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ALMOST SYMMETRIC NUMERICAL SEMIGROUPS AND ALMOST GORENSTEIN SEMIGROUP RINGS

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The theory of numerical semigroups is important to commutative ring theory via semigroup ring $k[H]$ as well as to the theory of algebraic curves via the Weierstrass semigroup of a point on a compact Riemann surface. Among the numerical semigroups, symmetric semigroups played a central role.

The authors work in the field of commutative ring theory and the semigroup ring $k[H]$ over a field k of a numerical semigroup H is very useful when we need examples. The semigroup ring $k[H]$ is always Cohen-Macauly (we will write as **CM** after this), being a 1-dimensional integral domain and the notion of symmetric semigroup is particularly important because H is symmetric if and only if $k[H]$ is a Gorenstein ring. Besides this notion, we did not have any notions to classify the semigroups except the notion of type denoted by type(H), corresponding to the CM type of $k[H]$, until the notion of almost symmetric semigroups and almost Gorenstein rings were defined by Barucci and Fröberg in [BF]. Incidentally, J. Komeda [Ko] classified 4-generated almost symmetric numerical semigroups of type 2 (a numerical semigroup H is almost symmetric of type 2 if and only if H is pseudo-symmetric). Barucci and Fröberg defined almost Gorenstein rings only for 1-dimensional analytically irreducible CM rings ([BF], [GMP]). The notion of almost Gorenstein rings has been extended to higher dimension by S. Goto and his coworkers [GTT]. It is of interest to note that if P is a point of a compact Riemann surface X whose Weierstrass semigroup is H, then the normal graded ring $R(X,P) = \sum_{n\geq 0} H^0(X, \mathcal{O}_X(nP))t^n$ is almost Gorenstein if and only of H is almost symmetric ([GW], [GTT]). In particular, if genus of X is g and P is a general point of X, then $H = \{0, g+1, \rightarrow\}$ is almost symmetric of type g and hence $R(X,P)$ is almost Gorenstein.

This paper is a survey including some announcements of our results and the detailed version will be submitted to somewhere else.

Almost symmetric semigroups enjoy several symmetries and we believe we can make a very beautiful theory on them, that is, the reason why we are writing this article.

1. Preliminaries and Fundamental Properties

Let us begin with very elementary theory of numerical semigroups and semigroup rings. We refer to the basic concepts of numerical semigroups to the book [RG].

Definition 1.1. We always denote $\mathbb{N} = \{0, 1, 2, \ldots\}$ the set of non-negative integers.

(1) A *numerical semigroup* H is a subset of $\mathbb{N}$ closed by addition, $0 \in H$ and $\mathbb{N} \setminus H$ is a finite set. In the following, H will be a numerical semigroup. We denote H_+ to be the positive elements of H.

(2) We write $H = \langle n_1, \ldots, n_e \rangle$ if $H = \{\sum a_i n_i \mid a_i \in \mathbb{N}\}$ and we say that H is generated by $\{n_1, \ldots, n_e\}$. Furthermore, we denote by H_+ the strictly positive elements of H, namely $H_+ = H \setminus \{0\}$. If e is minimal with this property, then we write $\operatorname{emb}(H) = e$ and say H is generated by e elements or simply, "H is e-generated". Note that $\operatorname{emb}(H)$ is equal to the embedding dimension of $k[H]$.

(3) For any field k, let $k[H] = k[t^h \mid h \in H]$, where t is a variable over k and we call $k[H]$ the semigroup ring of H. We frequently write $R = k[H]$ and also R_+ for the maximal graded ideal of R, namely, R_+ is the ideal generated by elements of positive degree of R. If $H = \langle n_1, \ldots, n_e \rangle$, then we can define a surjective ring homomorphism π from a polynomial ring $S = k[X_1, \ldots, X_e]$ to $R = k[H]$, sending X_i to t^{n_i}. We denote $I_H = \operatorname{Ker}(\pi)$. The ideal I_H is called the defining ideal of $R = k[H]$. Note that I_H is generated by binomials of the form $\prod X_i^{a_i} - \prod X_j^{b_j}$ with $\sum a_i n_i = \sum b_j n_j$. We consider R as a graded ring in the sense of [GW] putting $R_n = kt^n$ for every $n \in H$. Note that I_H is generated by at least $e - 1$ elements and H is called a *complete intersection* (CI) if I_H is generated by exactly $e - 1$ elements.

