Stability of nonswirl axisymmetric solutions to the Navier-Stokes equations

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Abstract

The existence of global regular axisymmetric solutions to the Navier-Stokes equations without swirl and in a finite axisymmetric cylinder is proved. The solutions are such that norms bounded with respect to time are controlled by the same constant for all $t \in \mathbb{R}_+$. Assuming that the initial velocity and the external force are sufficiently close to the initial velocity and the external force of a nonswirl axisymmetric solutions, we prove existence of global regular axisymmetric solutions which remain close to the nonswirl axisymmetric solution for all time. In this sense we have stability of nonswirl axisymmetric solutions. However, to prove this we need a smallness condition on the aximuthal component of vorticity of the external force for the nonswirl solution.

AMS subject classification. Primary, 35Q30, 35B35, 76D03, 76D05, 76D10; Secondary,

Key words: axisymmetric solutions to the Navier-Stokes equations, stability of nonswirl solutions, global regular axisymmetric solutions, special slip boundary conditions

1 Introduction

In this paper we consider axially symmetric solutions to the Navier-Stokes equations

$$v_t + v \cdot \nabla v - \nu \Delta v + \nabla p = f, \tag{1.1}$$

$$\operatorname{div} v = 0, \tag{1.2}$$

where $v = (v_1(x,t), v_2(x,t), v_3(x,t)) \in \mathbb{R}^3$ is the velocity of the fluid, $p = p(x,t) \in \mathbb{R}$ is the pressure, $f = (f_1(x,t), f_2(x,t), f_3(x,t)) \in \mathbb{R}^3$ is the external force field, $\nu > 0$ is the constant viscosity ceofficient, $x = (x_1, x_2, x_3)$ are the Cartesian coordinates.

Equations (1.1), (1.2) are considered in an axisymmetric cylindrical bounded domain $\Omega \subset \mathbb{R}^3$ with the axis of symmetry equal to the x_3-axis. Let S be the boundary of Ω. On S we assume the following conditions

$$v \cdot \bar{n} = 0, \quad \text{on} \quad S, \tag{1.3}$$

(1.4) azimuthal component of vorticity vanishes on S,

(1.5) azimuthel component of velocity vanishes on S,

where $\bar{n}$ is the unit outward vector normal to S.
The boundary S is split two parts $S = S_1 \cup S_2$, where S_1 is parallel to x_3-axis and S_2 perpendicular. We have that $S_2 = S_2(-a) \cup S_2(a)$, where $a > 0$ is given and $S_2(b)$ meets the x_3-axis at $x_3 = b$, $b \in \{-a, a\}$.
Finally, we add the initial conditions

$$v|_{t=0} = v(0). \tag{1.6}$$

The aim of this paper is to prove stability of nonswirl axisymmetric solutions in a set of general axisymmetric solutions. Moreover, we have to prove global existence of regular nonswirl axisymmetric solutions bounded by constants independent of time. To examine axisymmetric solutions we introduce the cylindrical coordinates r, φ, z by the relations

$$x_1 = r\cos\varphi, \quad x_2 = r\sin\varphi, \quad x_3 = z. \tag{1.7}$$

Next, we use the orthonormal basis

$$\bar{e}_r = (\cos\varphi, \sin\varphi, 0), \quad \bar{e}_\varphi = (-\sin\varphi, \cos\varphi, 0), \quad \bar{e}_z = (0, 0, 1) \equiv \bar{e}_3. \tag{1.8}$$

Then the cylindrical coordinates of v, $\omega = \operatorname{rot} v$, f are defined by

$$v(r,z,t) = v_r(r,z,t)\bar{e}_r + v_\varphi(r,z,t)\bar{e}_\varphi + v_z(r,z,t)\bar{e}_z, \tag{1.9}$$

$$\begin{aligned} \bar{\omega}(r,z,t) &= \omega_r(r,z,t)\bar{e}_r + \omega_\varphi(r,z,t)\bar{e}_\varphi + \omega_z(r,z,t)\bar{e}_z \\ &= -v_{\varphi,z}\bar{e}_r + (v_{r,z} - v_{z,r})\bar{e}_\varphi + \left(v_{\varphi,r} + \frac{1}{r}v_\varphi\right)\bar{e}_z, \end{aligned} \tag{1.10}$$

$$f(r,z,t) = f_r(r,z,t)\bar{e}_r + f_\varphi(r,z,t)\bar{e}_\varphi + f_z(r,z,t)\bar{e}_z \tag{1.11}$$

