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LARGE DATA INCOMPRESSIBLE NONSTATIONARY FLOWS IN CYLINDRICAL DOMAINS

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ABSTRACT. We discuss the existence of solutions to the large data incompressible nonstationary flows in a cylindrical domain. The motion of the fluid is modeled using the Navier-Stokes system with the slip boundary conditions and prescribed inflow and outflow functions.
Key words: Navier-Stokes equations, motions in cylindrical domains, boundary slip conditions, global existence of regular solutions, large data.
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1. INTRODUCTION

We present the problem examined in papers [RZ1]-[RZ7], related to the nonstationary, incompressible motion in cylindrical domain. The results obtained in these papers will be the start point for further analysis of inflow-outflow problems and next, flows around some obstacle, with large velocities. The motion is modeled with Navier-Stokes system of equations, with slip boundary condition. The main goal is to obtain the existence result for the inflow-outflow problem with arbitrarily large flux, where the initial velocity does not change to much along the axis of cylinder and the inflow does not change much along directions perpendicular to this axis either with respect to time. The inflow-outflow problem is the subject in papers [RZ4], [RZ5], [RZ6] and also [RZ7]- for the reverse Y-shaped domain. In the paper [RZ1] we consider the problem with no inflow and in [RZ2] and [RZ3] we examine an auxiliary Poisson equation in order to obtain the weighted estimates in L_2 and L_p Sobolev weighted spaces, crucial for other results.

Introducing some large data, like the inflow, is a difficult question to establish global existence of regular solutions. Regularity of solutions for Navier-Stokes equations, even with no flux, requires some smallness conditions. We underline, that our restrictions admit much more general class of solutions than, for example, in [K1, K2, Z1, Z3] because in these papers the flux must converge to zero sufficiently fast or there is no flux. Many results of this type were proved assuming smallness of initial velocity, some restrictions on domain (so called thin domain, $\Omega = \Omega' \times (0,\varepsilon), \Omega' \in \mathbb{R}^2$ with small ε) or special structure of solutions (so that the solution is close to 2-dimensional solution). We mention here as some examples [M1], where the existence for large data is obtained under some geometrical constraints for 2d model in steady and evolutionary case; [M2], where steady Navier-Stokes equations in pipe-like domain are investigated and existence is shown for a class of cylindrical symmetric solutions and [Z4], where the problem of nonstationary

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flow in axially symmetric domain is examined and the result concerns the existence for solutions close to axially symmetric solutions and the inflow and outflow sufficiently close to homogeneous flux. In our results there is no restrictions on magnitude of flux, moreover, in the proof of the existence of global regular solutions we admit arbitrarily large L_2 norm of initial velocity. However, our data could not be arbitrary: if we were able to take any data then the regularity problem for the weak solutions to the Navier-Stokes would be solved. We assume smallness of derivatives along the axis of cylinder for inflow function and initial velocity.

Let us formulate the system of Navier-Stokes equations describing the motion in papers [RZ4]-[RZ6].

$$
\begin{aligned}
&v_t + v\cdot\nabla v - \operatorname{div}\mathbb{T}(v,p) = f && \text{in } \Omega^T = \Omega\times(0,T),\\
&\operatorname{div} v = 0 && \text{in } \Omega^T,\\
&v\cdot\bar n = 0 && \text{on } S_1^T,\\
&\nu\bar n\cdot\mathbb{D}(v)\cdot\bar\tau_\alpha + \gamma v\cdot\bar\tau_\alpha = 0,\quad \alpha=1,2, && \text{on } S_1^T,\\
&v\cdot\bar n = d && \text{on } S_2^T,\\
&\bar n\cdot\mathbb{D}(v)\cdot\bar\tau_\alpha = 0,\quad \alpha=1,2, && \text{on } S_2^T,\\
&v\big|_{t=0} = v(0) && \text{in } \Omega,
\end{aligned}
\tag{1.1}
$$

where $\Omega\subset\mathbb{R}^3$ is a cylindrical domain (Figure 1), $S=\partial\Omega$, $T<\infty$ is the existence time, v is the velocity of the fluid motion with $v(x,t)=(v_1(x,t),v_2(x,t),v_3(x,t))\in\mathbb{R}^3$, $p=p(x,t)\in\mathbb{R}^1$ denotes the pressure, $f=f(x,t)=(f_1(x,t),f_2(x,t),f_3(x,t))\in\mathbb{R}^3$ – the external force field, $x=(x_1,x_2,x_3)$ are the Cartesian coordinates, $\bar n$ is the unit outward vector normal to the boundary S and $\bar\tau_\alpha$, $\alpha=1,2$, are tangent vectors to S and the dot $\cdot$ denotes the scalar product in $\mathbb{R}^3$. $\mathbb{T}(v,p)$ is the stress tensor of the form

$$\mathbb{T}(v,p)=\nu\mathbb{D}(v)-p\mathbb{I},$$

where ν is the constant viscosity coefficient and $\mathbb{I}$ is the unit matrix. Next, $\gamma>0$ is the slip coefficient and $\mathbb{D}(v)$ denotes the dilatation tensor of the form

$$\mathbb{D}(v)=\{v_{i,x_j}+v_{j,x_i}\}_{i,j=1,2,3}.$$

In paper [RZ1] we consider the problem (1.1) with no inflow and no friction on the boundary, so boundary conditions on S are zero:

$$
\begin{aligned}
&v\cdot\bar n = 0 && \text{on } S^T,\\
&\bar n\cdot\mathbb{T}(v,p)\cdot\bar\tau_\alpha = 0,\quad \alpha=1,2, && \text{on } S^T.
\end{aligned}
$$

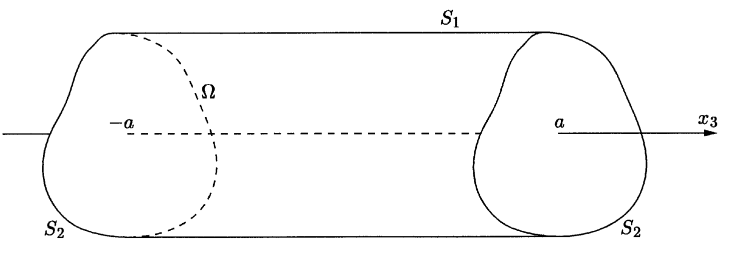

FIGURE 1. Domain Ω.

Domain $\Omega \subset \mathbb{R}^3$ as presented on the picture is a straight cylinder parallel to the x_3 axis with arbitrary cross section. We denote the boundary of Ω by S and set $S = S_1 \cup S_2$ where S_1 is parallel to the axis x_3 and S_2 is perpendicular to x_3. Consequently,

$$\begin{aligned} S_1 &= \{x \in \mathbb{R}^3 : \varphi_0(x_1, x_2) = c_0, \ -a < x_3 < a\}, \\ S_2(-a) &= \{x \in \mathbb{R}^3 : \varphi_0(x_1, x_2) < c_0, \ \ x_3 = -a\}, \\ S_2(a) &= \{x \in \mathbb{R}^3 : \varphi_0(x_1, x_2) < c_0, \ \ x_3 = a\} \end{aligned}$$

where a, c_0 are positive given numbers and $\varphi_0(x_1, x_2) = c_0$ describes a sufficiently smooth closed curve in the plane $x_3 = \text{const}$.

To define inflow and outflow we specify boundary condition $(1.1)_4$ introducing $d = (d_1, d_2)$ where

$$\begin{aligned} d_1 &= -v \cdot \bar{n}|_{S_2(-a)} \\ d_2 &= v \cdot \bar{n}|_{S_2(a)} \end{aligned}$$

with $d_i \geq 0, i = 1, 2$. Using $(1.1)_{2,3}$ and (1.2) we conclude the following compatibility condition

$$\Phi \equiv \int_{S_2(-a)} d_1 dS_2 = \int_{S_2(a)} d_2 dS_2, \tag{1.2}$$

where Φ is the flux.

Plan of analysis that leads to our global existence result:

- Existence of weak solutions with estimate A on interval $(0, T)$, paper [RZ4], based on weighted estimates from papers [RZ2] and [RZ3].
- A priori estimates on regular solutions on interval $(0, T)$ with constant $\mathcal{A}$ for large times T, paper [RZ5] and paper [RZ1] for the problem with no inflow.
- Existence of regular solutions on interval $(0, T)$ for large data, paper [RZ5] and paper [RZ1].
- Regular solutions on intervals $(kT, (k+1)T), k \in \mathbb{N}$, with estimate $\mathcal{A}_k$ for large times T, paper [RZ6].
- Controling initial data on intervals $(kT, (k+1)T)$ and prolongation of solutions to global ones, paper [RZ6].

In Sections 2-5, we focus on these issues.

