

2 Scale invariant structure of the critical Hardy inequality

In this section, we survey the relation between the standard dilation scaling $x \mapsto \lambda x$ for $\lambda > 0$ and the power-type scaling (1.3), which is pointed out in [19]. Actually, they are equivalent by virtue of the transformation

$$B_1(0) \setminus \{0\} \ni x \mapsto y = \left(\log \frac{1}{|x|}\right)^{-p} \frac{x}{|x|} \in \mathbb{R}^n \setminus \{0\} \tag{2.8}$$

and the associated transformation on functions:

$$T_p : C_0^\infty(\mathbb{R}^n \setminus \{0\}) \to C_0^\infty(B_1(0) \setminus \{0\}) \ ; \ T_p u(x) = u\left(\left(\log \frac{1}{|x|}\right)^{-p} \frac{x}{|x|}\right)$$

introduced by Horiuchi and Kumlin in [17, Definition 3.1 and 3.2]. Note that a transformation (2.8) sends a dilation $x \mapsto \lambda x$ to a power-type scaling $x \mapsto |x|^{\lambda-1}x$ as follows.

Proposition 2.1 ([19]). *Define*

$$D_\lambda u(x) := u(\lambda x), \quad u : \mathbb{R}^n \to \mathbb{R}, \quad \lambda > 0,$$
$$S_\lambda v(x) := v(|x|^{\lambda-1}x), \quad v : B_1(0) \to \mathbb{R}, \quad \lambda > 0.$$

Then there holds

$$T_p\left(D_{\lambda^{-p}} u\right)(x) = S_\lambda(T_p u)(x), \quad u : \mathbb{R}^n \to \mathbb{R}, \quad \lambda > 0, \quad 0 < p < \infty.$$

By using the relation in Proposition 2.1, we have the following equivalence between a standard Sobolev inequality with invariance under the dilation D_λ and a Sobolev type inequality on B_1 with invariance under a power-type scaling S_λ. Remark that $\partial_r u$ denotes a radial derivative defined by $\partial_r u = \frac{x}{|x|} \cdot \nabla u$.

Theorem 2.1 ([19]). *Let $1 < q < n$ and $\frac{1}{q^*} = \frac{1}{q} - \frac{1}{n}$. The following inequalities are equivalent:*

$$\text{(I)} \quad \left(\int_{B_1} \frac{|u(x)|^{q^*}}{|x|^n \left(\log \frac{1}{|x|}\right)^{pn+1}} dx \right)^{\frac{1}{q^*}}$$

$$\leq C_q p^{\frac{1}{n}} \left(\int_{B_1} |x|^{q-n} \left(\log \frac{1}{|x|}\right)^{pq-pn-1} \left| |\nabla u|^2 - |\partial_r u|^2 + \left(\frac{1}{p} \log \frac{1}{|x|}\right)^2 |\partial_r u|^2 \right|^{\frac{q}{2}} dx \right)^{\frac{1}{q}},$$

$$u \in C_0^\infty(B_1 \setminus \{0\}).$$

$$\text{(II)} \quad \left(\int_{\mathbb{R}^n} |v(y)|^{q^*} dy \right)^{\frac{1}{q^*}} \leq C_q \left(\int_{\mathbb{R}^n} |\nabla v|^q \, dy \right)^{\frac{1}{q}}, \quad v \in C_0^\infty(\mathbb{R}^n \setminus \{0\}), \tag{2.9}$$

where C_q is the Sobolev best constant given by

$$C_q = \pi^{-\frac{1}{2}} n^{-\frac{1}{q}} \left(\frac{q-1}{n-q}\right)^{1-\frac{1}{q}} \left(\frac{\Gamma\left(1+\frac{n}{2}\right)\Gamma(n)}{\Gamma\left(\frac{n}{q}\right)\Gamma\left(1+n-\frac{n}{q}\right)} \right)^{\frac{1}{n}}.$$

As a by-product of Theorem 2.1, one can obtain the Alvino inequality [2]

$$\sup_{x \in B_1} \frac{|u(x)|}{\left(\log \frac{1}{|x|}\right)^{\frac{1}{n'}}} \leq \pi^{-\frac{1}{2}} n^{-\frac{1}{n}} \left(\Gamma\left(1+\frac{n}{2}\right)\right)^{\frac{1}{n}} \left(\int_{B_1} |\nabla u|^n dx \right)^{\frac{1}{n}}, \quad u \in W^{1,n}_{0,\mathrm{rad}}(B_1), \tag{2.10}$$

