数理解析研究所講究録
第2008巻 2016年 121-128

Disk arrays and cyclic orderings

Tomoko Adachi
Department of Information Sciences, Toho University
2-2-1 Miyama, Funabashi, Chiba, 274-8510, Japan
E-mail: adachi@is.sci.toho-u.ac.jp

Abstract These disk array architectures are known as redundant arrays of independent disks (RAID) . Minimizing the number of disk operations when writing to consecutive disks leads to cyclic orderings. Using the special bipartite graph $H(h;t)$, Mueller et al.(2005) gave label in the case of $h=1,2$. Adachi and Kikuchi (2015) gave label in the case of $h=3$. In this paper, we give a new label in the case of $h=4$ and $t=1$, in order to investigate infinite family $H(4;t)$. And we obtain a cyclic ordering for the complete bipartite graph $K_{36,36}$.

1. Introduction

The desire to speed up secondary storage systems has lead to the development of *disk arrays* which achieve performance through disk parallelism. To avoid high rates of data loss in large disk arrays one includes redundant information stored on the check disks which allows the reconstruction of the original data stored on the information disks even in the presence of disk failures. These disk array architectures are known as *redundant arrays of independent disks* (RAID) (see [11] and [10]).

Optimal erasure-correcting codes using combinatorial framework in disk arrays are discussed in [11] and [9]. For an optimal ordering, there are [5] and [6]. Cohen et al. [8] gave a cyclic construction for a cluttered ordering of the complete graph. In the case of a complete graph, there are [7] and [3]. Furthermore, in the case of a complete bipartite graph, [12] and [2] gave a cyclic construction for a cluttered ordering of the complete bipartite graph by utilizing the notion of a wrapped Δ-labelling. In the case of a complete tripartite graph, we refer to [1].

In a RAID system disk writes are expensive operations and should therefore be minimized. In many applications there are writes on a small fraction of consecutive disks — say d disks — where d is small in comparison to k, the number of information disks. Therefore, to minimize the number of operations when writing to d consecutive information disks one has to minimize the number of check disks — say f — associated to the d information disks. Minimizing the number of disk operations when writing to consecutive disks leads to the concept of (d,f)-cluttered orderings which were introduced for the complete graph by Cohen et al. [8]. Mueller et al. [12] adapted the concept of wrapped Δ-labellings to the complete bipartite graph.

Using the special bipartite graph $H(h;t)$ in section 3, Mueller et al. [12] gave label in the case of $h = 1, 2$. Adachi and Kikuchi [2] gave label in the case of $h = 3$. In this paper, we give a new label in the case of $h = 4$ and $t = 1$, and give a cyclic ordering for the complete bipartite graph $K_{36,36}$.

2. A Cyclic Ordering

Let $G = (V, E)$ be a graph with $n = |V|$ and $E = \{e_0, e_1, \cdots, e_{m-1}\}$. Let $d \leq m$ be a positive integer, called a *window* of G, and π a permutation on $\{0, 1, \cdots, m-1\}$, called an *edge ordering* of G. Then, given a graph G with edge ordering π and window d, we define $V_i^{\pi,d}$ to be the set of vertices which are connected by an edge of $\{e_{\pi(i)}, e_{\pi(i+1)}, \cdots, e_{\pi(i+d-1)}\}$, $0 \leq i \leq m-1$, where indices are considered modulo m. The cost of accessing a subgraph of d consecutive edges is measured by the number of its vertices. An upper bound of this cost is given by the *d-maximum access cost* of G defined as $\max_i |V_i^{\pi,d}|$. An ordering π is a *(d, f)-cluttered ordering*, if it has d-maximum access cost equal to f. We are interested in minimizing the parameter f.

