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Independent definition of reticulations on residuated lattices

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1 Introduction

A notion of reticulation which provides topological properties on algebras has introduced on commutative rings in 1980 by Simmons in [5]. For a given commutative ring A, a pair (L, λ) of a bounded distributive lattice and a mapping $\lambda : A \to L$ satisfying some conditions is called a reticulation on A, and the map λ gives a homeomorphism between the topological space $Spec(A)$ consisting of prime filters of A and the topological space $Spec(L)$ consisting of prime filters of L. The concept of reticulation are generalized to non-commutative rings, MV-algebras ([1]), BL-algebras ([3]), quantale ([2]) and so on. Since these algebras are axiomatic extensions of residuated lattices which are algebraic semantics of so-called fuzzy logic, it is natural to consider properties of reticulations on residuated lattices. In 2008, Mureşan has published a paper about reticulations on residuated lattices and she has provided an axiomatic definition of reticulations on residuated lattices, in which five conditions are needed. In this short note, we show that only two independent conditions of reticulation are enough to axiomatize reticulations on residuated lattices and also prove that reticulations on residuated lattices can be considered as homomorphisms between residuated lattices and bounded distributive lattices.

2 Residuated lattices and reticulations

An algebraic system $(A, \wedge, \vee, \odot, \to, 0, 1)$ is called a *residuated lattice* if

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(1) $(A, \wedge, \vee, 0, 1)$ is a bounded lattice.

(2) $(A, \odot, 1)$ is a commutative monoid, that is, for all $a, b \in A$, $a \odot b = b \odot a$, $a \odot 1 = 1 \odot a = a$.

(3) For all $a, b, c \in A$,

$$a \odot b \leq c \Longleftrightarrow a \leq b \to c$$

We have basic results about residuated lattices.

Proposition 1. *Let A be a residuated lattice. For all $a, b, c \in A$, we have*

(1) $a \leq b \Longleftrightarrow a \to b = 1$

(2) $a \to (b \to c) = a \odot b \to c = b \to (a \to c)$

(3) $a \odot (a \to b) \leq b$

(4) $a \to b \leq (b \to c) \to (a \to c)$

(5) $a \to b \leq (c \to a) \to (c \to b)$

(6) $(a \vee b)^{m+n} \leq a^m \vee b^n$, *where* $a^m = \underbrace{a \odot a \odot \cdots \odot a}_{m}$

Proof. We only show the case (6): $(a \vee b)^{m+n} \leq a^m \vee b^n$. We have

$$\begin{aligned}
(a \vee b)^{m+n} &= \underbrace{(a \vee b) \odot \cdots \odot (a \vee b)}_{m+n} \\
&= a^{m+n} \vee (a^{m+n-1} \odot b) \vee \cdots \vee (a^m \odot b^n) \\
&\vee (a^m \odot b^n) \vee \cdots \vee (a \odot b^{m+n-1}) \vee b^{m+n} \\
&\leq a^m \vee a^m \vee \cdots \vee a^m \vee b^n \vee \cdots \vee b^n \vee b^n \\
&= a^m \vee b^n.
\end{aligned}$$

□

A non-empty subset $F \subseteq A$ of a residuated lattice A is called a *filter* if

(F1) If $a, b \in F$ then $a \odot b \in F$.

(F2) If $a \in F$ and $a \leq c$ then $c \in F$.

For an element $a \in A$, we set

$$[a) = \{b \in A \,|\, (\exists n \in N) \ s.t. \ a^n \leq b\}$$

and it is called a *principal filter*. By $\mathcal{F}(A)$ (or $\mathcal{PF}(A)$), we mean the set of all filters (or principal filters, respectively) of A.

Moreover, a filter $P(\neq A)$ is called a *prime filter* if it satisfies a condition that $a \in P$ or $b \in P$ when $a \vee b \in P$. We denote the set of all prime filters of A by $Spec(A)$.

For a bounded lattice L, a non-empty subset F of L is called a *lattice filter* if

(LF1) If $x, y \in F$ then $x \wedge y \in F$.

(LF2) If $x \in F$ and $x \leq y$ then $y \in F$.

A lattice filter $F(\neq L)$ is called *prime* if it satisfies the condition that if $x \vee y \in F$ then $x \in F$ or $y \in F$. By $Spec(L)$ we mean the set of all prime lattice filters of L.

It is trivial that every filter is also a lattice filter.