Let us recall that the swirl is defined by

$$u_0 = rv_\varphi. \tag{1.12}$$

Nonswirl axisymmetric solutions satisfy (see [1]) ($v_\varphi = 0$)

$$\overset{1}{\omega}_{,t} + \overset{1}{v} \cdot \nabla\overset{1}{\omega} - \nu\left(\nabla^2 - \frac{1}{r^2}\right)\overset{1}{\omega} - \frac{1}{r}\overset{1}{v}_r\overset{1}{\omega} = \overset{1}{F}_\varphi \quad \text{in} \quad \Omega_+ = \Omega \times \mathbb{R}_+, \tag{1.13}$$

(1.14)
$$-\left(\nabla^2-\frac{1}{r^2}\right)\overset{1}{\psi}=\overset{1}{\omega}\quad\text{in }\ \Omega_+,$$

where $\overset{1}{F}_\varphi=\overset{1}{f}_{r,z}-\overset{1}{f}_{z,r}$, $\overset{1}{\omega}=\overset{1}{\omega}_\varphi$, $\overset{1}{\psi}$ is the stream function, which implies the radial and axial components of velocity

(1.15)
$$\overset{1}{v}_r=-\overset{1}{\psi}_{,z},\quad \overset{1}{v}_z=\frac{1}{r}(r\overset{1}{\psi})_{,r}.$$

From (1.3)–(1.5) we have

(1.16)
$$\overset{1}{\omega}|_S=0,\quad \overset{1}{v}_r|_{S_1}=0,\quad \overset{1}{v}_z|_{S_2}=0.$$

In view of (1.15) boundary conditions $(1.16)_{2,3}$ are satisfied if

(1.17)
$$\overset{1}{\psi}|_S=0.$$

The axisymmetric solutions with nonvanishing swirl to the Navier-Stokes equations satisfy the problem

(1.18)
$$\overset{2}{u}_{,t}+\overset{2}{v}\cdot\nabla\overset{2}{u}-\nu\left(\nabla^2-\frac{1}{r^2}\right)\overset{2}{u}+\frac{1}{r}\overset{2}{v}_r\overset{2}{u}=\overset{2}{f}_\varphi\quad\text{in }\ \Omega_+,$$

(1.19)
$$\overset{2}{\omega}_{,t}+\overset{2}{v}\cdot\nabla\overset{2}{\omega}-\nu\left(\nabla^2-\frac{1}{r^2}\right)\overset{2}{\omega}-\frac{1}{r}\overset{2}{v}_r\overset{2}{\omega}-\frac{2}{r}\overset{2}{u}\overset{2}{u}_{,z}=\overset{2}{F}_\varphi\quad\text{in }\ \Omega_+,$$

(1.20)
$$-\left(\nabla^2-\frac{1}{r^2}\right)\overset{2}{\psi}=\overset{2}{\omega}\quad\text{in }\ \Omega_+,$$

where $\overset{2}{\omega}=\overset{2}{\omega}_\varphi$, $\overset{2}{F}_\varphi=\overset{2}{f}_{r,z}-\overset{2}{f}_{z,r}$, $\overset{2}{u}=\overset{2}{v}_\varphi$ and $\overset{2}{\psi}$ implies the radial and axial components of velocity

(1.21)
$$\overset{2}{v}_r=-\overset{2}{\psi}_{,z},\quad \overset{2}{v}_z=\frac{1}{r}(r\overset{2}{\psi})_{,r}$$

From (1.3)–(1.5) we have

(1.22)
$$\overset{2}{\omega}|_S=0,\quad \overset{2}{u}|_S=0,\quad \overset{2}{v}_r|_{S_1}=0,\quad \overset{2}{v}_z|_{S_2}=0.$$

The last two boundary conditions in (1.22) are satisfied in view of the assumption