2. Weak solutions and weighted estimates

In order to formulate the weak solutions existence theorem we define a space natural for the study of weak solutions to the Navier-Stokes equations:

$$V_2^0(\Omega^T) = \{u : ||u||_{V_2^0(\Omega^T)} \equiv \operatorname{ess} \sup_{t \in (0,T)} ||u||_{L_2(\Omega)} + \left(\int_0^T ||\nabla u||^2_{L_2(\Omega)} dt \right)^{\frac{1}{2}} < \infty\}.$$

We use as well Lebesque and Sobolev spaces:

- isotropic and anisotropic Lebesgue spaces:

$$\begin{aligned} &L_p(Q), &&Q \in \{\Omega^T, S^T, \Omega, S\}, \ \ p \in [1, \infty], \\ &L_q(0, T; L_p(Q)), &&Q \in \{\Omega, S\}, \ \ p, q \in [1, \infty]; \end{aligned}$$

- Sobolev spaces

$$W_q^s(Q), \quad Q \in \{\Omega, S\}, \quad s, q \in [1, \infty),$$

- anisotropic Sobolev spaces:

$$W_q^{s,s/2}(Q^T), \quad Q \in \{\Omega, S\}, \quad s = 2m, \ m \in \mathbb{N}, \quad q \in [1, \infty),$$

with the norm

$$\|u\|_{W_q^{s,s/2}(Q^T)} = \left(\sum_{|\alpha|+2a\le s} \int_{Q^T} |D_x^\alpha \partial_t^a u|^q dxdt \right)^{\frac{1}{q}},$$

where

$$D_x^\alpha = \partial_{x_1}^{\alpha_1}\partial_{x_2}^{\alpha_2}\partial_{x_3}^{\alpha_3}, |\alpha| = \alpha_1 + \alpha_2 + \alpha_3, \quad a, \alpha_i \in \mathbb{Z}_+ \cup \{0\}.$$

In the special case $q = 2$,

$$H^s(Q) = W_2^s(Q), \quad Q \in \{\Omega, S\}, \quad s \in \mathbb{Z}_+ \cup \{0\},$$

with the norm

$$\|u\|_{H^s(Q)} = \left(\sum_{|\alpha|\le s} \int_Q |D_x^\alpha u|^2 dx \right)^{\frac{1}{2}}$$

Obviously, we need as well the definition of weak solution to the system (1.1), however, since for this we have to introduce some auxiliary functions in order to homogenize the inflow boundary condition, we are going to present the corresponding construction in the next section.

Theorem 1. *Assume the compatibility condition (1.2). Assume that* $v(0) \in L_2(\Omega)$; $f \in L_2(0,T; L_{6/5}(\Omega))$; $d_i \in L_\infty(0,T; W_p^{s-1/p}(S_2)) \cap L_2(0,T; W_2^{1/2}(S_2))$; $\frac{3}{p} + \frac{1}{3} \le s, p > 3$ *or* $p = 3, s > \frac{4}{3}$; *and* $d_{i,t} \in L_2(0,T; W_{6/5}^{1/6}(S_2)), i = 1,2$. *Then there exists a weak solution* v *to problem (1.1) such that* v *is weakly continuous with respect to* t *in* $L^2(\Omega)$ *norm and* v *converges to* v_0 *as* $t \to 0$ *strongly in* $L^2(\Omega)$ *norm. Moreover,* $v \in V_2^0(\Omega^T)$, $v \cdot \bar\tau_\alpha \in L_2(0,T; L_2(S_1)), \alpha = 1,2$, *and* v *satisfies, for all* $t \le T$

$$\begin{aligned} &\|v\|^2_{V_2^0(\Omega^t)} + \gamma \sum_{\alpha=1}^2 \int_0^t \|v \cdot \bar\tau_\alpha\|^2_{L_2(S_1)} \le 2\|f\|^2_{L_2(0,t;L_{\frac{6}{5}}(\Omega))} \\ &+\varphi\left(\sup_{\tau\le t}\|d\|_{W_3^{s-\frac{1}{p}}(S_2)}\right)\left(\|d\|^2_{L_2(0,t;W_2^{\frac{1}{2}}(S_2))} + \|d_t\|^2_{L_2(0,t;W_{\frac{6}{5}}^{\frac{1}{6}}(S_2))}\right) + \|v(0)\|^2_{L_2(\Omega)} \equiv A^2 \end{aligned} \tag{2.1}$$

where φ *is a nonlinear positive increasing function.*

To show the existence theorem, we need to obtain an energy type estimate, and for this purpose, we have to make the Dirichlet boundary condition $(1.1)_5$ homogeneous.

To this end, we extend the functions corresponding to the inflow and outflow so that

$$\tilde d_i|_{S_2(a_i)} = d_i, \ i = 1,2, \ a_1 = -a, \ a_2 = a$$

We use the Hopf construction (see papers of Hopf [H1] and Ladyshenskaya [L]) and introduce the function η :

$$\eta(\sigma; \varepsilon, \rho) = \begin{cases} 1, & 0 \le \sigma \le \rho e^{-1/\varepsilon} \equiv r, \\ -\varepsilon \ln \dfrac{\sigma}{\rho}, & r < \sigma \le \rho, \\ 0, & \rho < \sigma < \infty. \end{cases}$$

We define (locally) functions η_i on the neighborhood of S_2 (inside Ω) by setting:

$$\eta_i = \eta(\sigma_i; \varepsilon, \rho), \; i = 1, 2,$$

where σ_i denote local coordinates defined on a small neighborhood of $S_2(a_i)$:

$$\sigma_1 = a + x_3, \; \sigma_2 = a - x_3$$

and we set

$$\begin{aligned} \alpha &= \sum_{i=1}^{2} \tilde{d}_i \eta_i, \\ b &= \alpha \bar{e}_3, \; \bar{e}_3 = (0, 0, 1). \end{aligned}$$

We set

$$u = v - b.$$

Therefore,

$$\begin{aligned} \operatorname{div} u &= -\operatorname{div} b = -\alpha_{x_3} \quad \text{in } \Omega, \\ u \cdot \bar{n} &= 0 \quad \text{on } S. \end{aligned}$$

Now we notice that the boundary condition for u is homogeneous, but the function u is not ideal as the new variable: it is not divergence free. Let us rewrite the compatibility condition

$$\int_\Omega \alpha_{,x_3} dx = -\int_{S_2(-a)} \alpha|_{x_3=-a} dS_2 + \int_{S_2(a)} \alpha|_{x_3=a} dS_2 = 0.$$

We need to correct the function u, so we define φ as a solution to the Neumann problem

$$\begin{aligned} \Delta\varphi &= -\operatorname{div} b \quad \text{in } \Omega, \\ \bar{n} \cdot \nabla\varphi &= 0 \quad \text{on } S, \\ \int_\Omega \varphi dx &= 0. \end{aligned} \tag{2.2}$$

Next, we set

$$w = u - \nabla\varphi = v - (b + \nabla\varphi) \equiv v - \delta.$$

Consequently, (w, p) is a solution to the following problem:

$$\begin{aligned} & w_t + w \cdot \nabla w + w \cdot \nabla\delta + \delta \cdot \nabla w - \operatorname{div} \mathbb{T}(w, p) \\ & \quad = f - \delta_t - \delta \cdot \nabla\delta + \nu \operatorname{div} \mathbb{D}(\delta) \equiv F(\delta, t) \quad \text{in } \Omega^T, \\ & \operatorname{div} w = 0 \quad \text{in } \Omega^T, \\ & w \cdot \bar{n} = 0 \quad \text{on } S^T, \\ & \nu\bar{n} \cdot \mathbb{D}(w) \cdot \bar{\tau}_\alpha + \gamma w \cdot \bar{\tau}_\alpha \\ & \quad = -\nu\bar{n} \cdot \mathbb{D}(\delta) \cdot \bar{\tau}_\alpha - \gamma\delta \cdot \bar{\tau}_\alpha \equiv B_{1\alpha}(\delta), \quad \alpha = 1, 2, \quad \text{on } S_1^T, \\ & \bar{n} \cdot \mathbb{D}(w) \cdot \bar{\tau}_\alpha = -\bar{n} \cdot \mathbb{D}(\delta) \cdot \bar{\tau}_\alpha \equiv B_{2\alpha}(\delta), \quad \alpha = 1, 2, \quad \text{on } S_2^T, \\ & w\big|_{t=0} = v(0) - \delta(0) = w(0) \quad \text{in } \Omega, \end{aligned} \tag{2.3}$$

where $\operatorname{div}\delta = 0$. Since Dirichlet boundary conditions for w are homogeneous and w is divergence free, we can define weak solutions to the problem (2.3)