In the following, let A be a residuated lattice and L be a bounded distributive lattice. For any subset $S \subseteq A$, we define

$$D(S) = \{P \in Spec(A) \,|\, S \not\subseteq P\}.$$

It is easy to show that

Proposition 2. *$\tau_A = \{D(S) \,|\, S \subseteq A\}$ is a topology on $Spec(A)$ and $\{D(a)\}_{a \in A}$, where $D(a) = \{P \in Spec(A) \,|\, a \notin P\}$, forms a base of the topology τ_A.*

Similarly, we also define a topology on $Spec(L)$ for a bounded distributive lattice L as follows. For any subset $S \subseteq L$, we define

$$D(S) = \{P \in Spec(L) \,|\, S \not\subseteq P\},$$

then

Proposition 3. *$\sigma_L = \{D(S) \,|\, S \subseteq L\}$ is a topology on $Spec(L)$ and $\{D(x)\}_{x \in L}$, where $D(x) = \{P \in Spec(L) \,|\, x \notin P\}$, forms a base for σ_L.*

According to [4], we define a reticulation. A pair (L, λ) of a bounded distributive lattice L and a map $\lambda : A \to L$ is called a *reticulation* on a residuated lattice A if the map satisfies the five conditions

(R1) $\lambda(a \odot b) = \lambda(a) \wedge \lambda(b)$

(R2) $\lambda(a \vee b) = \lambda(a) \vee \lambda(b)$

(R3) $\lambda(0) = 0,\ \lambda(1) = 1$

(R4) $\lambda : A \to L$ is surjective.

(R5) $\lambda(a) \leq \lambda(b)$ if and only if there exists $n \in N$ such that $a^n \leq b$.

Proposition 4 ([4]). *Let (L, λ) be a reticulation of A. Then we have*

(1) *λ is order-preserving, that is, if $a \leq b$ then $\lambda(a) \leq \lambda(b)$*

(2) $\lambda(a \wedge b) = \lambda(a) \wedge \lambda(b)$

(3) *For all $n \in N$, $\lambda(a^n) = \lambda(a)$*

(4) $\lambda(a) = \lambda(b) \Longleftrightarrow [a) = [b)$

We also have the following results.

Proposition 5. *Let (L, λ) be a reticulation of A. Then we have*

(1) $\lambda(a \wedge b) = \lambda(a \odot (a \to b))$

(2) $\lambda[a) = [\lambda a)$

The next fundamental result about reticulation is very important.

Theorem 1 (Mureşan [4]). *For a reticulation (L, λ) of A,*

(a) *$Spec(A)$ and $Spec(L)$ are topological spaces.*

(b) *$\lambda^* : Spec(L) \to Spec(A)$ is a homeomorphism, where λ^* is defined by $\lambda^*(P) = \lambda^{-1}(P)$ $(P \in Spec(L))$.*

(c) *If (L_1, λ_1) and (L_2, λ_2) are reticulations of a residuated lattice A, then there exists an isomorphism $f : L_1 \to L_2$ such that $\lambda_1 \circ f = \lambda_2$.*

(d) *$(\mathcal{PF}(A), \eta)$ is a reticulation on A, where $\eta : A \to \mathcal{PF}(A)$ is a map defined by $\eta(a) = [a)$.*

3 Simple characterization of reticulation

In this section we prove that the conditions (R1)-(R3) of reticulations can be proved from the rest (R4) and (R5), that is, reticulation can be defined by only two conditions (R4) and (R5). We note that the condition (R4) is independent from the conditions (R1)-(R3) and (R5) is also independent from (R1)-(R4). It follows from our result that the conditions (R4) and (R5) are independent to each other. Let A be a residuated lattice and L be a bounded distributive lattice. Let $f : A \to L$ be a map satisfying the following conditions

(R4) $f : A \to L$ is surjective.

(R5) $f(a) \leq f(b) \Longleftrightarrow (\exists n \in N)$ s.t. $a^n \leq b$

We have next results about the map.

Lemma 1. (1) $a \leq b \Longrightarrow f(a) \leq f(b)$

(2) $f(a \wedge b) = f(a \odot b)$

(3) (R1) $f(a \wedge b) = f(a) \wedge f(b)$

(4) (R2) $f(a \vee b) = f(a) \vee f(b)$

(5) (R3) $f(0) = 0,\ f(1) = 1$

Proof. (1) If $a \leq b$, since $a = a^1$, then we have $a = a^1 \leq b$ and thus $f(a) \leq f(b)$ by (R5).

