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Initial value conditions for the Navier-Stokes equations in the weighted Serrin class

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1 Introduction

This manuscript contains a summary of [3] and new related references. For details we refer the reader to [3] and [2]. In [3] we consider the initial value problem

$$\partial_t u - \Delta u + u\cdot\nabla u + \nabla p = f,\ \ \operatorname{div} u = 0\ \text{ in } (0,T)\times\Omega \qquad (1.1)$$
$$u|_{\partial\Omega} = 0,\ u(0) = u_0$$

in a bounded domain $\Omega\subset\mathbb{R}^3$ with boundary $\partial\Omega$ of class $C^{2,1}$ and a time interval $[0,T)$, $0<T\le\infty$. We define a new type of a strong solution, the "$L^s_\alpha(L^q)$-strong solution" as follows.

Definition 1.1. *Let* $u_0\in L^2_\sigma(\Omega)$ *be an initial value and let* $f=\operatorname{div} F$ *with* $F=(F_{ij})^3_{i,j=1}\in L^2(0,T;L^2(\Omega))$ *be an external force. A vector field*

$$u\in L^\infty(0,T;L^2_\sigma(\Omega))\cap L^2(0,T;W^{1,2}_0(\Omega)) \qquad (1.2)$$

is called a weak solution (in the sense of Leray-Hopf) of the Navier-Stokes system (1.1) *with data* u_0, f, *if the relation*

$$-\langle u,w_t\rangle_{\Omega,T} + \langle\nabla u,\nabla w\rangle_{\Omega,T} - \langle uu,\nabla w\rangle_{\Omega,T} = \langle u_0,w(0)\rangle_\Omega - \langle F,\nabla w\rangle_{\Omega,T} \qquad (1.3)$$

holds for each test function $w\in C^\infty_0([0,T);C^\infty_{0,\sigma}(\Omega))$, *and if the energy inequality*

$$\frac12\|u(t)\|_2^2 + \int_0^t \|\nabla u\|_2^2\,\mathrm{d}\tau \le \frac12\|u_0\|_2^2 - \int_0^t (F,\nabla u)\,\mathrm{d}\tau \qquad (1.4)$$

is satisfied for $0 \le t < T$.

A weak solution u *of* (1.1) *is called an* $L^s_\alpha(L^q)$*-strong solution with exponents* $2 < s < \infty$, $3 < q < \infty$ *and weight* τ^α *in time,* $0 < \alpha < \frac{1}{2}$, *where* $\frac{2}{s} + \frac{3}{q} = 1 - 2\alpha$ *such that additionally the weighted Serrin condition*

$$u \in L^s_\alpha(0,T;L^q(\Omega)), \quad i.e. \quad \int_0^T (\tau^\alpha \|u(\tau)\|_q)^s \,\mathrm{d}\tau < \infty \tag{1.5}$$

is satisfied. If in (1.5) $\alpha = 0$ *and* $\frac{2}{s} + \frac{3}{q} = 1$, *then* u *is called a strong solution (*$L^s(L^q)$*-strong solution).*

The existence of at least one weak solution u of (1.1) is well-known since the pioneering work of [7, 9]. The existence of a strong solution u of (1.1) could be shown up to now at least in a sufficiently small interval $[0,T), 0 < T \le \infty$, and under additional smoothness conditions on the initial data u_0 and the external force f. The first sufficient condition on the initial data for a bounded domain seems to be due to [8], yielding a solution class of so-called local strong solutions. Since then many results on sufficient initial value conditions for the existence of local strong solutions have been developed. Recent results in [4, 5] yield sufficient and necessary conditions, also written in terms of (solenoidal) Besov spaces $\mathbb{B}^{-\frac{2}{s}}_{q,s}(\Omega) = \mathbb{B}^{-1+\frac{3}{q}}_{q,s}(\Omega)$ where $\frac{2}{s} + \frac{3}{q} = 1$. In this work, we are interested in a weighted Serrin condition with respect to time and $L^s_\alpha(L^q)$-strong solutions. Our result in [3] yields a sufficient condition on initial data and external force to guarantee the existence of local $L^s_\alpha(L^q)$-strong solutions. The motivation for this approach is an extension of the results in [4, 5] where $\frac{2}{s} + \frac{3}{q} = 1$ to the case $u_0 \notin \mathbb{B}^{-1+\frac{3}{q}}_{q,s}(\Omega)$, i.e.,