Definition 1. *We call w a weak solution to problem (2.3) if for any sufficiently smooth function ψ such that*

$$\operatorname{div}\psi|_\Omega = 0, \quad \psi\cdot\bar{n}|_S = 0$$

the integral identity

$$\begin{aligned}&\int_{\Omega^T} w_t\cdot\psi dxdt + \int_{\Omega^T} H(w)\cdot\psi dxdt + \nu\int_{\Omega^T}\mathbb{D}(v)\cdot\mathbb{D}(\psi)dxdt\\ &+\gamma\sum_{\alpha=1}^{2}\int_{S_1^T} w\cdot\bar{\tau}_\alpha\psi\cdot\bar{\tau}_\alpha dS_1dt - \sum_{\alpha,\sigma=1}^{2}\int_{S_\sigma^T} B_{\sigma\alpha}\psi\cdot\bar{\tau}_\alpha dS_\sigma dt = \int_{\Omega^T} F\cdot\psi dxdt\end{aligned}$$

holds, where

$$H(w) = w\cdot\nabla w + w\cdot\nabla\delta + \delta\cdot\nabla w.$$

In order to obtain the energy estimate we use $\psi = w$ as a test function: thus, we multiply the first equation in (2.3) by ψ, integrate by parts on Ω and apply the definition of F, therefore

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|w\|^2_{L_2(\Omega)} + \int_\Omega(w\cdot\nabla\delta\cdot w + \delta\cdot\nabla w\cdot w)dx - \int_\Omega \operatorname{div}\mathbb{T}(w+\delta,p)\cdot wdx\\ = \int_\Omega (f-\delta_t-\delta\cdot\nabla\delta)\cdot wdx.\end{aligned}$$

We use the boundary conditions on S_1 and S_2 in (1.1) and apply the Korn inequality to reformulate this equality and obtain:

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|w\|^2_{L_2(\Omega)} + \nu\|w\|^2_{H^1(\Omega)} + \gamma\sum_{\alpha=1}^{2}\|w\cdot\bar{\tau}_\alpha\|^2_{L_2(S_1)}\\ &\le -\int_\Omega(w\cdot\nabla\delta\cdot w + \delta\cdot\nabla w\cdot w)dx + c\sum_{\alpha=1}^{2}\|\delta\cdot\bar{\tau}_\alpha\|^2_{L_2(S_1)}\\ &\quad + c\|\mathbb{D}(\delta)\|^2_{L_2(\Omega)} + \int_\Omega(f-\delta_t-\delta\cdot\nabla\delta)wdx.\end{aligned}$$

The most difficult terms are those caused by nonlinearity $w\cdot\nabla w$. Let us look closer at some example and focus on the integral

$$\begin{aligned}\int_\Omega\delta\cdot\nabla w\cdot wdx &= \int_\Omega(b+\nabla\varphi)\cdot\nabla w\cdot wdx\\ &= \int_\Omega b\cdot\nabla w\cdot wdx + \int_\Omega\nabla\varphi\cdot\nabla w\cdot wdx = I_1 + I_2.\end{aligned}$$

We can estimate I_1 using the Hölder inequality and, moreover, the local support of b yields some smallness:

$$\begin{aligned}|I_1| &\le \|\nabla w\|_{L_2(\Omega)}\|w\|_{L_6(\Omega)}\|b\|_{L_3(\Omega)} \le c\|w\|^2_{H^1(\Omega)}\|b\|_{L_3(\widetilde{S}_2(\rho))}\\ &\le c\rho^{1/6}\|w\|^2_{H^1(\Omega)}\|b\|_{L_6(\widetilde{S}_2(\rho))} \le c\rho^{1/6}\|w\|^2_{H^1(\Omega)}\|\delta\|_{L_6(\Omega)}\\ &\le c\rho^{1/6}\|w\|^2_{H^1(\Omega)}\|\tilde{d}\|_{H^1(\Omega)}\end{aligned}$$

where

$$\widetilde{S}_2(\rho) = \{x\in\Omega : x_3\in(-a,-a+\rho)\cup(a-\rho,a)\} = \widetilde{S}_2(\rho,a_1)\cup\widetilde{S}_2(\rho,a_2).$$

We estimate I_2 as follows:

$$|I_2| = \left| \int_\Omega \nabla\varphi \cdot \nabla w \cdot w dx \right| \leq \|\nabla\varphi\|_{L_3(\Omega)} \|w\|_{L_6(\Omega)} \|\nabla w\|_{L_2(\Omega)}$$

To extract some small parameter we use the result of [RZ3] on the Poisson problem (2.2) in weighted Sobolev spaces

$$\begin{aligned} \|\nabla\varphi\|_{L_3(\Omega)} &\leq c\|\nabla\varphi\|_{L_{3,-\mu'}(\Omega)} \leq c\|\nabla_{x_3}\nabla\varphi\|_{L_{3,1-\mu'}(\Omega)} \leq c\|\varphi\|_{L^2_{3,1-\mu'}(\Omega)} \\ &\leq c\|\operatorname{div} b\|_{L_{3,1-\mu'}(\Omega)} \end{aligned}$$

where we denote

$$\|u\|_{L^k_{p,\mu}(\Omega)} = \left(\sum_{|\alpha|=k} \int |D^\alpha_x u|^p_{\min_{i=1,2}} |(\operatorname{dist}(x, S_2(a_i))|^{p\mu} dx \right)^{1/p}, \mu \in \mathbb{R}, p \in (1,\infty).$$

Let us emphasize, that in L_2 approach we use the explicit form of solutions, and for the existence result we apply the regularizer technique. In $L_p, p > 2$ weighted spaces (since we need the result for $p = 3$), we introduce some auxiliary problem and more subtle techniques to compare norms of solutions even for different weights. Note as well, that weight of the form $x_3^\mu, p \in (1,\infty)$ is not the Muckenhaupt weight so the results of Coifmann-Feffermann [CF] can not be applied.

To estimate the last norm, we choose $\frac{2}{3} \leq 1 - \mu' \leq 1$. With $\mu = 1 - \mu'$ we have

$$\begin{aligned} c\|\operatorname{div} b\|_{L_{3,\mu}(\Omega)} &\leq c\varepsilon \left(\sum_{i=1}^2 \int_{\tilde{S}_2(a_i)} |\tilde{d}_i|^3 \frac{\sigma_i^{3\mu}}{\sigma_i^3} dx \right)^{1/3} + \left(\sum_{i=1}^2 \int_{\tilde{S}_2(a_i)} |\tilde{d}_{i,x_3}|^3 |\rho(x)|^{3\mu} dx \right)^{1/3} \\ \leq c \sum_{i=1}^2 \varepsilon &\left(\sup_{x_3} \int_{S_2(a_i)} |\tilde{d}_i|^3 dx' \int_r^\rho \frac{\sigma_i^{3\mu}}{\sigma_i^3} d\sigma_i \right)^{1/3} + \sum_{i=1}^2 \left(\sup_{x_3} \int_{S_2(a_i)} |\tilde{d}_{i,x_3}|^3 dx' \int_0^\rho \sigma_i^{3\mu} d\sigma_i \right)^{1/3} \\ &\leq c\varepsilon\rho^{\mu-2/3} \sup_{x_3} \|\tilde{d}\|_{L_3(S_2)} + c\rho^{\mu+1/3} \sup_{x_3} \|\tilde{d}_{,x_3}\|_{L_3(S_2)} \end{aligned}$$

where $\sigma_i = \operatorname{dist}\{S_2(a_i), x\}, x \in S_2(a_i, \rho)$. We note that the last bound holds for $\mu > \frac{2}{3}$ since for $\mu = \frac{2}{3}$ the r.h.s. takes the form

$$c \sup_{x_3} \|\tilde{d}\|_{L_3(S_2)} + c\rho \sup_{x_3} \|\tilde{d}_{,x_3}\|_{L_3(S_2)},$$

which cannot be made small for large $\tilde{d}$. Then,

$$|I_2| \leq c \left[\varepsilon\rho^{\mu-2/3} \sup_{x_3} \|\tilde{d}\|_{L_3(S_2)} + \rho^{\mu+1/3} \sup_{x_3} \|\tilde{d}_{,x_3}\|_{L_3(S_2)} \right] \|w\|^2_{H^1(\Omega)}.$$

Let us notice, that in order to estimate $\sup_{x_3} \|\tilde{d}\|_{L_3(S_2)}$ and $\sup_{x_3} \|\tilde{d}_{,x_3}\|_{L_3(S_2)}$ by some W^s_p norm we apply the Sobolev anisotropic imbedding (see [BIN], Ch.3, Section 10) which reads

$$\begin{aligned} 2\left(\frac{1}{p} - \frac{1}{3}\right)\frac{1}{s} + \frac{1}{p} \cdot \frac{1}{s} + \frac{1}{s} \leq 1 \quad \text{for} \quad p > 3, \\ 2\left(\frac{1}{p} - \frac{1}{3}\right)\frac{1}{s} + \frac{1}{p} \cdot \frac{1}{s} + \frac{1}{s} < 1 \quad \text{for} \quad p = 3. \end{aligned}$$