In the following, $H = (U, E)$ always denotes a bipartite graph with vertex set U which is partitioned into two subsets denoted by V and W. Any edge of the edge set E contains exactly one point of V and W respectively. Let $\ell = |E|$, then a *Δ-labelling* of H with respect to V and W is defined to be a map $\Delta : U \to Z_\ell \times Z_2$ with $\Delta(V) \subset Z_\ell \times \{0\}$ and $\Delta(W) \subset Z_\ell \times \{1\}$, where each element of Z_ℓ occurs exactly once in the difference list

$$\Delta(E) := \Big(\pi_1(\Delta(v) - \Delta(w))\Big| v \in V, w \in W, (v, w) \in E\Big). \tag{2.1}$$

Here, $\pi_1 : Z_\ell \times Z_2 \to Z_\ell$ denotes the projection on the first component. In general, Δ-labellings are a well-known tool for the decomposition of graphs into subgraphs (see [4]). In this context a decomposition is understood to be a partition of the edge set of the graph. In case of the complete bipartite graph, one has the following proposition.

Proposition 2.1 *([12]) Let $H = (U, E)$ be a bipartite graph, $\ell = |E|$, and Δ be a Δ-labelling as defined above. Then there is a decomposition of the complete bipartite graph $K_{\ell,\ell}$ into isomorphic copies of H.*

Next, we define the concept of a (d, f)-movement which can easily be generalized to arbitrary set system.

Definition 2.1 Let G be a graph with edge set $E(G) = \{e_0, e_1, \ldots e_{n-1}\}$, where n is positive integer, and let $\Sigma_0, \Sigma_1 \subset E(G)$ with $d := |\Sigma_0| = |\Sigma_1|$. For a permutation σ on $\{0,1,\cdots,n-1\}$ define $V_i^{\sigma,d} := \bigcup_{j=0}^{d-1} e_{\sigma(i+j)}$ for $0 \le i \le n-d$. Then, for some given a positive integer f, and a map σ is called a (d,f)*-movement from* Σ_0 *to* Σ_1 if $\Sigma_0 = \{e_{\sigma(j)} | 0 \le j \le d-1\}$, $\Sigma_1 = \{e_{\sigma(j)} | n-d \le j \le n-1\}$, and $\max_i |V_i^{\sigma,d}| \le f$.

In order to assemble such (d,f)-movements of certain subgraphs to a (d,f)-cluttered ordering, we need some notion of consistency. Let $\varphi : \Sigma_0 \to \Sigma_1$ be any bijection, then a (d,f)-movement σ from Σ_0 to Σ_1 is called *consistent* with φ if

$$\varphi(e_{\sigma(j)}) = e_{\sigma(n-d+j)}, \quad \text{for } j = 0,1,\ldots,d-1. \tag{2.2}$$

Now, for each $j \in Z_\ell$ one gets an automorphism τ_j of the bipartite graph $K_{\ell,\ell}$ defined by cyclic translation of the vertex set:

$$\tau_j : Z_\ell \times Z_2 \to Z_\ell \times Z_2, \quad \tau_j((u,b)) := (u+j, b), \tag{2.3}$$

$(u,b) \in Z_\ell \times Z_2$. Obviously, τ_j induces in a natural way an automorphism of the edge set of $K_{\ell,\ell}$ which we also denote τ_j. Then, $\tau_j(E^{(i)}) = E^{(i+j)}$ and $\tau_j(\Sigma_0^{(i)}) = \Sigma_0^{(i+j)}$, $i \in Z_\ell$. Next, we define a subgraph $G^{(0)} \subset K_{\ell,\ell}$ by specifying its edge set $E(G^{(0)}) := E^{(0)} \cup \Sigma_0^{(\kappa)}$. Let $E(G^{(0)}) = \{e_0^{(0)}, e_1^{(0)}, \ldots e_{n-1}^{(0)}\}$, $n = \ell + d$, where we fix some arbitrary edge ordering. We denote the restriction of the cyclic translation τ_κ to $\Sigma_0^{(0)}$ by $\varphi_\kappa^{(0)}$ which defines a bijection $\varphi_\kappa^{(0)} : \Sigma_0^{(0)} \to \Sigma_0^{(\kappa)}$.

Definition 2.2 With above notation, a (d,f)-movement of $G^{(0)}$ from $\Sigma_0^{(0)}$ to $\Sigma_0^{(\kappa)}$ consistent with $\varphi_\kappa^{(0)}$ will be denoted as (d,f)*-movement from* $\Sigma_0^{(0)}$ *consistent with the translation parameter* κ.