$$e^{-\tau A}u_0 \notin L^s(0,T;L^q(\Omega)), \quad \text{but} \quad \int_0^T \left(\tau^\alpha \|e^{-\tau A}u_0\|_q\right)^s \mathrm{d}\tau < \infty, \ \ \frac{2}{s} + \frac{3}{q} = 1 - 2\alpha$$

with some $\alpha > 0$. More precisely, for the case $\alpha = 0$ (classical Serrin class), the condition $e^{-\tau A}u_0 \in L^{s(q,0)}(0,T;L^q(\Omega))$ with $\frac{2}{s(q,0)} + \frac{3}{q} = 1$ is equivalent to $u_0 \in \mathbb{B}^{-1+\frac{3}{q}}_{q,s(q,0)}(\Omega)$, whereas for α with $0 < \alpha < \frac{1}{2}$ (weighted Serrin class) the condition $e^{-\tau A}u_0 \in L^{s(q,\alpha)}_\alpha(0,T;L^q(\Omega))$ with $\frac{2}{s(q,\alpha)} + \frac{3}{q} = 1 - 2\alpha$ is equivalent to $u_0 \in \mathbb{B}^{-1+\frac{3}{q}}_{q,s(q,\alpha)}(\Omega)$. Since $s(q,\alpha) > s(q,0)$, by embedding theorems we know $\mathbb{B}^{-1+\frac{3}{q}}_{q,s(q,0)}(\Omega) \subset \mathbb{B}^{-1+\frac{3}{q}}_{q,s(q,\alpha)}(\Omega)$. Therefore, the spaces to yield strong solutions are larger than the classical Serrin class discussed in the literature, and the theory of [4, 5] is extended to the scale of Besov spaces $\mathbb{B}^{-1+\frac{3}{q}}_{q,s(q,\alpha)}(\Omega)$ filling the gap between $\mathbb{B}^{-1+\frac{3}{q}}_{q,s(q,0)}(\Omega)$ and $\mathbb{B}^{-1+\frac{3}{q}}_{q,\infty}(\Omega)$.

We state our main result in [3] in a more precise way as follows.

Theorem 1.2. ([3, Theorem 1.2]) *Let* $\Omega \subseteq \mathbb{R}^3$ *be a bounded domain with boundary* $\partial\Omega$ *of class* $C^{2,1}$, *and let* $0 < T \le \infty$, $2 < s < \infty$, $3 < q < \infty$, $0 < \alpha < \frac{1}{2}$ *with* $\frac{2}{s} + \frac{3}{q} =$

$1-2\alpha$ *be given. Consider the Navier-Stokes equation with initial value* $u_0 \in L^2_\sigma(\Omega)$ *and an external force* $f = \operatorname{div} F$ *where* $F \in L^2(0,T;L^2(\Omega)) \cap L^{s/2}_{2\alpha}(0,T;L^{q/2}(\Omega))$. *Then there exists a constant* $\epsilon_* = \epsilon_*(q,s,\alpha,\Omega) > 0$ *with the following property: If*

$$\|e^{-\tau A}u_0\|_{L^s_\alpha(0,T;L^q)} + \|F\|_{L^{s/2}_{2\alpha}(L^{q/2})} \leq \epsilon_*, \tag{1.6}$$

then the Navier-Stokes system (1.1) *has a unique* $L^s_\alpha(L^q)$*-strong solution with data* u_0, f *on the interval* $[0,T)$.

Theorem 1.3. ([3, Theorem 1.3]) *Let* Ω *be as in Theorem 1.2, let* $2 < s < \infty$, $3 < q < \infty$, $0 < \alpha < \frac{1}{2}$ *with* $\frac{2}{s} + \frac{3}{q} = 1 - 2\alpha$ *be given, and let* $u_0 \in L^2_\sigma(\Omega)$ *and an external force* $f = \operatorname{div} F$ *where* $F \in L^2(0,\infty;L^2(\Omega)) \cap L^{s/2}_{2\alpha}(0,\infty;L^{q/2}(\Omega))$.

(1) The condition

$$\int_0^\infty (\tau^\alpha \|e^{-\tau A}u_0\|_q)^s \,\mathrm{d}\tau < \infty \tag{1.7}$$

is sufficient and necessary for the existence of a unique $L^s_\alpha(L^q)$*-strong solution* $u \in L^s_\alpha(0,T;L^q)$ *of the Navier-Stokes system* (1.1), *with data* u_0, f *in some interval* $[0,T)$, $0 < T \leq \infty$.