Thus, we find that p, s satisfy

$$\frac{3}{p}+\frac{1}{3}\le s, \quad p>3 \quad \text{or} \quad p=3,\, s>\frac{4}{3}.$$

We deal with other terms in the integral inequality and apply Korn inequality, Schwartz inequality, Sobolev imbedding to obtain

$$\frac{1}{2}\frac{d}{dt}\|w\|^2_{L_2(\Omega)}+\nu\|w\|^2_{H^1(\Omega)}+\gamma\sum_{\alpha=1}^{2}\|w\cdot\bar{\tau}_\alpha\|^2_{L_2(S_1)}$$

$$\le \varphi_1(\rho,\varepsilon,\mu,\|\tilde{d}\|_{W^s_p(\Omega)})\|w\|^2_{H^1(\Omega)}+\|f\|^2_{L_{6/5}(\Omega)}+\varphi_2(\|\tilde{d}\|_{W^s_p(\Omega)})\left(\|\tilde{d}\|^2_{W^1_2(\Omega)}+\|\tilde{d}_t\|^2_{W^1_{6/5}(\Omega)}\right)$$

where $\varphi_i, i=1,2$ is a nonlinear positive increasing function and φ_1 is small for small values of ε,ρ,μ. Thus, we can choose parameters $\mu>\frac{2}{3},\rho,\varepsilon$ in dependence on ν and $\|\tilde{d}\|_{W^s_p(\Omega)}$ so that

$$\varphi_1(\rho,\varepsilon,\mu,\|\tilde{d}\|_{W^s_p(\Omega)})\le\frac{\nu}{2}.$$

Therefore, we can 'consume' the term $\varphi_1\|w\|^2_{H^1(\Omega)}$ by $\nu\|w\|^2_{H^1(\Omega)}$ on the left hand side and in consequence, we have only data terms on the right hand side. Next, integrating with respect to time, we obtain the estimate of the form

$$\|w\|^2_{V^0_2(\Omega^t)}+\gamma\sum_{\alpha=1}^{2}\int_0^t\|w\cdot\bar{\tau}_\alpha\|^2_{L_2(S_1)}dt\le 2\|f\|^2_{L_2(0,t;L_{6/5}(\Omega))}$$

$$+\varphi(\sup_\tau\|\tilde{d}\|_{W^s_p(\Omega)})\left(\|\tilde{d}\|^2_{L_2(0,t;W^1_2(\Omega))}+\|\tilde{d}_t\|^2_{L_2(0,t;W^1_{6/5}(\Omega))}\right)+\|w(0)\|^2_{L_2(\Omega)}.$$

With Galerkin method we prove the existence of weak solutions and so the Theorem 1. With the a priori estimate, we show also existence of global weak solutions.

Theorem 2. *Assume the compatibility condition (1.2). Let* $f\in L_2(kT,(k+1)T;L_{6/5}(\Omega))$, $d_i\in L_\infty(\mathbb{R}^+;W_p^{s-1/p}(S_2))\cap L_2(kT,(k+1)T;W_2^{1/2}(S_2))$, *where* $\frac{3}{p}+\frac{1}{3}\le s, p>3$ *or* $p=3, s>\frac{4}{3}$, *and* $d_{i,t}\in L_2(kT,(k+1)T;W^{1/6}_{6/5}(S_2)), i=1,2$. *Then there exists a global weak solution* v *to (1.1) such that*

$$v\in V^0_2(\Omega\times(kT,(k+1)T))\ \ \forall k\in\mathbb{N}_0=\mathbb{N}\cup\{0\}.$$

Moreover, the following global estimate hold:

$$\|v(kT)\|^2_{L_2(\Omega)} \le \frac{1}{1-e^{-\nu T}}l_0^2+\|v(0)\|^2_{L_2(\Omega)}\equiv\mathbf{a}^2,$$

$$\|v\|^2_{V^0_2(\Omega\times(kT;t))}\le l_0^2+\|v(kT)\|^2_{L_2(\Omega)} \le l_0^2+\mathbf{a}^2=A(k,T),\quad t\in[kT;(k+1)T],$$

where

$$l_0^2=\frac{2}{\nu}\|f\|^2_{L_2(kT,(k+1)T;L_{6/5}(\Omega))}+\frac{1}{\nu}\varphi(\sup_t\|d\|_{W_3^{s-1/p}(S_2)})\sup_t\left(\|d\|^2_{W_2^{1/2}(S_2)}+\|d_t\|^2_{W^{1/6}_{6/5}(S_2)}\right),$$

and φ *is an increasing function.*

The next step is to work with weak solutions in order to increase the regularity.

We mention here as well the paper [RZ7], where we consider the inflow-outflow problem in the reverse Y-shaped domain (Figure 2), with one inflow and two outflows. The motion of the fluid is described by the Navier-Stokes equations with boundary slip conditions. First, we prove some a priori energy type estimates, next part is devoted to the proof of existence of weak solutions to the problem by the Galerkin method. We examine also the properties of solutions near the neighborhood of artificial boundaries D_2 and D_3, where $D_i = \Omega_1 \cap \Omega_i, i = 2, 3$, dilatation tensor $\mathbb{D}(v) = \{v^i_{,x_j} + v^j_{,x_i}\}_{i,j=1,2,3}$ and

$$\begin{aligned} &v_1 \cdot \bar{n}_1 = v_i \cdot \bar{n}_1 \\ &\bar{n}_1 \cdot \mathbb{D}(v_1) \cdot \bar{\tau}_j = \bar{n}_1 \cdot \mathbb{D}(v_i) \cdot \bar{\tau}_j, \quad \text{on } D_i,\ i = 2, 3,\ j = 1, 2. \end{aligned}$$

The problem can be treated as a simple model of the blood flow in veins or arteries.

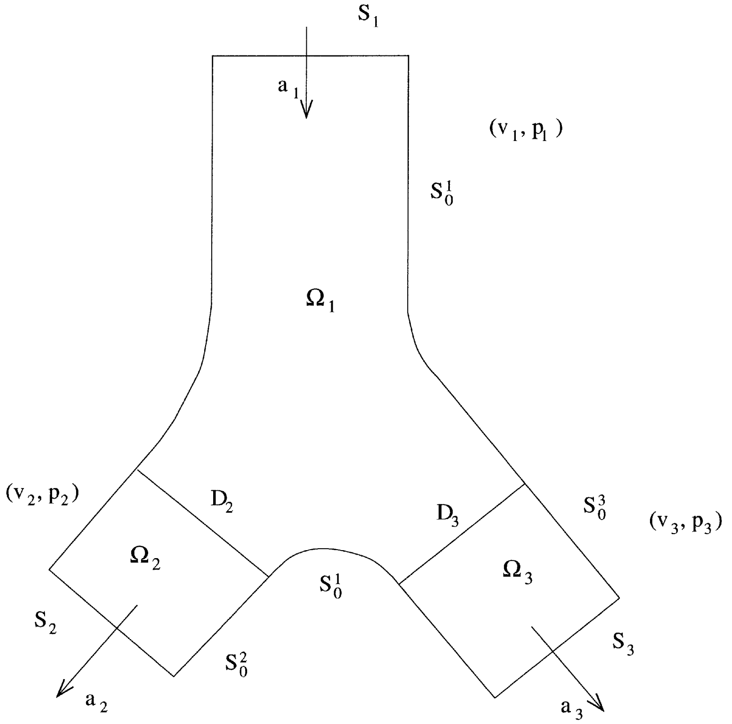

FIGURE 2. Y-shaped domain

3. A priori estimates on regular solutions

In [RZ5] and [RZ1] we have been proved the existence of possibly large solutions (with respectively large data). In our case there is no restrictions on the magnitudes of the initial velocity $v(0)$, the external force either, in paper [RZ5], on the flux d. Therefore we prove the existence of large regular solutions to (1.1). However, our data could not be arbitrary: if we were able to take any data then the regularity problem for the weak solutions to the Navier-Stokes would be solved. We assume smallness of the following quantities

$$\|v_{,x_3}(0)\|_{L_2(\Omega)}, \quad \|f_{,x_3}\|_{L_2(0,T;L_{6/5}(\Omega))} \tag{3.1}$$

and, in paper [RZ5]

$$\|d_t\|_{L_2(0,T;H^1(S_2))}, \quad \|d_{x'}\|_{L_2(0,T;H^1(S_2))\cap L_\infty(0,T;H^1(S_2))}. \tag{3.2}$$

It points that the initial velocity and the external force does not change much along the cylinder. This could mean that we are looking for solutions close to 2-dimensional solutions which are located in the cross-section of the cylinder on the plane perpendicular to its axis. However, this possibility does not occur: we prove the existence of solutions with not small $v_{x_3x_3}$ either p_{x_3} (Theorem 2). This property indicates that we consider indeed 3-dimensional problem where all auxiliary problems, applied theorems of imbeddings and interpolations are 3-dimensional. The smallness restrictions for quantities (3.1) and (3.2) are necessary to obtain the a priori estimate and help us to overcome the difficulties that appear in regularity problem to the Navier-Stokes equations. Finally, let us note that smallness of norm (3.2) implies that the flux does not change much with respect to time and in S_2.