(4) We denote $\mathrm{F}(H) = \max\{n \in \mathbb{Z} \mid n \notin H\}$ and call it the *Frobenius number* of H. We call $c(H) = \mathrm{F}(H) + 1$ the *conductor* of H. Note that $c(H)$ is the minimal n such that $n + \mathbb{N} \subset H$. Note that $\mathrm{F}(H) - h \notin H$ for any $h \in H$.

(5) We denote $g(H) = \sharp[\mathbb{N} \setminus H]$ and call it the *genus* of H.

(6) H is called *symmetric* if for every $n \in \mathbb{Z}$ one has that $n \notin H$ if and only if $\mathrm{F}(H)-n \in H$. It is easy to see that for every H one has that $c(H) \leq 2g(H)$, and that H is symmetric if and only if $c(H) = 2g(H)$. We say that H is *pseudo-symmetric* if we have $c(H) = 2g(H) - 1$. Note that H is pseudo-symmetric if and only if $\mathrm{F}(H)$ is even and for every $n \in \mathbb{Z}, n \neq \mathrm{F}(H)/2$,one has that $n \notin H$ if and only if $\mathrm{F}(H) - n \in H$. These notions are particularly important because a semigroup H is *irreducible* (i.e. if $H = H_1 \cap H_2$, then $H = H_1$ or $H = H_2$) if and only if H is symmetric or pseudo-symmetroc.

(7) For $n_1, n_2 \in \mathbb{Z}$, we define $n_1 \geq_H n_2$ if $n_1 - n_2 \in H$. We put $\mathrm{PF}(H)$ the set of maximal elements of $\mathbb{Z} \setminus H$ with respect to the order $\geq_H$. We call $\mathrm{PF}(H)$ the set of pseudo-Frobenius numbers. Thus $n \in \mathrm{PF}(H)$ if and only if $n \notin H$ and for every $h \in H_+, n+h \in H$. Note that H is symmetric if and only if $\mathrm{PF}(H) = \{\mathrm{F}(H)\}$ and H is pseudo-symmetric if and only if $\mathrm{PF}(H) = \{\mathrm{F}(H), \mathrm{F}(H)/2\}$. We define $\mathrm{type}(H) = \sharp \mathrm{PF}(H)$. For the convenience, we denote $\mathrm{PF}'(H) = \mathrm{PF}(H) \setminus \{\mathrm{F}(H)\}$.

(8) We denote by K_R the graded R-module generated by $\{t^{-n} \mid n \notin H\}$. Thus $t^{-\mathrm{F}(H)}$ is a generator of K_R of the minimum degree. It is clear that K_R is generated by $\{t^{-n} \mid n \in \mathrm{PF}(H)\}$. Since in commutative ring theory, the CM type $\mathrm{type}(R)$ is defined by the number of minimal generators of K_R, we see that $\mathrm{type}(H) = \mathrm{type}(k[H])$.

Now we can define almost symmetric numerical semigroups.

Definition 1.2 ([BF]). We say that H almost symmetric (AS) if for every $f \in \mathrm{PF}'(H)$ and $h \in H_+$, $-f + h + \mathrm{F}(H) \in H$. In other words, the factor R-module $K_R/t^{-\mathrm{F}(H)}R$ is killed by R_+. (The R-module $K_R/t^{-\mathrm{F}(H)}R$ is a "finite dimensional vector space".)

Note that by this definition "H is AS and $\mathrm{type}(H) = 2$" is equivalent to say that H is pseudo-symmetric.

We define almost Gorenstein rings.

Definition 1.3. (1) [BF] Assume $(R, \mathfrak{m})$ is a one-dimensional analytically unramified CM local ring with canonical module K_R. Then we call R is almost Gorenstein (AG) if for a general element ω of K_R, $K_R/R\omega$ is a vector space (killed by $\mathfrak{m}$). Thus by the remark above, $R = k[H]$ is AG if and only if H is AS.

(2) [GTT] Assume $(R, \mathfrak{m})$ is a CM local ring of dimension $d \geq 1$ with canonical module K_R. Then we say that R is AG if there is an injective R homomorphism $\kappa : R \to K_R$ such that $K_R/\kappa(R)$ is a Ulrich R-module of dimension $d-1$ (a CM R-module M of pure dimension d is called an Ulrich module if $\mu(M) = e(M)$, where $\mu(M), e(M)$ denote the number of minimal generators of M and the multiplicity of M, respectively.)

Notation 1.4. In the following, we abbreviate almost symmetric as AS and almost Gorenstein as AG. Also, when we mention about AS or AG, we assume that H is not symmetric ($k[H]$ is not Gorenstein).

There are several beautiful symmetries in the theory of AS semigroups. We can distinguish a given H is AS or not by looking at $\mathrm{PF}(H)$.