(2) Let u *be a weak solution of the system* (1.1) *in* $[0,\infty) \times \Omega$ *with data* u_0, f, *and let*

$$\int_0^\infty (\tau^\alpha \|e^{-\tau A}u_0\|_q)^s \,\mathrm{d}\tau = \infty. \tag{1.8}$$

Then the weighted Serrin's condition $u \in L^s_\alpha(0,T;L^q(\Omega))$ *does not hold for each* $0 < T \leq \infty$. *Moreover, the system* (1.1) *does not have a* $L^s_\alpha(L^q)$*-strong solution with data* u_0, f *and weighted Serrin exponents* s, q, α *in any interval* $[0,T)$, $0 < T \leq \infty$.

Besides, we also prove a restricted Serrin's uniqueness theorem in [3]. A weak-strong uniqueness theorem in the sense of the classical Serrin Uniqueness Theorem seems to be out of reach for $L^s_\alpha(L^q)$-strong solutions within the full class of weak solutions satisfying the energy inequality. The reason is based on the algebraic identities and sharp use of norms and Hölder estimates in the proof of Serrin's Theorem, cf. [10, Ch. V, Sect. 1.5]. However, we prove uniqueness within the *subclass of well-chosen weak solutions* describing weak solutions constructed by concrete approximation procedures. We refer to Assumptions 1.5, 1.8 and Remarks 1.6, 1.7 for precise definitions.

Theorem 1.4. ([3, Theorem 1.4]) *Let* $\Omega \subset \mathbb{R}^3$ *be a bounded domain with boundary of class* $C^{2,1}$ *and let* $2 < s < \infty$, $3 < q < \infty$, $0 < \alpha < \frac{1}{2}$ *with* $\frac{2}{s} + \frac{3}{q} = 1 - 2\alpha$ *be given. Moreover, suppose that* $u_0 \in L^2_\sigma(\Omega) \cap \mathbb{B}^{-1+\frac{3}{q}}_{q,s}$ *and an external force* $f = \operatorname{div} F$ *where* $F \in L^2(0,\infty;L^2(\Omega)) \cap L^{s/2}_{2\alpha}(0,\infty;L^{q/2}(\Omega))$ *are given. Then the unique* $L^s_\alpha(L^q)$*-strong solution* $u \in L^s_\alpha(0,T;L^q(\Omega))$ *is unique on a time interval* $[0,T')$, $T' > 0$, *in the class of all well-chosen weak solutions.*

Assumption 1.5. *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with boundary of class $C^{2,1}$.*

(1) Given $u_0 \in L^2_\sigma(\Omega)$ and an external force $f = \operatorname{div} F$ where $F \in L^2(0,\infty; L^2(\Omega))$ we assume the existence of approximating sequences $(u_{0n}) \subset L^2_\sigma(\Omega)$ of u_0 such that

$$u_{0n} \to u_0 \ \text{ in } \ L^2_\sigma(\Omega)$$

and $(F_n) \subset L^2(0,\infty; L^2(\Omega))$ of F such that

$$F_n \to F \ \text{ in } \ L^2(0,\infty; L^2(\Omega)) \ \text{ as } \ n \to \infty.$$

(2) Let (J_n) denote a family of bounded operators in $\mathcal{L}(L^q_\sigma(\Omega), D(A_q^{1/2}))$ such that for each $1 < q < \infty$ there exists a constant $C_q > 0$ such that

$$\|J_n\|_{\mathcal{L}(L^q_\sigma)} + \|\frac{1}{n} A_q^{1/2} J_n\|_{\mathcal{L}(L^q_\sigma)} \le C_q \quad \text{ and } \quad J_n u \to u \text{ in } L^q_\sigma(\Omega) \text{ as } n \to \infty.$$

(3) For each $n \in \mathbb{N}$ let u_n denote the weak solution of the approximate Navier-Stokes system

$$\begin{gathered} \partial_t u_n - \Delta u_n + (J_n u_n)\cdot \nabla u_n + \nabla p_n = \operatorname{div} F_n, \ \ \operatorname{div} u_n = 0 \ \text{ in } \ (0,T)\times\Omega \\ u_n|_{\partial\Omega} = 0, \ u_n(0) = u_{n0} \end{gathered} \tag{1.9}$$