To formulate results of [RZ5] we introduce some quantities

Definition 2. *Let A be the estimate for weak solutions to (1.1) established in Theorem 1. We set*

$$\begin{aligned} D_0 &= \|d_1\|^6_{L_\infty(0,T;L_3(S_2(-a)))} + A^2 + 1, \\ \Lambda &= \|d_t\|^2_{L_2(0,T;H^1(S_2))} + \|d_{,x'}\|^2_{L_2(0,T;H^1(S_2))} + \|d_{,x'}\|^2_{L_\infty(0,T;H^1(S_2))} \\ &\quad + \|f_3\|^2_{L_2(0,T;L_{4/3}(S_2))} + \|f_{,x_3}\|^2_{L_2(0,T;L_{6/5}(\Omega))} + \|v_{,x_3}(0)\|^2_{L_2(\Omega)}, \\ \Gamma &= \|f\|_{L_2(0,T;L_2(\Omega))} + \|v(0)\|_{H^1(\Omega)}. \end{aligned}$$

In the case with no inflow, such parameters could be written in shorter form, which suggests that paper [RZ1] is just a particular case of more general problem. However, the techniques that lead to a priori estimates have the source in the paper [RZ1]. Therefore, paper [RZ1] should be considered, also from the chronological point of view, as the pioneer paper: the results proved there made possible to apply and generalize methods for more complex inflow problem.

Condition 1. *Let quantities Λ, Γ, D_0 be finite. Assume that Λ is so small that there exists a constant $\mathcal{A}$ satisfying*

$$\begin{aligned} D_0^2\Lambda^2(\mathcal{A} + \mathcal{A}^2 + \Gamma)\exp[TD_0 + D_0(\mathcal{A}+\Gamma) + \mathcal{A}^2 + \Gamma] \\ + \|f_{,x_3}\|^2_{L_2(0,T;L_2(\Omega))} + \|v_{,x_3}(0)\|^2_{H^1(\Omega)} \le \mathcal{A}^2. \end{aligned} \tag{3.3}$$

Let us notice, that from the Condition 1 it follows that the time existence T is inversely proportional to Λ. Thus, the quantity Λ is going to be an important smallness parameter. Further, we should point out that it depends on derivatives of inflow but not on the function d, which agrees with our assumptions on flow structure.

Theorem 3. *(a priori estimates) Assume that Condition 1 holds. Then a solution to problem (1.1) satisfies the following estimate*

$$\begin{gathered} \|v_{,x_3}\|_{W_2^{2,1}(\Omega^T)} \le \mathcal{A}, \\ \|v\|_{W_2^{2,1}(\Omega^T)} + \|\nabla p\|_{L_2(\Omega^T)} \le \varphi(\mathcal{A},\Lambda,\Gamma), \\ \|\nabla p_{,x_3}\|_{L_2(\Omega^T)} \le \varphi(\mathcal{A},\Lambda,\Gamma), \end{gathered}$$

where φ is an increasing positive function.

To omit restrictions on magnitudes of $v(0)$, f and d, we carry out the proof of Theorem 3 in such a way that some smallness of $v_{,x_3}(0)$ and $f_{,x_3}$ in L_2 norms and on derivatives $d_t, d_{x'}$ are instead imposed. The crucial idea is hypothesis, justified by physics, that derivative of velocity along the axis of the cylinder, assuming sufficiently small initial values, should remain stable and then large initial velocity $v(0)$ either large flux do not change that phenomena. Motivated by this splitting, we need to analyze problems for $h = v_{,x_3}$, $q = p_{,x_3}$. We differentiate equations (1.1) with respect to x_3 to verify that h, q satisfy the following system of equations:

$$
(3.4)\qquad
\begin{aligned}
h_{,t} - \operatorname{div} \mathbb{T}(h,q) &= -v \cdot \nabla h - h \cdot \nabla v + g && \text{in } \Omega^T,\\
\operatorname{div} h &= 0 && \text{in } \Omega^T,\\
\bar{n} \cdot h = 0, \quad \bar{n} \cdot \mathbb{D}(h) \cdot \bar{\tau}_\alpha + \gamma h \cdot \bar{\tau}_\alpha &= 0, \quad \alpha = 1,2 && \text{on } S_1^T,\\
h_i = -d_{x_i}, \quad i = 1,2, \quad h_{3,x_3} &= \Delta' d && \text{on } S_2^T,\\
h\big|_{t=0} &= h(0) && \text{in } \Omega,
\end{aligned}
$$

where $g = f_{,x_3}$, $\Delta' = \partial^2_{x_1} + \partial^2_{x_2}$.

Let us observe here, that the $h-$ system, written in fact as the Stokes system, depends, obviously, on velocity v through nonlinear terms, but as a data on the right hand side we have only *derivatives*: derivatives of external force and initial velocity- with respect to axis of cylinder and derivatives of inflow with respect to directions perpendicular to the axis. Thus, if we assumed the smallness of such data and if we were able to deal with nonlinearities, then the flow with possibly large data (like velocity, force, inflow) and with small variations of discussed derivatives, would be stable with respect to h. Consequently, we could deduce some smallness of h in corresponding norms.

The relation, that let us estimate h in terms of v has the form:

$$\|h\|_{W_2^{2,1}(\Omega^T)} \le \varphi_0(\|v\|_{W_2^{2,1}(\Omega^T)})\|h\|_{L_2(\Omega^T)} + \varphi(data).$$

where φ_0, φ are positive, increasing functions and *data* denotes the appropriate norms of data functions. Since we are able to estimate L_2 norm of h with the small parameter Λ (see Definition 2, Condition 1):

$$\|h\|_{L_2(\Omega^T)} \le \varphi(data, \|v\|_{W_2^{2,1}(\Omega^T)})\Lambda,$$

and the norm $W_2^{2,1}(\Omega^T)$ of velocity v through the same norm of h and data:

$$\|v\|_{W_2^{2,1}(\Omega^T)} \le c\|h\|_{W_2^{2,1}(\Omega^T)} + \varphi(data),$$

then, we combine the three inequalities to obtain

$$(*)\qquad \|h\|_{W_2^{2,1}(\Omega^T)} \le \varphi_1(data, \|h\|_{W_2^{2,1}(\Omega^T)})\Lambda + \varphi(data),$$

where φ_1 is positive, increasing function. Then, for constant $\mathcal{A}$ respectively greater than *data* and sufficiently small parameter Λ, we infer Theorem 3. Let us point out, that relation (*) yields Condition 1, which defines the smallness of Λ with respect to $\mathcal{A}$, time T and other data.

We now more precisely discuss the three inequalities above.

The energy inequality for h, deduced through the system (3.4), has the following form:

$$(3.5)\qquad \|h\|_{V_2^0(\Omega^T)} \le c(\|h\|_{L_2(\Omega^T)} + \|h\|_{L_\infty(0,T;L_3(\Omega))}) + \varphi(data)$$

Therefore, we observe that it does not involve any small parameter that we could use in the existence proof. Thus, we will need some more refined relation and this can be achieved by improving the regularity through the system for vorticity component χ. For this reason, let us introduce rotation and $\chi = (\operatorname{rot} v)_3 = (\operatorname{rot} v)_3 = v_{2,x_1} - v_{1,x_2}$ and note that $v' = (v_1, v_2)$ and h are related as follows by the rot-div problem

$$\begin{aligned} v_{1,x_2} - v_{2,x_1} &= \chi && \text{in } \Omega', \\ v_{1,x_1} + v_{2,x_2} &= -h_3 && \text{in } \Omega', \\ v' \cdot \bar{n}' &= 0 && \text{on } S_1', \end{aligned}$$

where $S_1' = S_1 \cap \{\text{plane } x_3 = \text{const} \in (-a, a)\}$, and x_3, t are treated as parameters.

The function χ satisfies

$$\begin{aligned} &\chi_{,t} + v \cdot \nabla\chi - h_3\chi + h_2 v_{3,x_1} - h_1 v_{3,x_2} - \nu\Delta\chi = F_3 && \text{in } \Omega^T, \\ &\chi = v_i(n_{i,x_j}\tau_{1j} + \tau_{1i,x_j}n_j) + v \cdot \bar{\tau}_1(\tau_{12,x_1} - \tau_{11,x_2}) \\ &+ \frac{\gamma}{\nu} v_j\tau_{1j} \equiv \chi_* && \text{on } S_1^T, \\ &\chi_{,x_3} = 0 && \text{on } S_2^T, \\ &\chi\big|_{t=0} = \chi(0) && \text{in } \Omega, \end{aligned} \tag{3.6}$$

where $F_3 = f_{2,x_1} - f_{1,x_2}$.