(1.23)
$$\overset{2}{\psi}|_S=0.$$

To complete the above problems we assume the following initial conditions

(1.24)
$$\overset{1}{\psi}|_{t=0}=\overset{1}{\psi}(0),\quad \overset{1}{\omega}|_{t=0}=\overset{1}{\omega}(0)$$

and

$$\overset{2}{\psi}|_{t=0} = \overset{2}{\psi}(0), \quad \overset{2}{\omega}|_{t=0} = \overset{2}{\omega}(0), \quad \overset{2}{u}|_{t=0} = \overset{2}{u}(0). \tag{1.25}$$

To examine the above problems and show stability we introduce the quantities with lower index 1 by the following relations

$$\overset{k}{u} = r\overset{k}{u}_1, \quad \overset{k}{\omega} = r\overset{k}{\omega}_1, \quad \overset{k}{\psi} = r\overset{k}{\psi}_1, \quad \overset{k}{f}_\varphi = r\overset{k}{f}_{\varphi 1}, \quad \overset{k}{F}_\varphi = r\overset{k}{F}_{\varphi 1}, \quad k = 1, 2, \tag{1.26}$$

where $\overset{1}{u}_1 = 0$, $\overset{1}{f}_{\varphi 1} = 0$. Hence, the functions with upper index k are solutions to the problems

$$\begin{aligned}
&\overset{k}{u}_{1,t} + \overset{k}{v}\cdot\nabla\overset{k}{u}_1 - \nu\Big(\Delta\overset{k}{u}_1 + \frac{2}{r}\overset{k}{u}_{1,r}\Big) = 2\overset{k}{u}_1\overset{k}{\psi}_{1,z} + \overset{k}{f}_{\varphi 1} \quad \text{in } \Omega_+,\\
&\overset{k}{\omega}_{1,t} + \overset{k}{v}\cdot\nabla\overset{k}{\omega}_1 - \nu\Big(\Delta\overset{k}{\omega}_1 + \frac{2}{r}\overset{k}{\omega}_{1,r}\Big) = 2\overset{k}{u}_1\overset{k}{u}_{1,z} + \overset{k}{F}_{\varphi 1} \quad \text{in } \Omega_+,\\
&-\Big(\Delta\overset{k}{\psi}_1 + \frac{2}{r}\overset{k}{\psi}_{1,r}\Big) = \overset{k}{\omega}_1 \quad \text{in } \Omega_+,\\
&\overset{k}{\omega}_1|_{S_+} = 0, \quad \overset{k}{\psi}_1|_{S_+} = 0, \quad \overset{k}{u}_1|_{S_+} = 0,\\
&\overset{k}{\omega}_1|_{t=0} = \overset{k}{\omega}_1(0), \quad \overset{k}{\psi}_1|_{t=0} = \overset{k}{\psi}_1(0), \quad \overset{k}{u}_1|_{t=0} = \overset{k}{u}_1(0),
\end{aligned} \tag{1.27}$$

where $k = 1, 2$, and $\overset{1}{u}_1 = 0$, $\overset{1}{f}_{\varphi 1} = 0$, $S_+ = S \times \mathbb{R}_+$, $S_{i+} = S_i \times \mathbb{R}_+$, $i = 1, 2$.
To prove stability of nonswirl axisymmetric solutions we introduce the differences

$$\omega = \overset{2}{\omega} - \overset{1}{\omega}, \quad \psi = \overset{2}{\psi} - \overset{1}{\psi}, \quad u = \overset{2}{u} - \overset{1}{u}, \quad v = \overset{2}{v} - \overset{1}{v} \tag{1.28}$$

and

$$\omega = r\omega_1, \quad \psi = r\psi_1, \quad u = ru_1. \tag{1.29}$$

The functions u_1, ω_1, ψ_1 are solutions to the problem

$$\begin{aligned}
&u_{1,t} + v\cdot\nabla u_1 + \overset{1}{v}\cdot\nabla u_1 - \nu\Big(\Delta u_1 + \frac{2}{r}u_{1,r}\Big) = 2u_1\psi_{1,z} + u_1\overset{1}{\psi}_{1,z} + f_{\varphi 1},\\
&\omega_{1,t} + (v + \overset{1}{v})\cdot\nabla\omega_1 + v\cdot\nabla\overset{1}{\omega}_1 - \nu\Big(\Delta\omega_1 + \frac{2}{r}\omega_{1,r}\Big) = 2u_1u_{1,z} + F_{\varphi 1},\\
&-\Big(\Delta\psi_1 + \frac{2}{r}\psi_{1,r}\Big) = \omega_1,\\
&\omega_1|_{S_+} = 0, \quad \psi_1|_{S_+} = 0, \quad u_1|_{S_+} = 0,\\
&\omega_1|_{t=0} = \omega_1(0), \quad \psi_1|_{t=0} = \psi_1(0), \quad u_1|_{t=0} = u_1(0),\\
&v\cdot\bar{n}|_{S_+} = 0, \quad \overset{1}{v}\cdot\bar{n}|_{S_+} = 0.
\end{aligned} \tag{1.30}$$