Remark 1.6. *A typical example of operators (J_n) in Assumption 1.5 is given by the family of Yosida operators $J_n = \left(I + \frac{1}{n} A_q^{1/2}\right)^{-1}$. It is well known that this family of operators is uniformly bounded on $L^q_\sigma(\Omega)$ as well as on $D(A_q^{1/2})$ for each $1 < q < \infty$. Moreover, $J_n u \to u$ in $L^q_\sigma(\Omega)$ as $n \to \infty$. By analogy, the operators $J_n = e^{-A_q^{1/2}/n}$ have the same properties.*

We know from [10, Ch. V, Thm. 2.5.1] (with a minor modification in the case of $J_n = e^{-A_q^{1/2}/n}$) that there exists a unique weak solution $u_n \in \mathcal{L}H_T := L^\infty(L^2) \cap L^2(H^1_0)$ of (1.9) satisfying the uniform estimate

$$\begin{aligned} \|u_n\|_{L^\infty(L^2)} + \|u_n\|_{L^2(H^1)} &\le C(\|u_{0n}\|_2 + \|F_n\|_{L^2(L^2)}) \\ &\le C(\|u_0\|_2 + \|F\|_{L^2(L^2)} + 1) \end{aligned}$$

for all sufficiently large $n \in \mathbb{N}$. Therefore, there exists $v \in \mathcal{L}H_T$ and a subsequence (u_{n_k}) of (u_n) such that

$$u_{n_k} \rightharpoonup v \text{ in } L^2(H^1_0), \quad u_{n_k} \overset{*}{\rightharpoonup} v \text{ in } L^\infty(L^2), \quad u_{n_k} \to v \text{ in } L^2(L^2).$$

From the last convergence we also conclude that $u_{n_k}(t_0) \to v(t_0)$ in $L^2(\Omega)$ for a.a. $t_0 \in (0,T)$. Actually, $v \in \mathcal{L}H_T$ is a weak solution of (1.1).

Remark 1.7. *(1) Since we do not know whether weak solutions of (1.1) are unique, v may depend on the subsequence (u_{n_k}) chosen above. In this case, we say that*

$$v \quad \text{is a well-chosen weak solution of (1.1)}. \tag{1.10}$$

Note that a well-chosen weak solution v is always related to a concrete approximation procedure as in Assumption 1.5 and the choice of an adequate (weakly–) convergent subsequence of a sequence of approximate solutions (u_n).*

(2) The question whether solutions constructed by the Galerkin method fall into the scope of a modified Assumption 1.5 and yield uniqueness in the sense of Theorem 1.4 has not been settled. A similar question concerning the property to be a suitable weak solution, cf. H. Beirão da Veiga [1, p.321], has been answered in the affirmative, see J.-L. Guermond [6].

Assumption 1.8. *Under the assumptions of Assumption 1.5 additionally let $2 < s < \infty$, $3 < q < \infty$, $0 < \alpha < \frac{1}{2}$ with $\frac{2}{s} + \frac{3}{q} = 1 - 2\alpha$ be given. Suppose that even $u_0, u_{0n} \in \mathbb{B}_{q,s}^{-1+\frac{3}{q}}$ and $F, F_n \in L^{s/2}_{2\alpha}(0,\infty; L^{q/2}(\Omega))$ such that also*

$$u_{0n} \to u_0 \ \ in \ \mathbb{B}_{q,s}^{-1+\frac{3}{q}}, \quad F_n \to F \ \ in \ L^{s/2}_{2\alpha}(0,\infty; L^{q/2}(\Omega)) \ \ as \ \ n \to \infty.$$

From now on by a well-chosen weak solution of (1.1) we also assume that the approximation satisfies Assumption 1.8 as well as Assumption 1.5.

Remark 1.9. *In [2], the assumptions on well-chosen weak solutions had been weaken to improve or extend the restricted Serrin's uniqueness theorem of [3].*

In the next section, for reader's convenience, we summary the proof for the main theorems in [3].

2 Proof of Theorems 1.2 and 1.3

Now we are in the position to state the proof of the main theorem in [3].