Theorem 1.5. *([Na]) $H = \langle n_1, \ldots, n_e \rangle$ with* $\mathrm{type}(H) = t$ *and put* $\mathrm{PF}(H) = \{f_1, f_2, \ldots . f_{t-1}, F(H)\}$ *so that $f_1 < f_2 < \ldots < f_{t-1} < F(H)$. Then H is AS if and only if for every i, $1 \leq i \leq t-1$, $f_i + f_{t-i} = F(H)$.*

Thanks to the package "numerical semigroups" of GAP, we can get $\mathrm{PF}(H)$ once we input the generators of H. So, we can determine if H is AS or not, instantly.

Also there is a beautiful structure theorem of AS numerical semigroups with respect to irreducible numerical semigroups due to Rosales and Garcia-Sánchez.

Theorem 1.6. [RG2] *Assume that H is AS with $F = \mathrm{F}(H)$. Then there is a unique irreducible (symmetric or pseudo-symmetric) numerical semigroup H_1 with* $\mathrm{F}(H_1) = F(H)$ *and a subset A of the minimal generating set of H_1 satisfying the following condition (†) so that $H = H_1 \setminus A$.*

(†) *For every $x \in A, x > \mathrm{F}(H)/2$ and for every $x, y \in A, x + y - \mathrm{F}(H) \notin H$.*

In this case, $\mathrm{type}(H) = \mathrm{type}(H_1) + 2\sharp A$.

Let $H = \langle n_1, \ldots, n_e \rangle$, $S = k[X_1, \ldots, X_e]$ and $\pi : S \to R = k[H]$ defined by $\pi(X_i) = t^{n_i}$. The minimal free resolution of R over S is very important in the commutative ring theory.

Let $(F_\bullet, d_\bullet)$ be a minimal free resolution of $R = k[H]$ as a graded S module. We denote $F_i = \oplus S(-b_{ij})$ and $\beta_i = \mathrm{rank}(F_i)$.

Later we will explain a special property of $(F_\bullet, d_\bullet)$ when H is AS.

2. Review of some known results when H has small embedding dimension

If emb(H) is small, then it is easy to describe H. We recall some know facts.

Facts 2.1. *(1) If* $e = 2$, *then* $H = \langle a, b\rangle$ *with* $(a, b) = 1$. *In this case* $R = k[X, Y]/(X^b - Y^a)$ *and* $c(H) = (a-1)(b-1)$.

(2) If $e = 3$ *and* H *is symmetric, then* H *is a CI (complete intersection). For any* e, *it is shown by D. Delorme that if* H *is a CI, then* H *can be obtained by successive gluings (see* [De], [RG] *for the detail).*

(3) ([He], [NNW]*) If* $e = 3$ *and* H *is not symmetric, then* type$(H) = 2$ *and* I_H *is generated by the* 2 *by* 2 *minors of a matrix* $\begin{pmatrix} X^{\alpha} & Y^{\beta} & Z^{\gamma} \\ Y^{\beta'} & Z^{\gamma'} & X^{\alpha'} \end{pmatrix}$. *In this case,* H *is AS (pseudo-symmetri) if and only if either* $\alpha = \beta = \gamma = 1$ *or* $\alpha' = \beta' = \gamma' = 1$.

(3) ([Br]*) Assume* $e = 4$, H *is symmetric and not CI. Then it was shown by H. Bresinsky that* I_H *is generated by* 5 *elements and the structure of* I_H *is given, too.*

(4) ([Ko]*) Assume* $e = 4$ *and* H *is pseudo-symmetric. Then it was shown by J. Komeda that* I_H *is generated by* 5 *elements and the structure of* I_H *was given, too. Later we will discuss about the description of* I_H *using Moscariello's RF matrix.*

(5) The structure of the minimal free resolutions of case (3), (4) is given in [BFS].

(6) T. Numata conjectured in [Nu1] *that if* $e = 4$ *and* H *is AS then* type$(H) \le 3$. *He proved it in the case* emb$(H_1) \le 3$ *in the expression of 1.6. This conjecture was proved by A. Moscariello (*[Mo]*) and we will talk about his methods.*

3. The Apery set and the invariant α_i

In the following, we will discuss the structure of H which is AS. For that purpose, we review the methods of Komeda and Moscariello so that we can see the structure of H clearer.