(2) Since $a \odot b \leq a \wedge b$, we get $f(a \odot b) \leq f(a \wedge b)$. Moreover, since $(a \wedge b)^2 = (a \wedge b) \odot (a \wedge b) \leq a \odot b$, we also have $f(a \wedge b) \leq f(a \odot b)$ by (R5). This implies that $f(a \wedge b) = f(a \odot b)$.

(3) It is trivial that $f(a \wedge b) \leq f(a), f(b)$, that is, $f(a \wedge b)$ is a lower bound of a set $\{f(a), f(b)\}$. For any lower bound l of $\{f(a), f(b)\}$, since f is surjective, there is an element $c \in A$ such that $f(c) = l$. This implies that $f(c) \leq f(a), f(b)$ and hence that $c^m \leq a, c^n \leq b$ for some $m, n \in N$ by (R5). Since $c^{m+n} = c^m \odot c^n \leq a \odot b$, we get from (R5) that $l = f(c) \leq f(a \odot b) = f(a \wedge b)$. Therefore, $f(a \wedge b) = \inf_L\{f(a), f(b)\} = f(a) \wedge f(b)$.

(4) It is obvious that $f(a), f(b) \leq f(a \vee b)$. For any $u \in L$, if $f(a), f(b) \leq u$, since $u = f(d)$ for some $d \in A$ by (R4), then we have $f(a), f(b) \leq f(d)$, It follows from (R5) that there exist $m, n \in N$ such that $a^m \leq d, b^n \leq d$. Since $(a \vee b)^{m+n} \leq a^m \vee b^n \leq d \vee d = d$, we get that $f(a \vee b) \leq f(d) = u$ and hence that $f(a \vee b) = \sup_L\{f(a), f(b)\} = f(a) \vee f(b)$.

(5) For every $x \in L$, since f is surjective, there is a element $a \in A$ such that $f(a) = x$. It follows from $0 \leq a$ that $f(0) \leq f(a) = x$. If we take $x = 0$ then we have $f(0) = 0$. Similarly, we have $f(1) = 1$. $\square$

The result means that the definition of reticulation is given only two conditions (R4) and (R5).

4 Reticulation and homomorphism

Let A be a residuated lattice and (L, λ) its reticulation. As proved above, the map λ satisfies the following conditions:

(h1) $\lambda(0) = 0,\ \lambda(1) = 1$

(h2) $\lambda(a \wedge b) = \lambda(a \odot b) = \lambda(a) \wedge \lambda(b)$

(h3) $\lambda(a \vee b) = \lambda(a) \vee \lambda(b)$

This means that the map λ is an onto homomorphism from A to L of its reticulation with respect to the lattice operations. Let

$$\ker(\lambda) = \{(a, b) \mid \lambda(a) = \lambda(b), a, b \in A\}.$$

Proposition 6. $\ker(\lambda)$ *is a congruence on a residuated lattice A with respect to $\wedge, \vee, \odot$.*

Let

$$a/\ker(\lambda) = \{b \in A \mid (a, b) \in \ker(\lambda)\}$$
$$A/\ker(\lambda) = \{a/\ker(\lambda) \mid a \in A\}.$$

We define operators $\sqcap, \sqcup$ for $a/\ker(\lambda), b/\ker(\lambda) \in A/\ker(\lambda)$ and constants $\mathbf{0}, \mathbf{1}$ as follows:

$$\begin{aligned} a/\ker(\lambda) \sqcap b/\ker(\lambda) &= (a \wedge b)/\ker(\lambda) \\ &= (a \odot b)/\ker(\lambda) \\ a/\ker(\lambda) \sqcup b/\ker(\lambda) &= (a \vee b)/\ker(\lambda) \\ \mathbf{0} &= 0/\ker(\lambda), \\ \mathbf{1} &= 1/\ker(\lambda) \end{aligned}$$