According to Definition 1, such a (d,f)-movement is given by some permutation σ of the index set $\{0,1,\ldots,n-1\}$. By applying the cyclic translation τ_i one gets a graph $G^{(i)} := \tau_i(G^{(0)})$ with edge set $E(G^{(i)}) = E^{(i)} \cup \Sigma_0^{(i+\kappa)} = \{e_0^{(i)}, e_1^{(i)}, \ldots e_{n-1}^{(i)}\}$, $i \in Z_\ell$. We denote the restriction of τ_κ to $\Sigma_0^{(i)}$ by $\varphi_\kappa^{(i)}$ which defines a bijection

$$\varphi_\kappa^{(i)} : \Sigma_0^{(i)} \to \Sigma_0^{(i+\kappa)}, \qquad \varphi_\kappa^{(i)}(e^{(i)}) = e^{(i+\kappa)}, \ e^{(i)} \in \Sigma_0^{(i)}. \tag{2.4}$$

Then σ also defines a (d,f)-movement of $G^{(i)}$ from $\Sigma_0^{(i)}$ to $\Sigma_0^{(i+\kappa)}$ consistent with $\varphi_\kappa^{(i)}$. Using that $e_{\sigma(j)}^{(i)} \in \Sigma_0^{(i)}$, $0 \le j < d$, (see Defintion 1), we get, for $j = 0,1,\ldots,d-1$,

$$e_{\sigma(j)}^{(i+\kappa)} \overset{(2.4)}{=} \varphi_\kappa^{(i)}\left(e_{\sigma(j)}^{(i)}\right) \overset{(2.2)}{=} e_{\sigma(n-d+j)}^{(i)} = e_{\sigma(\ell+j)}^{(i)}. \tag{2.5}$$

Having such a consistent σ, it is easy to construct a (d,f)-cluttered ordering of $K_{\ell,\ell}$. In short, one orders the edges of $K_{\ell,\ell}$ by first arranging the subgraphs of the decomposition along $E^{(0)}, E^{(\kappa)}, E^{(2\kappa)}, \ldots, E^{((\ell-1)\kappa)}$ and then ordering the edges within each subgraph according to σ.

Proposition 2.2 *([12]) Let $H = (U, E)$, $\ell = |E|$, be a bipartite graph allowing some ρ-labelling, and let κ be a translation parameter coprime to ℓ. Furthermore, let $\Sigma_0 \subset E$, $d := |\Sigma_0|$. If there is a (d, f)-movement from Σ_0 consistent with κ, then there also is a (d, f)-cluttered ordering for the complete bipartite graph $K_{\ell,\ell}$.*

3. Labelling of $H(h;t)$

In this section, we define an infinite family of bipartite graphs which allow (d, f)-movements with small f. In order to ensure that these (d, f)-movements are consistent with some translation parameter κ, we impose an additional condition on the Δ-labellings also referred to as *wrapped-condition.*

Let h and t be two positive integers. For each parameter h and t, we define a bipartite graph denoted by $H(h;t) = (U, E)$. Its vertex set U is partitioned into $U = V \cup W$ and consists of the following $2h(t + 1)$ vertices:

$$\begin{aligned} V &:= \{v_i | 0 \leq i < h(t+1)\}, \\ W &:= \{w_i | 0 \leq i < h(t+1)\}. \end{aligned}$$

The edge set E is partitioned into subsets E_s, $0 \leq s < t$, defined by

$$\begin{aligned} E'_s &:= \{\{v_i, w_j\} | s \cdot h \leq i, j < s \cdot h + h\}, \\ E''_s &:= \{\{v_i, w_{h+j}\} | s \cdot h \leq j \leq i < s \cdot h + h\}, \\ E'''_s &:= \{\{v_{h+i}, w_j\} | s \cdot h \leq i \leq j < s \cdot h + h\}, \\ E_s &:= E'_s \cup E''_s \cup E'''_s, \quad \text{for } 0 \leq s < t, \\ E &:= \bigcup_{s=0}^{t-1} E_s. \end{aligned}$$