Definition 3.1. Let $a \in H$. Then we denote

$$\mathrm{Ap}(a, H) = \{h \in H \mid h - a \notin H\},$$

and call it the Apery set of a in H. It is clear that $\sharp\,\mathrm{Ap}(a, H) = a$, and that $0, n_i \in \mathrm{Ap}(a, H)$ for every i and that the largest element in $\mathrm{Ap}(a, H)$ is $a + \mathrm{F}(H)$.

If H is almost symmetric, then there is a duality on the Apery set. The proof is a consequence of the duality on $\mathrm{PF}(H)$, as given in Theorem 1.5.

Lemma 3.2. *Let $a \in H$ and $h \in \mathrm{Ap}(a,H)$. Then:*

(1) *If $h, h' \in H$ and if $h + h' \in \mathrm{Ap}(a,H)$, then $h, h' \in \mathrm{Ap}(a,H)$.*
(2) *Assume H is AG. If $h \in \mathrm{Ap}(a,H)$, then either $(a + \mathrm{F}(H)) - h \in \mathrm{Ap}(a,H)$ or $(a + \mathrm{F}(H)) - h \in \mathrm{PF}(H)$. In the latter case, we have $h - a \in \mathrm{PF}(H)$.*

Now, we put $H = \langle n_1, \ldots, n_e \rangle$ and define the invariant α_i for each n_i.

Definition 3.3. For every i, $1 \le i \le e$, we define α_i to be the minimal positive integer such that

$$\alpha_i n_i = \sum_{j=1, j \ne i}^{e} \alpha_{ij} n_j.$$

Note that α_{ij} may not be uniquely determined.

It is easy to see the following from the minimality of α_i.

Lemma 3.4. *For every $1 \le i, k \le e$, $i \ne k$, $(\alpha_i - 1)n_i \in \mathrm{Ap}(n_k, H)$.*

Combining these properties, we get the following, which will play an important role for the structure of AS semigroups.

Corollary 3.5. *If H is AS, then for every k and $i \ne k$, either $\mathrm{F}(H) + n_k - (\alpha_i - 1)n_i \in H$ or $(\alpha_i - 1)n_i = f + n_k$ for some $f \in \mathrm{PF}'(H)$.*

We give a short review on unique factorization of elements in H on the minimal generators of I_H.

Definition 3.6. Let H be a numerical semigroup minimally generated by $\{n_1, \ldots, n_e\}$.

(1) We say that $h = \sum_i a_i n_i$ has UF (Unique Factorization) if this expression is unique. It is obvious that h does not have UF if and only if $h \ge_H \deg(\phi)$ for some $\phi \in I_H$.
(2) We put $\mathrm{NUF}(H) = \{h \in H \mid h \text{ does not have UF } \} = \{\deg(\phi) | \phi \in I_H\}$. This is an ideal of H.
(3) We put $\mathrm{mNUF}(H) = \{h \in \mathrm{NUF}(H) \mid h \text{ is minimal with respect to } \le_H\}$. Note that if $\phi \in I_H$ and $\deg(\phi) \in \mathrm{mNUF}(H)$, then ϕ is a minimal generator of I_H. But the converse is not true in general. Hence $\sharp\,\mathrm{mNUF}(H) \le \mu(I_H)$.

Lemma 3.7. *Let $\phi = m_1 - m_2$ be a minimal generator of I_H, where m_1, m_2 are monomials on the X_i's. Then*

(1) *Take i, j so that $X_i | m_1$ and $X_j | m_2$. Then* $\deg\phi - n_i - n_j \notin H$ *and hence for some $f \in \mathrm{PF}(H)$,* $\deg(\phi) \leq_H f + n_i + n_j$.

(2) $\deg(\phi) = f + n_i + n_j$ *for some $f \in \mathrm{PF}'(H)$ if and only if* $\mathrm{F}(H) + n_i + n_j - \deg(\phi) \notin H$.

The following fact will play an important role in the classification of AS semigroups.

Lemma 3.8. *We assume that H is AS.*

(1) *If $\mathrm{F}(H) + n_k$ has UF for some k and assume that $\mathrm{F}(H) + n_k = \sum_{j \neq k} b_j n_j$ with $b_j < \alpha_j$ for every j. Then $n_k = \prod_{j \neq k}(b_j + 1) + \mathrm{type}(H) - 1$.*

(2) *If $e = 4$ and $\alpha_{ik} \geq 1$ for every $i \neq k$, then $\mathrm{F}(H) + n_k$ has UF.*

4. The Moscariello matrix $\mathrm{RF}(f)$ for $f \in \mathrm{PF}(H)$

A. Moscariello introduced the notion of RF (row factorization) matrices in his paper and we think this notion is very useful to describe the classification of AS semigroups.