Proof of Theorem 1.2. Let u be a weak solution of (1.1) with initial value $u_0 \in L^2_\sigma$ and external force $f = \operatorname{div} F$ where $F \in L^2(L^2) \cap L^{s/2}_{2\alpha}(L^{q/2})$. Furthermore, let E_{f,u_0} denote the solution of the Stokes problem

$$\begin{aligned} \partial_t v - \Delta v + \nabla p = f, \ \operatorname{div} v = 0 \\ v|_{\partial\Omega} = 0, \ v(0) = u_0, \end{aligned}$$

i.e.

$$\begin{aligned} E_{f,u_0}(t) &= e^{-tA}u_0 + \int_0^t A^{1/2} e^{-(t-\tau)A} A^{-1/2} P \operatorname{div} F(\tau)\, \mathrm{d}\tau \\ &=: E_{0,u_0}(t) + E_{f,0}(t). \end{aligned}$$

Assume $E_{0,u_0} \in L^s_\alpha(L^q)$, i.e. $\int_0^t \|\tau^\alpha e^{-\tau A} u_0\|_q^s \, \mathrm{d}\tau < \infty$. Since $u_0 \in L^2_\sigma$ and $F \in L^2(L^2)$, we know that $E_{f,u_0} \in C^0([0,T]; L^2) \cap L^2(H^1)$, satisfying the energy equality.

Moreover, by using the estimates (3.1) and (3.2) (see Appendix) with $2\beta + \frac{3}{q} = \frac{3}{q/2}$ with $q > 3$, i.e. $\beta = \frac{3}{2q} < \frac{1}{2}$,

$$\begin{aligned} \|E_{f,0}(t)\|_q &\leq c \int_0^t \|A^{\frac{1}{2}+\beta} e^{-(t-\tau)A} (A^{-\frac{1}{2}} P \operatorname{div}) F(\tau)\|_{\frac{q}{2}} \,\mathrm{d}\tau \\ &\leq c \int_0^t (t-\tau)^{-\beta-\frac{1}{2}} \|F(\tau)\|_{\frac{q}{2}} \,\mathrm{d}\tau. \end{aligned}$$

By applying the weighted Hardy-Littlewood-Sobolev inequality (see Lemma 3.1 in Appendix) with the exponents $s_2 = s$, $\alpha_2 = \alpha$, $s_1 = s/2$, $\alpha_1 = 2\alpha$, $\lambda = \beta + \frac{1}{2} \in (0,1)$, $-\frac{2}{s} < 2\alpha < 1 - \frac{2}{s}$ and $-\frac{1}{s} < \alpha < 1 - \frac{1}{s}$, we have

$$\|E_{f,0}\|_{L^s_\alpha(L^q)} \leq c \|F\|_{L^{s/2}_{2\alpha}(L^{q/2})} \tag{2.1}$$

provided $\frac{2}{s} + (\frac{3}{2q} + \frac{1}{2} + 2\alpha - \alpha) = 1 + \frac{1}{s}$ (which is equivalent to $\frac{2}{s} + \frac{3}{q} = 1 - 2\alpha$). We then set $\tilde{u} = u - E_{f,u_0}$ which solves the (Navier-)Stokes system

$$\begin{aligned} \partial_t \tilde{u} - \Delta \tilde{u} + u \cdot \nabla u + \nabla p = 0, \ \ \operatorname{div} \tilde{u} = 0 \\ \tilde{u}|_{\partial\Omega} = 0, \ \ \tilde{u}(0) = 0. \end{aligned}$$

So we can write at least formally

$$\begin{aligned} \tilde{u}(t) &= -\int_0^t e^{-(t-\tau)A} P \operatorname{div}(u \otimes u)(\tau) \,\mathrm{d}\tau \\ &= -\int_0^t A^{1/2} e^{-(t-\tau)A} (A^{-1/2} P \operatorname{div})(u \otimes u)(\tau) \,\mathrm{d}\tau. \end{aligned} \tag{2.2}$$

With $\beta = \frac{3}{2q}$ as above we get

$$\begin{aligned} \|\tilde{u}(t)\|_q &\leq c \int_0^t \|A^{\frac{1}{2}+\beta} e^{-(t-\tau)A}\| \|A^{-\frac{1}{2}} P \operatorname{div}\| \|(u \otimes u)\|_{\frac{q}{2}} \,\mathrm{d}\tau \\ &\leq c \int_0^t (t-\tau)^{-\frac{1}{2}-\beta} \|u\|_q^2 \,\mathrm{d}\tau \end{aligned} \tag{2.3}$$