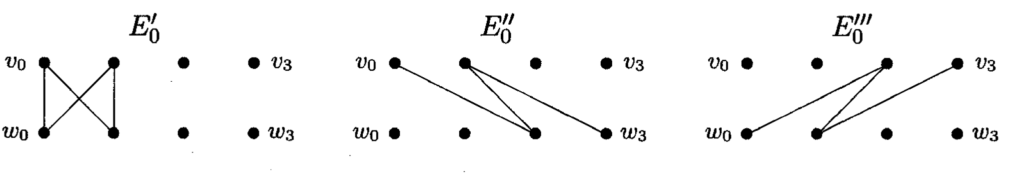

Figure 3.1: Partition of the edge set of $H(2;1)$.

Fig. 3.1 shows the edge partition of $H(2;1)$. For the number of edges holds $|E| = t \cdot (h^2 + \frac{h(h+1)}{2} + \frac{h(h+1)}{2}) = th(2h+1)$. The t subgraphs defined by the edge sets E_s, $0 \leq s < t$, and its respective underlying vertex sets are isomorphic to $H(h;1)$. Intuitively speaking, the bipartite graph $H(h;t)$ consists of t consecutive copies of $H(h;1)$, where the last h vertices of V and W respectively of one copy are identified with the first h vertices of V and W respectively of the next copy. Traversing these copies with increasing s will define a (d,f)-movement of $H(h;t)$ with small parameter f as is shown in the next proposition.

Proposition 3.3 *([12]) Let h,t be pogitive integers. Let $H(h;t)=(U,E)$, $t\geq 2$, be the bipartite graph as defined above. Then, there is a (d,f)-movement of $H(h;t)$ from E_0 to E_{t-1} with $d=h(2h+1)$ and $f=4h$.*

By Proposition 2.1 a Δ-labelling of the graph $H(h;t)$ will lead to a decomposition of the complete bipartite graph $K_{\ell,\ell}$ into ℓ isomorphic copies of $H(h;t)$, where $\ell = th(2h+1)$. However, in general there is no (d,f)-movement consistent with some translation parameter κ. To this means, we impose an additional condition on the Δ-labelling. The following definition generalizes and adapts the notion of a *wrapped Δ-labelling* to the bipartite case, which was introduced in [8] for certain subgraphs of the complete graph.

Definition 3.1 Let $H=(U,E)$, $\ell=|E|$, denote a bipartite graph and let $X,Y\subset U$ with $|X|=|Y|$. A Δ-labelling Δ is called a *wrapped Δ-labelling* of H relative to X and Y if there exists a $\kappa\in Z$ coprime to ℓ such that

$$\Delta(Y)=\Delta(X)+(\kappa,0) \tag{3.1}$$

as multisets in $Z_\ell\times Z_2$. The parameter κ is also referred to as *translation parameter* of the wrapped Δ-labelling.

For the graphs $H=H(h;t)$, we define $X:=\{v_i,w_i|0\leq i<h\}$ and $Y:=\{v_i,w_i|ht\leq i<h(t+1)\}$. Furthermore, in the following we only consider wrapped Δ-labellings relative to X and Y for which the stronger condition

$$\Delta(v_{i+ht})=\Delta(v_i)+(\kappa,0) \quad\text{and}\quad \Delta(w_{i+ht})=\Delta(w_i)+(\kappa,0), \tag{3.2}$$

hold for $0\leq i<h$. Suppose we have such labelling Δ satisfying condition (3.2). Now, $E^{(i)}$, $i\in Z_\ell$, are isomorphic copies of $H(h;t)$. Furthermore, $\Sigma_0^{(\kappa)}$ is isomorphic to $H(h;1)$ consisting of the first d edges of $E^{(\kappa)}$. From condition (3.2) follows that the graph $G^{(0)}\subset K_{\ell,\ell}$ with edge set $E(G^{(0)}):=E^{(0)}\cup\Sigma_0^{(\kappa)}$ can obviously identified with $H(h;t+1)$. In addition, one easily checks that the (d,f)-movement of $G^{(0)}=H(h;t+1)$ from Proposition 3.3 is consistent with the translation parameter κ.