Moreover,

$$v_r = -\psi_{,z}, \quad v_z = \frac{1}{r}(r\psi)_{,r} \tag{1.31}$$

The considered in this paper boundary conditions are more restrictive than the slip boundary conditions on S

$$v \cdot \bar{n} = 0 \tag{1.32}$$

$$\bar{n} \cdot \mathbb{D}(v) \cdot \bar{\tau}_\alpha = 0, \quad \alpha = 1,2, \tag{1.33}$$

where $\mathbb{D}(v) = \nabla v + \nabla v^T$ is the dilatation tensor, $\bar{\tau}_\alpha$, $\alpha = 1,2$, is the tangent vector to S. In view of [2, Ch. 4] (1.32), (1.33) imply the following boundary conditions

$$v_r|_{S_1} = 0, \quad v_z|_{S_2} = 0, \quad \omega|_S = 0, \quad u_{1,r}|_{S_1} = 0, \quad u_{1,z}|_{S_2} = 0. \tag{1.34}$$

Now, we formulate the main results of this paper. From Lemma 3.4 and (4.1), (4.2) we have

Theorem 1.1. *Assume that* $\overset{1}{\omega}_1(0) \in L_2(\Omega)$, $\overset{1}{F}_{\varphi 1} \in L_2(kT,(k+1)T; L_{6/5}(\Omega))$, $k \in \mathbb{N}_0$. *Then there exists a nonswirl axisymmetric solution to problem (1.1)–(1.6) such that*

$$v' \in L_\infty(\mathbb{R}_+; H^1(\Omega)) \cap L_2(kT,(k+1)T; H^2(\Omega)), \quad k \in \mathbb{N}_0,$$

where $v' = (v_r, v_z)$.

From Lemma 4.1 it follows

Theorem 1.2. *Let the assumptions of Theorem 1.1 hold. Let* $\gamma \in (0,\gamma_*]$, *where* γ_* *is so small that* $\nu_1 - \frac{c_0}{\nu_1}\gamma_* \ge \frac{c_*}{2}$, $c_* \in (0,\nu_1]$, c_0 *appears in (4.11) and* ν_1 *in Lemma 3.1. Let* $\|\omega_1(0)\|^2_{L_2(\Omega)} + \|u_1(0)\|^4_{L_4(\Omega)} \le \gamma$, *where* $\omega_1 = \omega/r$, $u_1 = u/r$ *and* ω, u *are introduced in (1.28). Let* $c_0(\|F_{\varphi 1}(t)\|^2_{L_{6/5}(\Omega)} + \|f_{\varphi 1}(t)\|^4_{L_{4/3}(\Omega)}) \le \frac{c_*}{4}\gamma$ *for* $t \in [kT,(k+1)T]$, $c_0 \int_{kT}^{(k+1)T}(\|F_{\varphi_1}(t)\|^2_{L_{6/5}(\Omega)} + \|f_{\varphi 1}(t)\|^4_{L_{4/3}(\Omega)})dt \le \alpha\gamma$, $k \in \mathbb{N}_0$.
Let $\bar{A}_1(k) = \frac{c_0}{\nu_1^2}\int_{kT}^{(k+1)T}\|\overset{1}{F}_{\varphi 1}(t)\|^2_{L_{6/5}(\Omega)}dt + \frac{1}{\nu_1}(\|\overset{1}{\omega}_1(kT)\|^2_{L_2(\Omega)} - \|\overset{1}{\omega}_1((k+1)T)\|_{L_2(\Omega)})$, *and* $\bar{A}_1^2 = \sup_{k\in\mathbb{N}_0}\bar{A}_1^2(k)$. *Let* $\alpha\exp(\bar{A}_1^2) + \exp\big(-\frac{c_*}{4}T\big) \le 1$. *Then*

$$\|\omega_1(t)\|^2_{L_2(\Omega)} + \|u_1(t)\|^4_{L_4(\Omega)} \le \gamma \quad \text{for any} \quad t \in \mathbb{R}_+. \tag{1.35}$$