Definition 4.1. ([Mo]) Let $f \in \mathrm{PF}(H)$. Then an $e \times e$ matrix $A = (a_{ij})$ is an **RF-matrix** (short for row-factorization matrix) if $a_{ii} = -1$ for every i, $a_{ij} \in \mathbb{N}$ if $i \neq j$ and for every $i = 1, \dots, e$,

$$\sum_{j=1}^{e} a_{ij} n_j = f.$$

In this case, we denote $A = \mathrm{RF}(f)$. Note that $\mathrm{RF}(f)$ need not be determined uniquely.

The most important property of the RF-matrix $\mathrm{RF}(f)$ is the following.

Lemma 4.2. *([Mo], Lemma 4) Let $f, f' \in \mathrm{PF}(H)$ with $f + f' = \mathrm{F}(H)$. If we put $\mathrm{RF}(f) = A = (a_{ij})$ and $\mathrm{RF}(f') = B = (b_{ij})$, then either $a_{ij} = 0$ or $b_{ji} = 0$ for every pair $i \neq j$. In particular, If we put $\mathrm{RF}(\mathrm{F}(H)/2) = (a_{ij})$, then either $a_{ij} = 0$ or $a_{ji} = 0$ for every $i \neq j$.*

Proof. By our assumption, $f + n_i = \sum_{k \neq i} a_{ik} n_k$ and $f' + n_j = \sum_{l \neq j} b_{jl} n_l$. If $a_{ij} \geq 1$ and $b_{ji} \geq 1$, then summing up these equations, we get

$$\mathrm{F}(H) = f + f' = (b_{ji} - 1)n_i + (a_{ij} - 1)n_j + \sum_{s \neq i,j} (a_{is} + b_{js}) n_s \in H,$$

a contradiction! □

Example 4.3. *A nice property of* $\mathrm{RF}(f)$ *is that we can get generators of* I_H *from the set of matrices* $\{\mathrm{RF}(f) \mid f \in \mathrm{PF}(H)\}$ *by 3.7. Namely, take any 2 rows* $\boldsymbol{a}_i, \boldsymbol{a}_j$ *of* $\mathrm{RF}(f)$ *and write* $\boldsymbol{a}_i - \boldsymbol{a}_j$ *as* $\boldsymbol{b}_+ - \boldsymbol{b}_-$*, which corresponds to an element of* I_H*. We will explain this by 2 examples. In the following, we use variables* x, y, z, w *instead of* $X_1, \dots, X_4$.

(1) *Let* $H = \langle 12, 17, 31, 40 \rangle$ *with* $\mathrm{PF}(H) = \{45, 90\}$*. Since* $90 = 2 \cdot 45$*, we know that* H *is pseudo-symmetric. We compute* $\mathrm{RF}(45) = \begin{pmatrix} -1 & 1 & 0 & 1 \\ 0 & -1 & 2 & 0 \\ 3 & 0 & -1 & 1 \\ 0 & 5 & 0 & -1 \end{pmatrix}$ *and in this case* $I_H = (z^5 - x^3yw, y^6 - z^2w, xz^2 - y^2w, w^2 - xy^4, x^4 - yz)$*. The generators of* I_H *corresponds to* $\boldsymbol{a}_1 - \boldsymbol{a}_3, \boldsymbol{a}_4 - \boldsymbol{a}_2, \boldsymbol{a}_2 - \boldsymbol{a}_1, \boldsymbol{a}_1 - \boldsymbol{a}_4, \boldsymbol{a}_3 - \boldsymbol{a}_1$*, respectively.*

(2) *Let* $H = \langle 18, 21, 23, 26 \rangle$ *with* $\mathrm{PF}(H) = \{31, 66, 97\}$ *and* $I_H = (xw - yz, y^5 - x^2z^3, xz^4 - y^4w, z^5 - y^3w^2, x^2y^2 - w^3, x^3y - zw^2, x^4 - z^2w)$*. We can check that* H *is AS of type* 3 *since* $31 + 66 = 97$ *and we compute*

$$\mathrm{RF}(31) = \begin{pmatrix} -1 & 0 & 1 & 1 \\ 0 & -1 & 0 & 2 \\ 3 & 0 & -1 & 0 \\ 2 & 1 & 0 & -1 \end{pmatrix}, \quad \mathrm{RF}(66) = \begin{pmatrix} -1 & 4 & 0 & 0 \\ 1 & -1 & 3 & 0 \\ 0 & 3 & -1 & 1 \\ 0 & 0 & 4 & -1 \end{pmatrix}.$$

We see that the equations $y^5 - x^2z^3, x^2y^2 - w^3, x^3y - zw^2, x^4 - z^2w$ *are obtained from* $\mathrm{RF}(31)$*,* $xz^4 - y^4w, z^5 - y^3w^2$ *from* $\mathrm{RF}(66)$ *and* $xw - yz$ *from both matrices.*

Moscariello proves a nice property of an RF-matrix. But his result can be improved a little more.