Proposition 3.4 *([12]) Let h,t be positive integers. From any wrapped Δ-labelling of $H(h;t)$, satisfying condition (3.2), one gets a (d,f)-cluttered ordering of the complete bipartite graph $K_{\ell,\ell}$ with $\ell=th(2h+1)$, $d=h(2h+1)$, and $f=4h$.*

Now, we construct some infinite families of such wrapped Δ-labellings. By applying Proposition 2.2 we get explicite (d, f)-cluttered orderings of the corresponding bipartite graphs.

Theorem 3.1 *([12]) Let t be a positive integer. For all t there is a (d, f)-cluttered ordering of the complete bipartite graph $K_{3t,3t}$ with $d = 3$ and $f = 4$.*

Theorem 3.2 *([12]) Let t be a positive integer. For all t there is a (d, f)-cluttered ordering of the complete bipartite graph $K_{10t,10t}$ with $d = 10$ and $f = 8$.*

Theorem 3.3 *([2]) Let t be a positive integer. For all t there is a (d, f)-cluttered ordering of the complete bipartite graph $K_{21t,21t}$ with $d = 21$ and $f = 12$.*

Here, we define a wrapped Δ-labelling of $H(4;1)$. $H(4;1) = (U, E)$ has 16 vertices and 36 edges. For a fixed t, a labelling Δ is a map $\Delta : U \to Z_8 \times Z_2$ on the vertex set $U = V \cup W$. We specify the second component of Δ on the vertices $V = (v_0, v_1, \ldots, v_7)$ by

$$0, a, 2a, 3a, \kappa, a + \kappa, 2a + \kappa, 3a + \kappa, \tag{3.3}$$

and on the vertices $W = (w_0, w_1, \ldots, w_7)$ by

$$0, b, 2b, 3b, \kappa, b + \kappa, 2b + \kappa, 3b + \kappa. \tag{3.4}$$

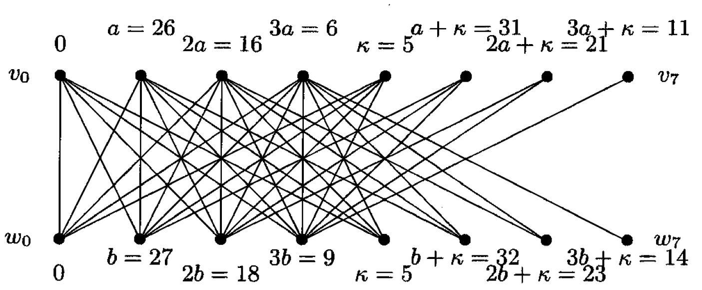

Figure 3.2: A wrapped Δ-labelling of $H(4;1)$, $|E| = 36$, $|V| = 16$, $\kappa = 5$.

Proposition 3.5 *As the values of a, b, κ of equations (3.3) and (3.4), we set*

$$a = 26, b = 27, \kappa = 5. \tag{3.5}$$

Then the differences of Δ using the notation from (2.1) cover all numbers in Z_{36} exactly once.

Proof. Suppose that we set equation (3.5). We now compute the differences of Δ using the notation from (2.1). All integers are considered modulo 36.

$$\begin{aligned}
\Delta(E_0') &= \{0,(a-b),2(a-b),3(a-b),a,2a,3a,-b,2a-b,3a-b,\\
&\quad -2b,a-2b,3a-2b,-3b,a-3b,2a-3b\}\\
&= \{0,-1,-2,-3,26,16,6,9,25,15,18,8,24,27,17,7\}\\
&= \{6,7,8,9\}\cup\{15,16,17,18\}\cup\{24,25,26,27\}\cup\{33,34,35,0\}\\
\Delta(E_0'') &= \{-\kappa,-\kappa+(a-b),-\kappa+2(a-b),-\kappa+3(a-b),\\
&\quad -\kappa+a,-\kappa+2a-b,-\kappa+3a-2b,-\kappa+2a,-\kappa+2a-b,-\kappa+3a\}\\
&= \{-5,-6,-7,-8,21,20,19,11,10,1\}\\
&= \{1\}\cup\{10,11\}\cup\{19,20,21\}\cup\{28,29,30,31\}\\
\Delta(E_0''') &= \{\kappa,\kappa+(a-b),\kappa+2(a-b),\kappa+3(a-b),\\
&\quad \kappa-b,\kappa+a-2b,\kappa+2a-3b,\kappa-2b,\kappa+a-2b,\kappa-3b\}\\
&= \{5,4,3,2,14,13,12,23,22,32\}\\
&= \{2,3,4,5\}\cup\{12,13,14\}\cup\{22,23\}\cup\{32\}.
\end{aligned}$$