Since ω and u describe a distance between swirl and nonswirl axisymmetric solutions to problem (1.1)–(1.6) we have

Theorem 1.3. *Let the assumptions of Theorems 1.1, 1.2 hold. Then there exists a global axisymmetric solution to problem (1.1)–(1.6) which remains close to nonswirl axisymmetric solutions for all time if they are sufficiently close at the initial time.*

There is a wide literature concerning stability of some special solutions to the Navier-Stokes equations. By the special solutions we mean either two-dimensional or nonswirl axisymmetric solutions. Stability ot two-dimensional solutions to the Navier-Stokes equations is considered in papers [5, 7, 8]. In [5, 7] the periodic boundary conditions are considered, so the fluid motion is located in a box. In [8] we consider the fluid motion in a cylindrical domain. Moreover, the Navier boundary conditions imply existence of two-dimensional solutions without any additional restrictions, which are bounded by a fixed constant independent of time.
More literature concerning stability of special solutions is cited in papers [5, 7, 8].

2 Notation and auxiliary results

Let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. By $L_p(\Omega)$, $p \in [1,\infty]$, $\Omega \subset \mathbb{R}^n$ we denote the Lebesgue space of integrable functions and by $H^s(\Omega)$, $s \in \mathbb{N}_0$, $\Omega \subset \mathbb{R}^n$, the Sobolev space of functions with the finite norm

$$\|u\|_{H^s(\Omega)} = \left(\sum_{|\alpha|\le s} \int_\Omega |D_x^\alpha u(x)|^2 dx\right)^{1/2},$$

where $D_x^\alpha = \partial_{x_1}^{\alpha_1} \dots \partial_{x_n}^{\alpha_n}$, $|\alpha| = \alpha_1 + \alpha_2 + \dots + \alpha_n$, $\alpha_i \in \mathbb{N}_0$, $i = 1, \dots, n$. Let $u = (u_1, \dots, u_n)$ be a vector. Then $|u| = \sqrt{u_2^2 + \dots + u_n^2}$.
The following Poincaré inequality holds

Lemma 2.1. *Let $u \in H^1(\Omega)$, $u|_S = 0$. Then there exists a constant c_p such that*

$$c_p\|u\|^2_{L_2(\Omega)} \le \|\nabla u\|^2_{L_2(\Omega)}. \tag{2.1}$$

3 Solutions without swirl

In this Section we prove the existence of regular global soltuions to problem (1.13)–(1.17), (1.24). We apply the energy method. We restrict our considerations to obtain necessary estimates only because existence follows from the Faedo-Galerkin method.

Estimates in this section are performed in the sense of a priori. We assume that there exist sufficiently regular solutions to problem (1.13)–(1.17), (1.24). Then after getting the estimates and performing the closure procedure we have estimates for solutions with regularity described by these estimates.

Lemma 3.1. *Let us consider (1.13). Let*

$$\begin{aligned} A_1^2 &= \frac{c_s}{\nu_1} \sup_k \int_{kT}^{(k+1)T} \|\overset{1}{F}_{\varphi 1}(t)\|^2_{L_{6/5}(\Omega)} dt < \infty, \\ A_2^2 &= \frac{A_1^2}{1 - e^{-\nu_1 T}} + \|\overset{1}{\omega}_1(0)\|^2_{L_2(\Omega)} < \infty, \end{aligned} \tag{3.1}$$

$\nu_1 = \frac{\nu c_1}{2}$, $c_1 = \min\{1, c_p\}$, c_p is the Poincaré constant from (2.1) and c_s is the constant from the imbedding $H^1(\Omega) \subset L_6(\Omega)$. Then

$$\|\overset{1}{\omega}_1(t)\|^2_{L_2(\Omega)} + \nu_1 \int_{kT}^t \|\overset{1}{\omega}_1(t')\|^2_{H^1(\Omega)} dt' \le A_1^2 + A_2^2, \tag{3.2}$$

where $t \in (kT, (k+1)T]$, $k \in \mathbb{N}_0$.