Lemma 4.4. *Assume $e = 4$. Assume $f \in \mathrm{PF}(H), f \neq \mathrm{F}(H)$ and put $A = (a_{ij}) = \mathrm{RF}(f)$. Then for every j, there exists i such that $a_{ij} > 0$. Namely, any column of A should contain some positive component.*

Remark 4.5. Moscariello proves that if for some j and $a_{ij} = 0$ for every $i \neq j$, then $f = \mathrm{F}(H)/2$.

Combining Lemma 4.2 and Lemma 4.4, we get the following Corollary.

Corollary 4.6. *Assume H is AG and let $f \in \mathrm{PF}'(H)$. Then every row of $\mathrm{RF}(f)$ has at least one 0 entry.*

We can restate the structure theorem of Komeda by using $\mathrm{RF}(\mathrm{F}(H)/2)$.

Theorem 4.7. *([Ko]) Let $H = \langle n_1, n_2, n_3, n_4\rangle$ be pseudo-symmetric.* [1]

(1) *For a suitable permutation of $\{1,2,3,4\}$, $\mathrm{F}(H)/2 + n_k$ has UF for every k (that is, $\mathrm{RF}(\mathrm{F}(H)/2)$ is uniquely determined) and $\mathrm{RF}(\mathrm{F}(H)/2)$ is in the following form*

$$\mathrm{RF}(\mathrm{F}(H)/2) = \begin{pmatrix} -1 & \alpha_2 - 1 & 0 & 0 \\ 0 & -1 & \alpha_3 - 1 & 0 \\ \alpha_1 - 1 & 0 & -1 & \alpha_4 - 1 \\ \alpha_1 - 1 & \alpha_2 - 1 - \alpha_{12} & 0 & -1 \end{pmatrix}$$

(2) $\mathrm{F}(H) + n_2$ *has UF and we have* $n_2 = \alpha_1\alpha_4(\alpha_3 - 1) + 1$.

(3) *Every generator of I_H is obtained from $\mathrm{RF}(\mathrm{F}(H)/2)$ as in the Example 4.3. Namely, $I_H = (x_2^{\alpha_2} - x_1x_3^{\alpha_3-1}, x_1^{\alpha_1} - x_2^{\alpha_{12}}x_4, x_3^{\alpha_3} - x_1^{\alpha_1-1}x_2x_4^{\alpha_4-1}, x_3^{\alpha_3-1}x_4 - x_1^{\alpha_1-1}x_2^{\alpha_2-\alpha_{12}}, x_4^{\alpha_4} - x_2^{\alpha_2-1-\alpha_{12}}x_3)$. (The difference of 1st and 3rd rows does not give a minimal generator of I_H.)*

Remark 4.8. The generators of I_H in [Ko] or [BFS] are obtained after the permutation $(1,2,3,4) \to (3,1,4,2)$. Namely, if we put

$$\mathrm{RF}(\mathrm{F}(H)/2) = \begin{pmatrix} -1 & 0 & 0 & \alpha_4 - 1 \\ \alpha_{21} & -1 & \alpha_3 - 1 & 0 \\ \alpha_1 - 1 & 0 & -1 & 0 \\ 0 & \alpha_2 - 1 & \alpha_3 - 1 & -1 \end{pmatrix}$$

then we get their equations.

Using $\mathrm{RF}(f)$, we can have a different proof of Moscariello's Theorem.

[1] Komeda uses the terminology "almost symmetric" for pseudo-symmetric

Theorem 4.9. [Mo] *If* $H = \langle n_1, \ldots, n_4\rangle$ *is* AG, *then* $\mathrm{type}(H) \le 3$.

We will not present the proof here but we list the lemmas which we use to prove this theorem.