We have to find the energy type estimates for quantities $h = v_{x_3}$ -see (3.4) and the third component of vorticity $\chi = (\operatorname{rot} v)_3$, see (3.6). To derive the boundary condition for χ we need the slip boundary condition in (1.1). In this case there are also appropriate the Navier conditions

$$v \cdot \bar{n}|_S = 0, \quad \bar{n} \times rot\ v|_S = 0$$

We have to underline that for the Dirichlet boundary condition we are not able to find boundary conditions for χ so Theorems 3 and 4 could not be proved.

To get the energy type estimates for solutions to problems (3.4) and (3.6) we need to make the nonhomogeneous Dirichlet boundary conditions homogeneous by an appropriate extensions. Otherwise we would not be able to integrate by parts. For this purpose we introduce corresponding functions $\tilde{h}$ and $\tilde{\chi}$ and derive appropriate estimates for these functions. Applying such auxiliary bounds we are able to find energy type estimates for h (see (3.5)) and for χ:

$$\begin{aligned} \|\chi\|_{V_2^0(\Omega^T)} \le c\|h\|_{L_\infty(0,T;L_3(\Omega))} + \varepsilon(\|v'\|_{L_\infty(0,t;H^1(\Omega))} + \|v'\|_{L_2(0,t;H^2(\Omega))}) \\ + c\|v'\|_{L_2(\Omega;H^{1/2}(0,t))} + \|\chi(0)\|_{L_2(\Omega)} + \varphi(1/\varepsilon, data) \end{aligned} \tag{3.7}$$

where $\varepsilon \in (0,1)$, φ is an increasing positive function and *data* denotes the appropriate norms of data functions.

With estimates (3.5) and (3.7) with small ε, we obtain for the rot-div problem the inequality

$$\|v'\|_{V_2^1(\Omega^T)} \le c(\|h\|_{L_2(\Omega^T)} + \|h\|_{L_\infty(0,T;L_3(\Omega))} + \|v'\|_{L_2(\Omega;H^{1/2}(0,t))}) + \varphi(data). \tag{3.8}$$

We apply the result of paper [A], related to Stokes-type system, to problem (1.1), which means that the first equation reads

$$v_{,t} - \operatorname{div}\mathbb{T}(v,p) = -v' \cdot \nabla v - v_3 h + f, \quad \text{in } \Omega^T$$

to get

$$\|v\|_{W_{5/3}^{2,1}(\Omega^T)} \le c(\|h\|_{L_2(\Omega^T)} + \|h\|_{L_\infty(0,T;L_3(\Omega))}) + \varphi(data). \tag{3.9}$$

To show this, we estimate the r.h.s. terms as follows:

$$\begin{aligned}\|v'\nabla v\|_{L_{5/3}(\Omega^T)} &\le \|v'\|_{L_{10}(\Omega^T)}\|\nabla v\|_{L_2(\Omega^T)} \le A\|v'\|_{L_{10}(\Omega^T)} \le cA\|v'\|_{V_2^1(\Omega^T)},\\ \|v_3 h\|_{L_{5/3}(\Omega^T)} &\le \|v_3\|_{L_{10/3}(\Omega^T)}\|h\|_{L_{10/3}(\Omega^T)} \le cA\|h\|_{L_{10/3}(\Omega^T)},\end{aligned}$$

where A is the bound for the weak solution to (1.1) from Theorem 1 and we used the imbedding (see [Z2])

$$\|v'\|_{L_{10}(\Omega^T)} \le c\|v'\|_{V_2^1(\Omega^T)}.$$

Next, we apply (3.8) and in view of the interpolation

$$\|v'\|_{L_2(\Omega;H^{1/2}(0,T))} \le \varepsilon\|v'\|_{W^{2,1}_{5/3}(\Omega^T)} + c(1/\varepsilon)A,$$

with small ε, we conclude (3.9). We can improve this to get

$$\|v\|_{W_2^{2,1}(\Omega^T)} \le cH + \varphi(data) \le c\|h\|_{W_2^{2,1}(\Omega^T)} + \varphi(data) \tag{3.10}$$

where $H = \|h\|_{L_2(\Omega^T)} + \|h\|_{L_\infty(0,T;L_3(\Omega))} + \|h\|_{L_{10/3}(\Omega^T)}$ and by the imbedding

$$H \le c\|h\|_{W_2^{2,1}(\Omega^T)}.$$

Next we apply [A] to the system for h, i.e. (3.4) and using some interpolation inequalities we get

$$\|h\|_{W_2^{2,1}(\Omega^T)} \le \varphi_0(\|v\|_{W_2^{2,1}(\Omega^T)})\|h\|_{L_2(\Omega^T)} + \varphi(data). \tag{3.11}$$

Finally, the crucial step is to find the bound for h in terms of v with a small parameter Λ. Therefore, we show

$$\|h\|_{V_2^0(\Omega^T)} \le \varphi(data, \|v\|_{W_2^{2,1}(\Omega^T)})\Lambda \tag{3.12}$$

We work with the relation (3.11): we combine (3.10) with estimate for $\|v\|_{W_2^{2,1}(\Omega^T)}$ and (3.12) to estimate $\|h\|_{L_2(\Omega^T)}$, to obtain for sufficiently small Λ and $\mathcal{A} > \varphi(data)$ the estimate

$$\|h\|_{W_2^{2,1}(\Omega^T)} \le \mathcal{A}.$$

In this way Theorem 3 is proved.

4. Existence of regular solutions

On the base of a priori estimates, we prove in papers [RZ5] and in [RZ1] the existence result.

Theorem 4. *(existence of regular solutions) Assume that Conditions 1 holds. Then there exists a solution to problem (1.1) such that*

$$v, v_{,x_3} \in W_2^{2,1}(\Omega^T), \nabla p, \nabla p_{,x_3} \in L_2(\Omega^T).$$

Moreover, if $v \in L_2(0,T;W_3^1(\Omega))$ then the solution of problem (1.1) is unique.

We establish the result on existence by the Leray-Schauder fixed point theorem, [LS]. We sketch the idea of the proof from paper [RZ1]. Let

$$\mathfrak{M}(\Omega^T) = \{h : \|h\|_{L_\infty(0,T;W_\eta^1(\Omega))} < \infty\}.$$

We construct the mapping $\Phi(\bar{h}) = h$,

$$\Phi : \mathfrak{M}(\Omega^T) \to W_\sigma^{2,1}(\Omega^T) \hookrightarrow \mathfrak{M}(\Omega^T),$$

considering the following Stokes-type problem

$$
\begin{array}{ll}
h_t - \operatorname{div} \mathbb{T}(h,q) = -\lambda[v(\bar{h},\bar{v}) \cdot \nabla \bar{h} + \bar{h} \cdot \nabla v(\bar{h},\bar{v})] + g & \text{in } \Omega^T, \\
\operatorname{div} h = 0 & \text{in } \Omega^T, \\
h \cdot \bar{n} = 0, \quad \bar{n} \cdot \mathbb{D}(h) \cdot \bar{\tau}_\alpha = 0, \quad \alpha = 1,2, & \text{on } S_1^T, \\
h_i = 0, \quad i = 1,2, \quad h_{3,x_3} = 0 & \text{on } S_2^T, \\
h\big|_{t=0} = h(0) & \text{in } \Omega,
\end{array}
$$

where $\lambda \in [0,1]$ and $\bar{h}, \bar{v}$ are treated as given functions.

In order to fulfill the assumptions of Leray-Schauder theorem, we show that Φ is uniformly continuous and compact in the product $\mathfrak{M}(\Omega^T) \times [0,1]$ for some parameters σ, η. Therefore, we apply [BIN], Chap. 2, and [S]. To have compact Φ we need compactness of imbedding

$$W_\sigma^{2,1}(\Omega^T) \hookrightarrow L_\infty(0,T; W_\eta^1(\Omega))$$

which is true for σ, η satisfying

$$\frac{5}{\sigma} - \frac{3}{\eta} - \frac{2}{\infty} < 1, \quad \sigma < \eta.$$

For the uniform continuity, we need more imbeddings. We need to find energy inequalities for problems, that we derive for differences $h_1 - h_2, v_1 - v_2$. To estimate nonlinear terms, we assume, with $j = 1,2$,

$$
\begin{gathered}
v_j \in W_2^{2,1}(\Omega^T) \hookrightarrow L_{\sigma\lambda_1}(\Omega^T), \text{ where } \frac{1}{\sigma\lambda_1} + \frac{1}{\eta} = \frac{1}{\sigma}, \\
\text{and } v_j \in W_2^{2,1}(\Omega^T) \hookrightarrow L_\sigma(0,T; W_\sigma^1(\Omega))
\end{gathered}
$$

and we derive the inequalities that parameters should fulfill. We combine all the relations to conclude $\frac{20}{7} < \sigma \le \frac{10}{3}$, $\eta > 4$.