Proof. Multiplying (1.13) by $\overset{1}{\omega}_1$ and integrating the result over Ω yields

$$\frac{1}{2}\frac{d}{dt}\|\overset{1}{\omega}_1\|^2_{L_2(\Omega)} + \nu\|\nabla\overset{1}{\omega}_1\|^2_{L_2(\Omega)} - 2\nu \int_\Omega \overset{1}{\omega}_{1,r}\overset{1}{\omega}_1 drdz = \int_\Omega \overset{1}{F}_{\varphi 1}\overset{1}{\omega}_1 dx. \tag{3.3}$$

The last term on the l.h.s. of (3.3) equals

$$-\nu\int_\Omega (\overset{1}{\omega}{}_1^2)_{,r}drdz = \nu\int_{-a}^{a}\overset{1}{\omega}{}_1^2|_{r=0}dz,$$

where we used that $\overset{1}{\omega}_1|_{r=R}=0$. Since the term is positive on the l.h.s. of (3.3), it can be omitted. By the Poincaré inequality (see Lemma 2.1) we have

$$\frac{d}{dt}\|\overset{1}{\omega}_1\|^2_{L_2(\Omega)}+\nu c_1\|\overset{1}{\omega}_1\|^2_{H^1(\Omega)}\le 2\int_\Omega \overset{1}{F}_{\varphi 1}\overset{1}{\omega}_1dx, \tag{3.4}$$

where $c_1=\min\{1,c_p\}$ and c_p is the constant from (2.1). Applying the Hölder and the Young inequalities to the r.h.s. of (3.4) we derive

$$\frac{d}{dt}\|\overset{1}{\omega}_1\|^2_{L_2(\Omega)}+\frac{\nu c_1}{2}\|\overset{1}{\omega}_1\|^2_{H^1(\Omega)}\le \frac{2c_s}{\nu c_1}\|\overset{1}{F}_{\varphi 1}\|^2_{L_{6/5}(\Omega)}, \tag{3.5}$$

where c_s is the constant from the Sobolev imbedding $H^1(\Omega)\subset L_6(\Omega)$. Setting $\nu_1=\frac{\nu c_1}{2}$ in (3.5) it follows

$$\frac{d}{dt}\|\overset{1}{\omega}_1\|^2_{L_2(\Omega)}+\nu_1\|\overset{1}{\omega}_1\|^2_{L_2(\Omega)}\le \frac{c_s}{\nu_1}\|\overset{1}{F}_{\varphi 1}\|^2_{L_{6/5}(\Omega)}. \tag{3.6}$$

Continuing, we have

$$\frac{d}{dt}(\|\overset{1}{\omega}_1\|^2_{L_2(\Omega)}e^{\nu_1 t})\le \frac{c_s}{\nu_1}\|\overset{1}{F}_{\varphi 1}\|^2_{L_{6/5}(\Omega)}e^{\nu_1 t} \tag{3.7}$$

Integrating (3.7) with respect to time from $t=kT$ to $t\in(kT,(k+1)T]$ we derive

$$\|\overset{1}{\omega}_1(t)\|^2_{L_2(\Omega)}\le \frac{c_s}{\nu_1}\int_{kT}^{t}\|\overset{1}{F}_{\varphi 1}(t')\|^2_{L_{6/5}(\Omega)}dt'+\exp(-\nu_1(t-kT))\|\overset{1}{\omega}_1(kT)\|^2_{L_2(\Omega)}. \tag{3.8}$$

Setting $t=(k+1)T$ inequality (3.8) implies

$$\|\overset{1}{\omega}_1((k+1)T)\|^2_{L_2(\Omega)}\le A_1^2+\exp(-\nu_1 T)\|\overset{1}{\omega}_1(kT)\|^2_{L_2(\Omega)}. \tag{3.9}$$

Hrence, iteration yields

$$\|\overset{1}{\omega}_1(kT)\|^2_{L_2(\Omega)}\le \frac{A_1^2}{1-e^{-\nu_1 T}}+e^{-\nu_1 kT}\|\overset{1}{\omega}_1(0)\|^2_{L_2(\Omega)}\le A_2^2. \tag{3.10}$$