Lemma 4.10. *We denote by* $\mathbf{e}_i$ *the i-th unit vector of* $\mathbb{Z}^4$. *Assume* $e = 4$ *and* H *is AS.*

(1) *There are* 2 *rows in* $\mathrm{RF}(\mathrm{F}(H)/2)$ *of the form* $(\alpha_i - 1)\mathbf{e}_i - \mathbf{e}_k$ *situated as the k-th row.*

(2) *If* $f \ne f' \in \mathrm{PF}(H)$ *with* $f + f' = \mathrm{F}(H)$, *then there are* 4 *rows in* $\mathrm{RF}(f)$ *and* $\mathrm{RF}(f')$ *of the form* $(\alpha_i - 1)\mathbf{e}_i - \mathbf{e}_k$ *situated as the k-th row.*

Lemma 4.11. *Assume* $e = 4$ *and* H *is AS. Then for any* $f, f' \in \mathrm{PF}'(H)$, $f \ne f', f + n_k \ne f' + n_l$ *for any* $1 \le k, l \le 4$.

The following question is asked in [Mo].

Question 4.12. Is $\mathrm{type}(H)$ bounded for a given e if H is AS ? If this is the case, what is the upper bound?

5. The free resolution of $k[H]$ and the mapping cone

Let as before $H = \langle n_1, \ldots, n_e\rangle$, $S = k[X_1, \ldots, X_e]$ and $\pi : S \to R = k[H]$ defined by $\pi(X_i) = t^{n_i}$. The minimal free resolution of R over S is very important in commutative ring theory.

Let $(F_\bullet, d_\bullet)$ be a minimal free resolution of $R = k[H]$ as a graded S module. Let $F_i = \oplus_j S(-b_{ij})$ and set $\beta_i = \mathrm{rank}(F_i)$. For example, the multiset $\{b_{ij}\}$ is the multiset of the degrees of the minimal generators of I_H and β_1 is the number of the minimal generators of I_H.

Note that $F_{e-1} \cong \oplus_{f\in \mathrm{PF}(H)} S(-f - N)$, where we put $N = \sum_{i=1}^{e} n_i$ and $K_R \cong \oplus_{n\in \mathbb{N}\setminus H} kt^{-n}$.

Let us recall that R is AG (or H is AS) if the cokernel of a natural R-module homomorphism

$$R \to K_R(-\mathrm{F}(H))$$

is annihilated by $\mathfrak{m} := R_+$. In other words, there is an exact sequence of graded S-modules

$$0 \to R \to K_R(-\mathrm{F}(H)) \to \oplus_{f\in \mathrm{PF}'(H)} k(-f) \to 0.$$

Note that we used the symmetry of $\mathrm{PF}(H)$ (Theorem 1.5) when H is aAS.

Since $K_S \cong S(-N)$, the minimal free resolution of K_R is given by $(F_\bullet)^\vee$, where $(\bullet)^\vee = \mathrm{Hom}_S(\bullet, S(-N))$. Now, the injection $R \to K_R(-\mathrm{F}(H))$ lifts to a morphism $\phi : F_\bullet \to (F_\bullet)^\vee(-\mathrm{F}(H))$ and the cokernel of $R \to K_R(-\mathrm{F}(H))$ is given by the mapping cone $MC(\phi)$ of ϕ.

On the other hand, the free resolution of the residue field k is given by the Koszul complex $Kos_\bullet = Kos_\bullet(X_1, \ldots, X_e)$. Hence we get

Lemma 5.1 (Key Lemma). *The mapping cone $MC(\phi)$ gives a (non-minimal) free resolution of $\oplus_{f \in \mathrm{PF}(H), f \neq \mathrm{F}(H)} k(-f)$. Hence, the minimal free resolution obtained from $MC(\phi)$ is isomorphic to $\oplus_{f \in \mathrm{PF}'(H)} Kos_\bullet(-f)$.*

What can we say from this lemma ? For example, we get the structure of F_1 and F_{e-2}.

Lemma 5.2. *$\beta_{e-2} \geq e(t(H) - 1)$ and if $\beta_{e-2} = e(t(H) - 1) + s$, then there exist minimal generators $g_1, \ldots, g_s$ of I_H such that the multisets $\{b_{e-2,j}\}_{j=1}^{\beta_{e-2}}$ and $\coprod_{f \in \mathrm{PF}'(H)} \{f + N - n_i\}_{i=1}^{4} \coprod \{\mathrm{F}(H) + N - \deg(g)\}_{i=1}^{s}$ coincide.*

The following statement for type 3 AS looks very probable but we do not have a proof yet.