5. Global solutions

Next, in the paper [RZ6] we show the existence of global regular solutions to problem (1.1) for arbitrary flux. To this end, we extend the local existence result proved in [RZ5] for time intervals $(kT,(k+1)T), k \in \mathbb{N}_0 \equiv \mathbb{N} \cup \{0\}$, step by step in time. For this purpose we need sufficiently large time of local existence to show that data at the beginning of each step do not increase in time in appropriate norms.

We consider solutions to problem (1.1) in time intervals $(kT,(k+1)T), k \in \mathbb{N}_0$ and define $\Omega^{(k+1)T} = \Omega \times (kT,(k+1)T), S_i^{(k+1)T} = S_i \times (kT,(k+1)T), i = 1,2$. Namely, we examine the system of problems

$$
(5.1)\qquad
\begin{aligned}
& v_t + v\cdot\nabla v - \operatorname{div}\mathbb{T}(v,p) = f && \text{in } \Omega^{(k+1)T},\\
& \operatorname{div} v = 0 && \text{in } \Omega^{(k+1)T},\\
& v\cdot\bar n = 0 && \text{on } S_1^{(k+1)T},\\
& \nu\bar n\cdot\mathbb{D}(v)\cdot\bar\tau_\alpha + \gamma v\cdot\bar\tau_\alpha = 0,\quad \alpha=1,2, && \text{on } S_1^{(k+1)T},\\
& v\cdot\bar n = d && \text{on } S_2^{(k+1)T},\\
& \bar n\cdot\mathbb{D}(v)\cdot\bar\tau_\alpha = 0,\quad \alpha=1,2, && \text{on } S_2^{(k+1)T},\\
& v|_{t=kT} = v(kT) && \text{in } \Omega.
\end{aligned}
$$

We apply results proved for $(0,T)$ on corresponding sets and work with the estimate for regular solutions. Thus, we need to reformulate quantities, assumptions and results crucial for existence of regular solutions on time interval $(0,T)$ to sets $(kT,(k+1)T)$.

Definition 3.

$$
\begin{aligned}
\Lambda(k,T) &= \|d_t\|^2_{L_2(kT,(k+1)T;H^1(S_2))} + \|d_{,x'}\|^2_{L_2(kT,(k+1)T;H^1(S_2))} + \|d_{,x'}\|^2_{L_\infty(kT,(k+1)T;H^1(S_2))}\\
&\quad + \|f_3\|^2_{L_2(kT,(k+1)T;L_{4/3}(S_2))} + \|g\|^2_{L_2(kT,(k+1)T;L_{6/5}(\Omega))} + \|h(kT)\|^2_{L_2(\Omega)},\\
\Gamma(k,T) &= \|f\|_{L_2(kT,(k+1)T;L_2(\Omega))} + \|v(kT)\|_{H^1(\Omega)},\\
D_0(k,T) &= \|d_1\|^6_{L_\infty(kT,(k+1)T;L_3(S_2(-a)))} + A^2(k,T) + 1,
\end{aligned}
$$

where $A^2(k,T) = l_0^2 + \frac{l_0^2}{1-e^{-\nu T}} + e^{-\nu kT}\|v(0)\|^2_{L_2(\Omega)}$, $k = 0,1,2,\dots$.

The smallness parameter $\Lambda(k,T)$ should fullfill analogous assumptions:

Condition 2. *Let quantities* $\Lambda = \Lambda(k,T), \Gamma = \Gamma(k,T), D_0 = D_0(k,T)$ *be finite. Assume that* Λ *is so small that there exists a constant* $\mathcal{A}_k$ *satisfying*

$$
(5.2)\qquad
\begin{aligned}
& D_0^2\Lambda^2(\mathcal{A}_k + \mathcal{A}_k^2 + \Gamma)\exp[TD_0 + D_0(\mathcal{A}_k+\Gamma) + \mathcal{A}_k^2 + \Gamma]\\
&\qquad + \|g\|^2_{L_2(kT,(k+1)T;L_2(\Omega))} + \|h(kT)\|^2_{H^1(\Omega)} \le \mathcal{A}_k^2
\end{aligned}
$$

Observe, that with given parameters Λ,Γ,D_0, constant $\mathcal{A}_k$ can be found as the implicit function. Therefore, we prove the existence of regular solutions on time interval $(kT,(k+1)T)$ with estimate $\mathcal{A}_k$.

Lemma 5.1. *Assume that* $g,f\in L_2(kT,(k+1)T;L_2(\Omega))$, $v(kT),h(kT)\in H^1(\Omega)$, *and quantities* $\Lambda=\Lambda(k,T),\Gamma=\Gamma(k,T),D_0=D_0(k,T)$ *are finite. Assume that* Λ *is so small that satisfies Condition 2. Then there exists a solution to problem (5.1) such that* $v,h\in W_2^{2,1}(\Omega\times(kT,(k+1)T))$, $\nabla p, \nabla p_{,x_3}\in L_2(\Omega\times(kT,(k+1)T))$, *and*

$$
\begin{gathered}
\|h\|_{W_2^{2,1}(\Omega^{kT})} \le \mathcal{A}_k,\\
\|v\|_{W_2^{2,1}(\Omega^{kT})} + \|\nabla p\|_{L_2(\Omega^{kT})} \le \mathcal{Q}(\mathcal{A}_k),\\
\|\nabla p_{,x_3}\|_{L_2(\Omega^{kT})} \le \mathcal{Q}(\mathcal{A}_k),
\end{gathered}
$$

where $\mathcal{Q}$ *is an increasing positive function, quadratic in* $\mathcal{A}_k$, *of the form*

$$
\mathcal{Q}(\mathcal{A}_k) = \mathcal{A}_k^2 + \mathcal{A}_k + \Lambda(k,T)^2 + \Gamma^2(k,T).
$$

We note, that Lemma 5.1 implies existence of solutions to problem (5.1) in the time interval $[kT,(k+1)T]$ if we know that $v(kT), h(kT) \in H^1(\Omega), k \in \mathbb{N}_0$. Consequently, we will show that the constant $\mathcal{A}_k$ in the above lemma does not depend on k, thus Condition 2 will be satisfied for any k if it holds for $k=0$ and consequently, $\mathcal{A}_k = \mathcal{A}$ in theorem on global solutions (Theorem 5 below). To this end, we have to control the initial condition at each time step in the set of problems (5.1)

The proof of global existence result is divided into the following steps. Assume that

$$\|v(kT)\|_{H^1(\Omega)} \le \alpha(k)$$
$$\|h_t(kT)\|_{L_2(\Omega)} + \|h(kT)\|_{H^1(\Omega)} \le \beta(k)$$

where $k \in \mathbb{N}_0$. Then there exists quantities $\mathcal{Q}_k = \mathcal{Q}(\mathcal{A}_k)$ (see Lemma 5.1 and (5.3)) and $\mathcal{B}_k = \mathcal{B}(\mathcal{A}_k)$ (where $\mathcal{B}$ is also polynomial, like $\mathcal{Q}$) that for sufficiently small Λ (see (5.2)) there exists a local solution to problem (5.1) in the time interval $[kT,(k+1)T]$ such that

$$\begin{aligned} &\|v\|_{L_2(kT,(k+1)T;H^2(\Omega))} \le \mathcal{Q}_k, \\ &\|v_t\|_{L_2(kT,k(T+1);H^1(\Omega))} \le \mathcal{B}_k \end{aligned} \tag{5.3}$$

Moreover, for $\Lambda(k,T)$ sufficiently small we can choose T as large as we want. Then the main result of this paper is to show that

$$\|v((k+1)T)\|_{H^1(\Omega)} \le \alpha(k)$$

and

$$\|h_t((k+1)T)\|_{L_2(\Omega)} + \|h((k+1)T)\|_{H^1(\Omega)} \le \beta(k).$$

Starting from $t=(k+1)T$ we can repeat the above considerations to prove existence in the interval $[(k+1)T,(k+2)T]$. In fact, there exists a constant $c_0 = c_0(k)$ such that

$$\|v((k+1)T)\|_{H^1(\Omega)} \le c_0\|v(kT)\|_{H^1(\Omega)},$$
$$\|h_t((k+1)T)\|_{L_2(\Omega)} + \|h((k+1)T)\|_{H^1(\Omega)} \le c_0(\|h_t(kT)\|_{L_2(\Omega)} + \|h(kT)\|_{H^1(\Omega)})$$

The constant $c_0(k)$ could grow as we repeat such local existence proof n_0 times for some finite $n_0 \in \mathbb{N}$. Then, we have local existence result in the interval $[0,n_0T]$ and we can define new local existence time T as equal to n_0T. Thus the constant c_0 after sufficiently many steps can be less or equal to one.