Integrating (3.5) with respect to time from $t=kT$ to $t\in(kT,(k+1)T]$ yields

$$\|\overset{1}{\omega}_1(t)\|^2_{L_2(\Omega)}+\nu_1\int_{kT}^{t}\|\overset{1}{\omega}_1(t')\|^2_{H^1(\Omega)}\le \frac{c_s}{\nu_1}\int_{kT}^{t}\|\overset{1}{F}_{\varphi 1}(t')\|^2_{L_{6/5}(\Omega)}dt'+\|\overset{1}{\omega}_1(kT)\|^2_{L_2(\Omega)}. \tag{3.11}$$

Using $(3.1)_1$ and (3.10) gives (3.2). This concludes the proof. □

Next, we consider the problem (see (1.14), (1.17) without upper index 1)

$$-\Delta\psi + \frac{\psi}{r^2} = \omega, \quad \psi|_S = 0. \tag{3.12}$$

Using the cylindrical coordinates problem (3.12) is expressed in the form

$$-\left(\psi_{,rr} + \psi_{,zz} + \frac{1}{r}\psi_{,r}\right) + \frac{\psi}{r^2} = \omega, \quad \psi|_S = 0. \tag{3.13}$$

From (3.12) the following problem for $\psi_1 = \psi/r$ follows,

$$-\left(\Delta\psi_1 + \frac{2}{r}\psi_{1,r}\right) = \omega_1, \quad \psi_1|_S = 0 \tag{3.14}$$

so in cylindrical coordinates it takes the form

$$-\left(\psi_{1,rr} + \psi_{1,zz} + \frac{3}{r}\psi_{1,r}\right) = \omega_1, \quad \psi_1|_S = 0. \tag{3.15}$$

Lemma 3.2. *Assume that* $\omega_1 \in L_2(\Omega)$. *Assume that solutions to (3.14) vanish sufficiently quickly near the axis of symmetry. Then solutions to (3.15) satisfy the estimate*

$$\|\psi_1\|^2_{H^2(\Omega)} + \int\limits_\Omega \frac{1}{r^2}|\psi^{(1)}_{1,r}|^2 dx \le c\|\omega_1\|^2_{L_2(\Omega)}, \tag{3.16}$$

where $\psi_1^{(1)}$ *is only different from zero in some neighborhood of the axis of symmetry.*

Proof. To obtain estimates for solutions to problem (3.12) we introduce the following partition of unity

$$\sum_{k=1}^{2} \varphi^{(k)}(r) = 1, \quad 0 \le r \le R,$$

such that $\varphi^{(1)}(r) = 1$ for $r \le r_0$, $\varphi^{(1)}(r) = 0$ for $r \ge 2r_0$, $\varphi^{(2)}(r) = 1$ for $r \ge 2r_0$, $\varphi^{(2)}(r) = 0$ for $r \le r_0$, $2r_0 < R$. Next we introduce the notation

$$\psi^{(k)} = \psi\varphi^{(k)}, \quad \omega^{(k)} = \omega\varphi^{(k)}, \quad k = 1, 2.$$

Multiplying (3.14) by ψ_1, integrating over Ω and using the boundary conditions we get

$$\int\limits_\Omega |\nabla\psi_1|^2 dx - \int\limits_\Omega (\psi_1^2)_{,r} dr dz = \int\limits_\Omega \omega_1\psi_1 dx. \tag{3.17}$$

Using the boundary conditions again and the Hölder, the Young and the Poincaré inequalities to the r.h.s. of (3.17) we obtain the estimate

$$\|\psi_1\|^2_{H^1(\Omega)} + \int\limits_{-a}^{a} \psi_1^2|_{r=0} dz \le c\|\omega_1\|^2_{L_2(\Omega)}. \tag{3.18}$$