From this one easily checks that above lists cover all numbers in Z_{36} exactly once.

(Q.E.D.)

Note that $|E| = 36$ and $\kappa = 5$ are coprime and that the wrapped-condition (3.2) is obviously fulfilled. By Proposition 3.5, the differences of Δ using the notation from (2.1) cover all numbers in Z_{36} exactly once. Thus, Δ defines a wrapped Δ-labelling. By applying Proposition 3.4 we get the following result.

Theorem 3.4 *There is a (d, f)-cluttered ordering of the complete bipartite graph $K_{36,36}$ with $d = 36$ and $f = 16$.*

Here we can obtain a wrapped Δ-labelling of $H(4;1)$. And we are investigating $H(4;2)$, $(4;3)$, $\cdots$, $H(4;t)$. Form the proofs of Theorem 3.1, 3.2 and 3.3, we have obtain a wrapped Δ-labelling of $H(1;t)$, $H(2;t)$ and $H(3;t)$. In the future, we will investigate $H(4;t)$, $H(5;t)$, $\cdots$, $H(h;t)$.

References

[1] Adachi T. (2007), Optimal ordering of the complete tripartite graph $K_{9,9,9}$. *Proceedings of the Fourth International Conference on Nonlinear Analysis and Convex Analysis*, Yokohama Publishers, Inc., 1–10.

[2] Adachi T. and Kikuchi D. (2015), Some sequence of wrapped Δ-labellings for the complete bipartite graph. *Applied Mathematics*, 5(1):195–205.

[3] Adachi T. and Uehara H. (2014), Construction of Wrapped ρ-Labellings for RAID. *Journal of Mathematics and System Science*, 4(12):750-754.

[4] Bosák J. (1990), *Decompositions of Graphs.*, Kluwer Academic Publishers, Dordrecht.

[5] Cohen M. and Colbourn C. (2000), Optimal and pessimal orderings of Steiner triple systems in disk arrays. *LATIN 2000*, Lect. Notes Comp. Sci. 1776, Springer-Verlag, 95–104.

[6] Cohen M. and Colbourn C. (2001), Ordering disks for double erasure codes. *ACM Symposium on Parallel Algorithms and Architectures*, Theory Comput. Syst. 34, Springer-Verlag, 229–236.

[7] Cohen M. and Colbourn C. (2004), Ladder orderings of pairs and RAID performance. *Discrete Applied Mathematics*, 138:35–46.

[8] Cohen M., Colbourn C. and Froncek D. (2001), Cluttered orderings for the complete graph. *Computing and Combinatorics: Proc. 7th annual international conference, COCOON 2001*, Lect. Notes Comp. Sci. 2108, Springer-Verlag, 420–431.

[9] Chee Y., Colbourn C. and Ling A. (2000), Asymptotically optimal erasure-resilient codes for large disk arrays. *Discrete Applied Mathematics*, 102:3–36.

[10] Chen P., Lee E., Gibson G., Katz R. and Ptterson D. (1994), RAID: High-performance, reliable secondary storage. *ACM Computing Surveys*, 26:145–185.

[11] Hellerstein L., Gibson G., Karp R., Katz R. and Patterson D. (1994), Coding techniques for handling failures in large disk arrays. *Algorithmica*, 12:182–208.

[12] Mueller M., Adachi T. and Jimbo M. (2005), Cluttered orderings for the complete bipartite graph. *Discrete Applied Mathematics*, 152:213–228.