Conjecture 5.3. *Assume that H is AS with $\langle n_1, n_2, n_3, n_4 \rangle$ and* $\mathrm{type}(H) = 3$ *with* $\mathrm{PF}(H) = \{f, f', \mathrm{F}(H)\}$ *with* $f + f' = \mathrm{F}(H)$. *Then I_H is minimally generated by* 6 *or* 7 *elements and* 6 *of minimal generators are obtained from* $\mathrm{RF}(f)$ *or* $\mathrm{RF}(f')$ *as in Remark 2.2 with no cancellation. If $\mu(I_H) = 7$, then $X_1X_4 - X_2X_3 \in I_H$.*

6. When is $H + m$ AS for infinitely many m?

Definition 6.1. For $H = \langle n_1, n_2, \ldots, n_e \rangle$, we put $H + m = \langle n_1 + m, n_2 + m, \ldots, n_e + m \rangle$ In this section, we always assume that $n_1 < n_2 < \ldots < n_e$ and put $s = n_e - n_1$.

Families of semigroups of the type $H + m$ have first been considered by Herzog and Srinivasan. They conjectured that the Betti numbers $\beta_i(I_{H+m})$ are periodic functions on m for $m \gg 0$. This conjecture was later proved by Than Vu [Vu]. It is a natural question whether the AS property behaves in the same way.

First, we give a lower bound for the Frobenius number of $H + m$.

Lemma 6.2. *If we put $n_e - n_1 = s$, then* $\mathrm{F}(H+m) \geq m^2/s$ *for all sufficiently large* m.

The following fact is trivial but very important in our argument.

Lemma 6.3. *If* $\phi = \prod_{i=1}^{e} X_i^{a_i} - \prod_{i=1}^{e} X_i^{b_i} \in I_H$ *is homogeneous, namely, if* $\sum_{i=1}^{e} a_i = \sum_{i=1}^{e} b_i$, *then* $\phi \in I_{H+m}$ *for every* m.

We define $\alpha_i(m)$ to be the minimal positive integer such that

$$\alpha_i(m)(n_i + m) = \sum_{j=1, j\neq i}^{e} \alpha_{ij}(m)(n_j + m),$$

similarly as in Definition 3.3.

Lemma 6.4. *Let* $H + m$ *be as in Definition 6.1. Then,* $\alpha_2(m), \ldots, \alpha_{e-1}(m)$ *is constant for* $m \gg 1$, *while* $\alpha_1(m), \alpha_e(m) \geq m/s$

If we recall the form of $\mathrm{RF}(\mathrm{F}(H)/2)$ in Theorem 4.7, the following result is easily proved.

Theorem 6.5. *Assume* $H + m = \langle n_1 + m, \ldots, n_4 + m\rangle$. *Then for large enough* m, $H + m$ *is not AG of type* 2.

We think the following will be true.

Conjecture 6.6. *If* $H+m = \langle n_1+m, n_2+m, \ldots, n_e+m\rangle$ *is AS for infinitely many* m, *then* $\mathrm{type}(H + m)$ *is odd if* $H + m$ *is AG.*

Unlike the case of type 2, there are infinite series of $H + m$, which are AG of type 3 for infinitely many m. The following example was given by T. Numata.

Example 6.7. *If* $H = \langle 10, 11, 13, 14\rangle$, *then* $H + 4m$ *is AS of type* 3 *for all integer* $m \geq 0$.

Example 6.8. *For the following* H, $H + m$ *is AS with type* 3 *if* m *is a multiple of* $s = n_4 - n_1$.

(1) $H = \langle 34, 37, 39, 42\rangle$,
(2) $H = \langle 14, 19, 21, 26\rangle$
(3) $H = \langle 18, 25, 27, 34\rangle$.

We think we can completely determine $H+m$ which are AS of type 3 for infinitely many m. But the following is what we get at this moment.

Proposition 6.9. *Assume $H+m$ is AS of type 3 for infinitely many m. We fix sufficiently big m such that $H+m$ is AS of type 3. If $\mathrm{PF}(H+m) = \{f(m), f'(m), \mathrm{F}(H+m)\}$ with $f(m) < f'(m)$ and $f(m)+f'(m) = \mathrm{F}(H+m)$, then we have the following facts.*

(1) *$\alpha_2 = \alpha_3 \geq 3$ and an odd integer. We will write $a = \alpha_2 = \alpha_3$ in the following.*

(2) $\mathrm{RF}(f(m)) = \begin{pmatrix} -1 & a-1 & 0 & 0 \\ 1 & -1 & a-2 & 0 \\ 0 & a-2 & -1 & 1 \\ 0 & 0 & a-1 & -1 \end{pmatrix}$.

(3) *Hence, we have the relations $(n_1+m)+f(m) = (a-1)(n_2+m)$ and $(n_4+m)+f(m) = (a-1)(n_3+m)$, in particular, $xy-zw, y^a-x^2z^{a-2}, z^a-y^{a-2}w^2 \in I_{H_m}$.*

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