Then the Hardy-Littlewood-Sobolev inequality as above implies that

$$\|\tilde{u}(t)\|_{L^s_\alpha(L^q)} \leq c \|(\|u\|_q^2)\|_{L^{s/2}_{2\alpha}} = c \|u\|^2_{L^s_\alpha(L^q)}. \tag{2.4}$$

Since $u = \tilde{u} + E_{f,u_0}$ we have

$$\|\tilde{u}\|_{L^s_\alpha(0,T;L^q)} \leq c \big(\|\tilde{u}\|_{L^s_\alpha(0,T;L^q)} + \|F\|_{L^{s/2}_{2\alpha}(0,T;L^{q/2})} + \|e^{-\tau A} u_0\|_{L^s_\alpha(0,T;L^q)} \big)^2. \tag{2.5}$$

As in [5, p. 99] there exists by Banach's Fixed Point Theorem an $\epsilon_* = \epsilon_*(q, s, \alpha, \Omega) > 0$ such that we get the existence of a unique fixed point $\tilde{u} \in L^s_\alpha(0,T;L^q)$ solving

$$\partial_t \tilde{u} - \Delta\tilde{u} + (\tilde{u} + E_{f,u_0}) \cdot \nabla(\tilde{u} + E_{f,u_0}) + \nabla p = 0, \ \operatorname{div} \tilde{u} = 0$$
$$\tilde{u}|_{\partial\Omega} = 0, \ \tilde{u}(0) = 0$$

provided (1.6) is satisfied, i.e. $\|e^{-\tau A}u_0\|_{L^s_\alpha(0,T;L^q)} + \|F\|_{L^{s/2}_{2\alpha}(L^{q/2})} \le \epsilon_*$. Hence $u = \tilde{u} + E_{f,u_0} \in L^s_\alpha(0,T;L^q)$.

Now we need to prove that this constructed mild solution u is indeed a weak solution under the following conditions, cf. the assumptions in Theorem 1.2 and some facts already proved:

$$u, \, \tilde{u} \in L^s_\alpha(L^q), \ \ u_0 \in L^2_\sigma, \ e^{-\tau A}u_0 \in L^s_\alpha(L^q), \ \ F \in L^2(L^2) \cap L^{s/2}_{2\alpha}(L^{q/2}).$$

To this aim we need the following lemmata which had been proved in [3].

Lemma 2.1. ([3, Lemma 3.1]) *The mild solution u constructed in the above procedure satisfies $\nabla u \in L^2(0,T;L^2(\Omega))$.*

Lemma 2.2. ([3, Lemma 3.2]) *Under the assumptions of Lemma 2.1 we have that $u \in L^{s_2}(0,T;L^{q_2}(\Omega))$ for all $\frac{2}{s_2} + \frac{3}{q_2} = \frac{3}{2}$, $2 \le s_2 \le \infty$, $2 \le q_2 \le 6$. Moreover, $\|\tilde{u}(t)\|_2 \to 0$ and $u(t) \to u_0$ in $L^2(\Omega)$ as $t \to 0+$.*

Lemma 2.3. ([3, Lemma 3.3]) *Under the assumptions of Lemma 2.1, $u \in L^4_{\alpha/(2+8\alpha)}(0,T;L^4(\Omega))$.*

By Lemma 2.3 we may use that $u \in L^4_{\alpha/(2+8\alpha)}(L^4)$. Hence $u \in L^4(\epsilon,T;L^4)$ for all $0 < \epsilon < T$. So, by [10, IV. Thm. 2.3.1, Lemma 2.4.2] and for a.a. $\epsilon \in (0,T)$, u is the unique weak solution in $L^4(\epsilon,T;L^4)$ on (ϵ,T) of the linear Stokes problem

$$\partial_t u - \Delta u + \nabla p = \operatorname{div} \tilde{F}, \ \ \operatorname{div} u = 0$$
$$u|_{\partial\Omega} = 0, \ \ u|_{t=\epsilon} = u(\epsilon)$$

with external force $\operatorname{div}\tilde{F}$, $\tilde{F} = F - u \otimes u \in L^2(\epsilon,T;L^2)$ and initial value $u(\epsilon) \in L^4(\Omega) \subset L^2(\Omega)$. Therefore, u satisfies the energy equality on (ϵ,T), i.e.