which is known as a critical Sobolev inequality with $p = n$ in the sense that (2.10) implies the best embedding of the Sobolev space $W_0^{1,n}(B_1)$ into Orlicz spaces framework (see [8]). Namely, let u be a radially symmetric function belonging to $C_0^\infty(B_1 \setminus \{0\})$. It follows from Theorem 2.1 with $p = \frac{q-1}{n-q}$ that

$$\left(\int_{B_1} \frac{|u(x)|^{q^*}}{|x|^n \left(\log \frac{1}{|x|}\right)^{\frac{q^*}{n'}}} dx \right)^{\frac{1}{q^*}}$$

$$\leq \pi^{-\frac{1}{2}} n^{-\frac{1}{q}} \left(\frac{n-q}{q-1}\right)^{\frac{n-q}{nq}} \left(\frac{\Gamma\left(1+\frac{n}{2}\right)\Gamma(n)}{\Gamma\left(\frac{n}{q}\right)\Gamma\left(1+n-\frac{n}{q}\right)} \right)^{\frac{1}{n}} \left(\int_{B_1} |x|^{q-n} |\nabla u|^q \, dx \right)^{\frac{1}{q}}. \tag{2.11}$$

Taking $q \to n$ in (2.11), we obtain

$$\sup_{x\in B_1} \frac{|u(x)|}{\left(\log \frac{1}{|x|}\right)^{\frac{1}{n'}}} \le \pi^{-\frac{1}{2}} n^{-\frac{1}{n}} \left(\Gamma\left(1+\frac{n}{2}\right)\right)^{\frac{1}{n}} \left(\int_{B_1} |\nabla u|^n dx\right)^{\frac{1}{n}},$$

since

$$\lim_{q\uparrow n} \left(\int_{B_1} \frac{|u(x)|^{q^*}}{|x|^n \left(\log \frac{1}{|x|}\right)^{\frac{q^*}{n'}}} dx \right)^{\frac{1}{q^*}} = \sup_{x\in B_1} \frac{|u(x)|}{\left(\log \frac{1}{|x|}\right)^{\frac{1}{n}}}, \quad u \in C_0^\infty(B_1 \setminus \{0\}). \tag{2.12}$$

In particular, the relation in Theorem 2.1 shows us that Moser sequences which are minimizers associated with the Alvino inequality (2.10) (or Trudinger-Moser inequality, see [23] and [7]) and Talenti functions which are minimizers associated with the Sobolev inequality (see [26]). Indeed, the best constant in the inequality (I) in Theorem 2.1 is characterized by the following scaled Talenti functions

$$u_{q,a,b}(x) = \left(a + \left(b \log \frac{1}{|x|}\right)^{-\frac{q}{n-q}}\right)^{1-\frac{n}{q}} = \frac{b \log \frac{1}{|x|}}{\left(a\left(b \log \frac{1}{|x|}\right)^{\frac{q}{n-q}} + 1\right)^{1-\frac{n}{q}}}, \quad a, b > 0$$

since the sharp constant of the inequality (II) in Theorem 2.1 is characterized by the Talenti functions

$$v_{q,a,b}(y) = \left(a + (b|y|)^{\frac{q}{q-1}}\right)^{1-\frac{n}{q}}, \quad a, b > 0.$$

Now taking the limit $q \uparrow n$, we obtain

$$\lim_{q\uparrow n} \left(a\left(b \log \frac{1}{|x|}\right)^{\frac{q}{n-q}} + 1\right)^{\frac{n-q}{q}} = \begin{cases} b \log \frac{1}{|x|}, & |x| < e^{-\frac{1}{b}}, \\ 1, & |x| \ge e^{-\frac{1}{b}}, \end{cases}$$

for any $a > 0$. Therefore, we have a limit function of $u_{q,a,b}$ independently on a,

$$u_b(x) := \lim_{q\uparrow n} u_{q,a,b}(x) = \begin{cases} 1, & |x| \le e^{-\frac{1}{b}}, \\ b \log \frac{1}{|x|}, & e^{-\frac{1}{b}} < |x| \le 1, \end{cases} \tag{2.13}$$

Normalizing (2.13) by $\|\nabla u_b\|_{L^n}$, we obtain the Moser sequence (m_b) with a parameter $b > 0$:

$$m_b(x) = \begin{cases} \omega_{n-1}^{-\frac{1}{n}} b^{\frac{1}{n}-1}, & |x| \le e^{-\frac{1}{b}}, \\ \omega_{n-1}^{-\frac{1}{n}} b^{\frac{1}{n}} \log \frac{1}{|x|}, & e^{-\frac{1}{b}} < |x| \le 1. \end{cases}$$

In this view, we see that the Moser sequence is derived from the Talenti functions via Horiuchi-Kumlin transformation (2.8). Proofs and further remarks of Section 3 can be found in [19].

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