Moreover, in [RZ6] we show that for some quantities D_1 and D_2 dependent on $\mathcal{Q}_k$ and $\mathcal{B}_k$ the following inequalities hold

$$\|v((k+1)T)\|_{H^1(\Omega)} \le e^{-\frac{1}{2}c_*T}\|v(kT)\|_{H^1(\Omega)} + D_1(k), \tag{5.4}$$

and

$$\begin{aligned} &\|h_t((k+1)T)\|_{L_2(\Omega)} + \|h((k+1)T)\|_{H^1(\Omega)} \\ &\le e^{-\frac{1}{2}c_*T}(\|h_t(kT)\|_{L_2(\Omega)} + \|h(kT)\|_{H^1(\Omega)}) + D_2(k). \end{aligned} \tag{5.5}$$

The constant c_* comes from the imbedding and the rot-div problem

$$c_*\|v\|_{H^1(\Omega)} \le \|\operatorname{div}\mathbb{T}(v,p)\|_{L_2(\Omega)}.$$

Here, $\mathbf{a}$ is global estimate for L_2 norm of weak solutions (Theorem 2) whereas l_2, l_3 are time integrals on $(kT,(k+1)T)$ of *data* in corresponding norms and φ_2 is the function of the form

$$\varphi_2 = \sup_{t\in(kT,(k+1)T)} \|v\|^{\frac{1}{2}}_{L_2(\Omega)}(\sup_t \|d_{x'}\|^2_{L_4(S_2)} + \sup_t \|d_t\|^2_{H^1(S_2)}) \le \mathbf{a}^{\frac{1}{2}}\varphi(data).$$

Therefore, with some assumptions on $D_1(k), D_2(k)$ it follows that there exists α_0 and β_0 such that

$$\|v(0)\|_{H^1(\Omega)} \le \alpha_0,$$
$$\|h_t(0)\|_{L_2(\Omega)} + \|h(0)\|_{H^1(\Omega)} \le \beta_0$$

and, as well

$$\|v(kT)\|_{H^1(\Omega)} \le \alpha_0,$$
$$\|h_t(kT)\|_{L_2(\Omega)} + \|h(kT)\|_{H^1(\Omega)} \le \beta_0.$$

We can assume that $D_1 = \sup_k D_1(k)$ to deduce from (5.4)

$$\|v(kT)\|_{H^1(\Omega)} \le \frac{D_1(1-e^{-\frac{1}{2}c_*kT})}{1-e^{-\frac{1}{2}c_*T}} + e^{-\frac{1}{2}c_*kT}\|v(0)\|_{H^1(\Omega)}$$

Then, for $\|v(0)\|_{H^1(\Omega)}$ respectively greater than D_1 the above inequality yields

$$\|v(kT)\|_{H^1(\Omega)} \le \|v(0)\|_{H^1(\Omega)}.$$

In similar way we argue with h to show that if $\|h_t(0)\|_{L_2(\Omega)} + \|h(0)\|_{H^1(\Omega)}$ is greater than $D_2 = \sup_k D_2(k)$ then

$$\|h_t(kT)\|_{L_2(\Omega)} + \|h(kT)\|_{H^1(\Omega)} \le \|h_t(0)\|_{L_2(\Omega)} + \|h(0)\|_{H^1(\Omega)}.$$

Observe, that additional restrictions on D_1 i D_2 are stronger than more general conditions with constants α_0, β_0.

One could think that L_2 norm of h_t appears accidentally, since it does not appear in definitions of parameters Λ, Γ, D_0 either in Condition 2, like H^1 norms of v and h. However, the presence of this term is the consequence of energy estimate for h, where the norm $\|h_{3t}\|_{L_2(\Omega)}$ appears and so, to complete the considerations, we need to analyze the problem for h_t as well.

We obtain the result, under some assumptions on derivative of external force and restrictions for derivatives of inflow.

Assumption A.1 *Assume that T is sufficiently large so that*

$$-\frac{1}{2}c_*T + l_1[T^{1/4}\mathcal{Q}_k^{3/2} + T^{1/2}\mathcal{Q}_k] \le 0$$

Assumptions A.2 *Assume that T is sufficiently large so that*

$$-\frac{1}{2}c_*T + \varphi_1 T^{1/4}\mathcal{Q}_k^{3/2} + \mathcal{B}_k + \|d_t\|^2_{H^1(S_2)} \le 0$$

Let us specify quantities used above. Since $\mathbf{a}$ is the global bound for L_2 norm of velocity v (Theorem 2), we set

$$\begin{aligned}
\varphi_1 &= \sup_{t\in(kT,(k+1)T)} \|v\|^{\frac{9}{2}}_{L_2(\Omega)} + \sup_{t\in(kT,(k+1)T)} \|v\|^{\frac{1}{2}}_{L_2(\Omega)} \le \mathbf{a}^{\frac{9}{2}} + \mathbf{a}^{\frac{1}{2}},\\
l_1 &= c(1+\sup_t \|d_{,x_\alpha}\|^4_{L_2(S_2)})\mathbf{a}^{\frac{1}{2}}(1+\mathbf{a}^{\frac{5}{2}}).
\end{aligned}$$

Then, we consider the system (1.1) for $t \in \mathbb{R}_+$ which reads as follows

$$
\begin{aligned}
& v_t + v \cdot \nabla v - \operatorname{div} \mathbb{T}(v,p) = f && \text{in } \Omega^{\mathbb{R}_+} = \Omega \times \mathbb{R}_+,\\
& \operatorname{div} v = 0 && \text{in } \Omega^{\mathbb{R}_+},\\
& v \cdot \bar{n} = 0 && \text{on } S_1^{\mathbb{R}_+},\\
& \nu \bar{n} \cdot \mathbb{D}(v) \cdot \bar{\tau}_\alpha + \gamma v \cdot \bar{\tau}_\alpha{}' = 0, \quad \alpha = 1,2, && \text{on } S_1^{\mathbb{R}_+},\\
& v \cdot \bar{n} = d && \text{on } S_2^{\mathbb{R}_+},\\
& \bar{n} \cdot \mathbb{D}(v) \cdot \bar{\tau}_\alpha = 0, \quad \alpha = 1,2, && \text{on } S_2^{\mathbb{R}_+},\\
& v\big|_{t=0} = v(0) && \text{in } \Omega,
\end{aligned}
\tag{5.6}
$$

Next, we define $\bar{\Gamma}$ and smallness parameter $\bar{\Lambda}$.

Definition 4.

$$
\begin{aligned}
\bar{\Gamma} &= \sup_k \|f\|_{L_2(kT,(k+1)T;L_2(\Omega))} + \|v(0)\|_{H^1(\Omega)},\\
\bar{\Lambda} &= \sup_k \Big(\|d_t\|^2_{L_2(kT,(k+1)T;H^1(S_2))} + \|d_{,x'}\|^2_{L_2(kT,(k+1)T;H^1(S_2))} + \|d_{,x'}\|^2_{L_\infty(kT,(k+1)T;H^1(S_2))}\\
&\quad + \|f_3\|^2_{L_2(kT,(k+1)T;L_{\frac{4}{3}}(S_2))} + \|g\|^2_{L_2(kT,(k+1)T;L_{\frac{6}{5}}(\Omega))} \Big) + \|h(0)\|^2_{L_2(\Omega)}
\end{aligned}
$$

Remark 5.2. *We assume that $\bar{\Lambda}$ is the small parameter and this in particular implies that $\|h(0)\|_{L_2(\Omega)}$ is small. However, we do not require that on velocity and so $v(0)$ and the inflow d can be arbitrarily large.*

Then, we formulate the global result:

Theorem 5. *Suppose Assumptions A.1 - A.2 hold and $(\operatorname{rot} v)_3(0) \in L_2(\Omega)$, $(\operatorname{rot} f)_3 \in L_2(kT,(k+1)T;L_2(\Omega))$, $d_1 \in L_\infty(kT,(k+1)T;L_3(S_2(-a)))$. Assume that the quantity $\bar{\Lambda}$ is sufficiently small. Then for the solution of (5.6) and sufficiently large time T there exists constant $\mathcal{A}$ such that*

$$
\begin{aligned}
&\|v\|_{W_2^{2,1}(\Omega\times(kT,(k+1)T))} + \|\nabla p\|_{L_2(\Omega\times(kT,(k+1)T))} \le \mathcal{Q}(\mathcal{A}),\\
&\|v_{,x_3}\|_{W_2^{2,1}(\Omega\times(kT,(k+1)T))} + \|\nabla p_{,x_3}\|_{L_2(\Omega\times(kT,(k+1)T))} \le \mathcal{A}
\end{aligned}
$$

for any $k \in \mathbb{N}_0$ where $\mathcal{Q}(\mathcal{A}) = \mathcal{A}^2 + \mathcal{A} + \bar{\Lambda}^2 + \bar{\Gamma}$.

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