$$\frac{1}{2}\|u(t)\|_2^2 + \int_\epsilon^t \|\nabla u\|_2^2 \, \mathrm{d}\tau = \frac{1}{2}\|u(\epsilon)\|_2^2 - \int_\epsilon^t (F, \nabla u) \, \mathrm{d}\tau$$

for all $t \in (\epsilon,T)$ and a.a. $\epsilon \in (0,T)$. Moreover, $u \in C^0([\epsilon,T);L^2)$ and hence $u \in C^0((0,T);L^2)$, see [10, IV 2.1-2.3]. Furthermore, since by Lemma 2.2 $u \in L^\infty((0,T);L^2)$, it also satisfies the energy equality on $[0,T)$. Hence u is a weak solution; this completes the proof of Theorem 1.2. □

3 Appendix

For the reader's convenience, we explain some well-known properties of the Stokes operator. Let Ω be as in Theorem 1.2, let $[0,T), 0 < T \leq \infty$, be a time interval and let $1 < q < \infty$. Then $P_q : L^q(\Omega) \to L^q_\sigma(\Omega)$ denotes the Helmholtz projection, and the Stokes operator $A_q = -P_q\Delta : D(A_q) \to L^q_\sigma(\Omega)$ is defined with domain $D(A_q) = W^{2,q}(\Omega) \cap W^{1,q}_0(\Omega) \cap L^q_\sigma(\Omega)$ and range $R(A_q) = L^q_\sigma(\Omega)$. Since $P_q v = P_\gamma v$ for $v \in L^q(\Omega) \cap L^\gamma(\Omega)$ and $A_q v = A_\gamma v$ for $v \in D(A_q) \cap D(A_\gamma)$, $1 < \gamma < \infty$, we sometimes write $A_q = A$ to simplify the notation if there is no misunderstanding. In particular, if $q = 2$, we always write $P = P_2$ and $A = A_2$. Furthermore, let $A^\alpha_q : D(A^\alpha_q) \to L^q_\sigma(\Omega)$, $-1 \leq \alpha \leq 1$, denote the fractional powers of A_q. It holds $D(A_q) \subseteq D(A^\alpha_q) \subseteq L^q_\sigma(\Omega)$, $R(A^\alpha_q) = L^q_\sigma(\Omega)$ if $0 \leq \alpha \leq 1$. We note that $(A^\alpha_q)^{-1} = (A^{-\alpha}_q)$ and $(A_q)' = A_{q'}$ where $\frac{1}{q} + \frac{1}{q'} = 1$.

Now we recall the embedding estimate

$$\|v\|_q \leq c\|A^\alpha_\gamma v\|_\gamma\,, \quad v \in D(A^\alpha_\gamma),\ 1 < \gamma \leq q,\ 2\alpha + \frac{3}{q} = \frac{3}{\gamma},\ 0 \leq \alpha \leq 1, \tag{3.1}$$

and the estimate

$$\|A^\alpha_q e^{-tA_q} v\|_q \leq ct^{-\alpha}e^{-\delta t}\|v\|_q\,, \quad v \in L^q_\sigma(\Omega),\ 0 \leq \alpha \leq 1,\ t > 0, \tag{3.2}$$

with constants $c = c(\Omega, q) > 0$, $\delta = \delta(\Omega, q) > 0$.

Then we recall a weighted version of the Hardy-Littlewood-Sobolev inequality. For $\alpha \in \mathbb{R}$ and $s \geq 1$ we consider the weighted L^s-space

$$L^s_\alpha(\mathbb{R}) = \Big\{u : \|u\|_{L^s_\alpha} = \Big(\int_{\mathbb{R}} (|\tau|^\alpha |u(\tau)|)^s \,\mathrm{d}\tau\Big)^{1/s} < \infty\Big\}.$$

Lemma 3.1. *Let* $0 < \lambda < 1$, $1 < s_1 \leq s_2 < \infty$, $-\frac{1}{s_1} < \alpha_1 < 1 - \frac{1}{s_1}$, $-\frac{1}{s_2} < \alpha_2 < 1 - \frac{1}{s_2}$ *and* $\frac{1}{s_1} + (\lambda + \alpha_1 - \alpha_2) = 1 + \frac{1}{s_2}$, $\alpha_2 \leq \alpha_1$. *Then the integral operator*

$$I_\lambda f(t) = \int_{\mathbb{R}} (t - \tau)^{-\lambda} f(\tau) \,\mathrm{d}\tau$$

is bounded as operator $I_\lambda : L^{s_1}_{\alpha_1}(\mathbb{R}) \to L^{s_2}_{\alpha_2}(\mathbb{R})$.

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