Then we have from the result above that

Theorem 2 (Homomorphism Theorem). $(A/\ker(\lambda), \sqcap, \sqcup, \mathbf{0}, \mathbf{1})$ *is a bounded distributive lattice. If we define a map* $\nu : A \to A/\ker(\lambda)$ *by* $\nu(a) = a/\ker(\lambda)$*, then the quotient structure* $(A/\ker(\lambda), \nu)$ *is a reticulation of a residuated lattice* A *and*

$$A/\ker(\lambda) \cong L.$$

Proof. We only show that $A/\ker(\lambda)$ is distributive. It follows from the sequence below: For all $a/\ker(\lambda), b/\ker(\lambda), c/\ker(\lambda) \in A/\ker(\lambda)$, we have

$$\begin{aligned} a/\ker(\lambda) \sqcap (b/\ker(\lambda) \sqcup c/\ker(\lambda)) &= (a \odot (b \vee c))/\ker(\lambda) \\ &= ((a \odot b) \vee (a \odot c))/\ker(\lambda) \\ &= (a \odot b)/\ker(\lambda) \vee (a \odot c)/\ker(\lambda) \\ &= (a/\ker(\lambda) \sqcap b/\ker(\lambda)) \sqcup (a/\ker(\lambda) \sqcap c/\ker(\lambda)). \end{aligned}$$

□

On the other hand, in [4] a binary relation $\equiv$ on A is defined by

$$a \equiv b \Longleftrightarrow D(a) = D(b),$$

where $D(a) = \{P \in Spec(A) \mid a \notin P\}$. Since the binary relation $\equiv$ is a congruence on A with respect to lattice operations $\wedge$ and $\vee$, we consider its quotient algebra by $\equiv$. We take $[a] = \{b \in A \mid a \equiv b\}, A/\equiv = \{[a] \mid a \in A\}$. For $[a], [b] \in A/\equiv$, if we define

$$\begin{aligned} [a] \vee [b] &= [a \vee b] \\ [a] \wedge [b] &= [a \wedge b], \end{aligned}$$

then $(A/\equiv, \wedge, \vee, [0], [1])$ is a bounded distributive lattice and $(A/\equiv, \eta)$ is a reticulation of A ([4]).

We have another view point, namely, if we note $\lambda(a) = \lambda(b) \Longleftrightarrow [a) = [b)$, then we have

$$\begin{aligned} a \equiv b &\Longleftrightarrow D(a) = D(b) \\ &\Longleftrightarrow a \notin P \text{ iff } b \notin P \ (\forall P \in Spec(A)) \\ &\Longleftrightarrow a \in P \text{ iff } b \in P \ (\forall P \in Spec(A)) \\ &\Longleftrightarrow [a) = [b) \\ &\Longleftrightarrow \lambda(a) = \lambda(b) \\ &\Longleftrightarrow (a, b) \in \ker(\lambda). \end{aligned}$$

This means that the binary relation $\equiv$ defined in [4] is the same as the kernel $\ker(\lambda)$ of the lattice homomorphism λ.

Moreover, we introduce an partial order $\sqsubseteq$ on the class $\mathcal{PF}(A)$ of all principal filters of A by

$$[a) \sqsubseteq [b) \Longleftrightarrow [b) \subseteq [a).$$

It is easy to show that

$$\begin{aligned} \inf_{\sqsubseteq}\{[a),[b)\} &= [a \vee b) \\ \sup_{\sqsubseteq}\{[a),[b)\} &= [a \wedge b) = [a \odot b) \\ \mathbf{0} &= [1) = \{1\} \\ \mathbf{1} &= [0) = A. \end{aligned}$$

Hence $\mathcal{PF}(A)$ is a bounded distributive lattice. Moreover if we define a map $\xi : A \to \mathcal{PF}(A)$ by $\xi(a) = [a)$, then $(\mathcal{PF}(A), \xi)$ is a reticulation of A. Since the reticulation is unique up to isomorphism ([4]), we see that

$$A/\ker(\lambda) \cong \mathcal{PF}(A).$$

5 Conclusion

In this short note, we show that a reticulation map f can be defined only two independent conditions:

(R4) $f : A \to L$ is surjective.

(R5) $f(a) \leq f(b) \Longleftrightarrow (\exists n \in N)$ s.t. $a^n \leq b$

Moreover, the reticulation map is only a lattice homomorphism from a (residuated) lattice A to a bounded distributive lattice L. Moreover, since the implication $\to$ does not play an role in the definition of reticulation, we note that the argument in this short note can be generalized to the algebra $(A, \wedge, \vee, \odot, 0, 1)$, where $(A, \wedge, \vee, 0, 1)$ is a bounded lattice and $(A, \odot, 1)$ is a commutative monoid satisfying the axiom $x \odot (y \vee z) = (x \odot y) \vee (x \odot z)$ for all $x, y, z \in A$,

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