Multiplying (3.14) by $\varphi^{(1)}$ we obtain the problem

$$(3.19)\qquad -\Delta\psi_1^{(1)}-\frac{2}{r}\psi_{1,r}^{(1)}+2\psi_{1,r}\dot\varphi^{(1)}+\psi_1\ddot\varphi^{(1)}+\frac{2}{r}\psi_1\dot\varphi^{(1)}=\omega_1^{(1)},\quad \psi_1^{(1)}|_{S_2}=0,$$

where the dot denotes the derivative with respect to r. Differentiating (3.19) with respect to r yields

$$(3.20)\qquad \begin{aligned}&-\Delta\psi_{1,r}^{(1)}-\frac{2}{r}\psi_{1,rr}^{(1)}+\frac{3}{r^2}\psi_{1,r}^{(1)}=-\left[2\psi_{1,r}\dot\varphi^{(1)}+\psi_1\ddot\varphi^{(1)}+\frac{2}{r}\psi_1\dot\varphi^{(1)}\right]_{,r}+\omega_{1,r}^{(1)},\\ &\psi_{1,r}^{(1)}|_{S_2}=0.\end{aligned}$$

Multiply (3.20) by $\psi_{1,r}^{(1)}$ and integrate over Ω. Then we get

$$(3.21)\qquad \begin{aligned}&\int_\Omega|\nabla\psi_{1,r}^{(1)}|^2dx-2\int_\Omega\psi_{1,rr}^{(1)}\psi_{1,r}^{(1)}drdz+3\int_\Omega\frac{1}{r^2}|\psi_{1,r}^{(1)}|^2dx\\ &=\int_\Omega\left[2\psi_{1,r}\dot\varphi^{(1)}+\psi_1\ddot\varphi^{(1)}+\frac{2}{r}\psi_1\dot\varphi^{(1)}\right](\psi_{1,r}^{(1)}r)_{,r}drdz\\ &\quad+\int_\Omega\omega_{1,r}^{(1)}\psi_{1,r}^{(1)}dx.\end{aligned}$$

The first integral on the r.h.s. of (3.21) is bounded by

$$\varepsilon(\|\psi_{1,rr}^{(1)}\|_{L_2(\Omega)}^2+\|\psi_{1,r}^{(1)}\|_{L_2(\Omega)}^2)+c/\varepsilon\|\psi_1\|_{H^1(\Omega)}^2.$$

The second integral on the r.h.s. of (3.21) can be expressed in the form

$$(3.22)\qquad \int_\Omega(\omega_1^{(1)}\psi_{1,r}^{(1)}r)_{,r}drdz-\int_\Omega\omega_1^{(1)}\psi_{1,rr}^{(1)}dx-\int_\Omega\omega_1^{(1)}\psi_{1,r}^{(1)}drdz.$$

Since $\int_\Omega(\omega_1^{(1)}\psi_{1,r}^{(1)}r)_{,z}drdz=0$ the first integral in (3.22) equals

$$\int_\Omega[(\omega_1^{(1)}\psi_{1,r}^{(1)}r)_{,r}+(\omega_1^{(1)}\psi_{1,r}^{(1)}r)_{,z}]drdz=\int_\Omega\operatorname{div}(\overline{\omega_1^{(1)}\psi_{1,r}^{(1)}})dx=0,$$

where $\overline{\omega_1^{(1)}\psi_{1,r}^{(1)}}=(\omega_1^{(1)}\psi_{1,r}^{(1)},\omega_1^{(1)}\psi_{1,r}^{(1)})$.
The last two integrals in (3.22) are bounded by

$$\varepsilon\left(\|\psi_{1,rr}^{(1)}\|_{L_2(\Omega)}^2+\left\|\frac{1}{r^2}\psi_{1,r}^{(1)}\right\|_{L_2(\Omega)}^2\right)+c/\varepsilon\|\omega_1^{(1)}\|_{L_2(\Omega)}^2.$$

Employing the above estimates in (3.21) and using that ε is sufficiently small we derive the inequality

$$(3.23)\qquad \begin{aligned}&\frac{1}{2}\|\nabla\psi_{1,r}^{(1)}\|_{L_2(\Omega)}^2+\left\|\frac{1}{r}\psi_{1,r}^{(1)}\right\|_{L_2(\Omega)}^2-2\int_\Omega\psi_{1,rr}^{(1)}\psi_{1,r}^{(1)}drdz\\ &\le c(\|\omega_1^{(1)}\|_{L_2(\Omega)}^2+\|\psi_1\|_{H^1(\Omega)}^2).\end{aligned}$$

Finally, the last term on the l.h.s. of (3.23) equals

$$\int_{-a}^{a} |\psi_{1,r}^{(1)}|^2|_{r=0} dz.$$

Employing this and (3.18) in (3.23) yields

$$\|\nabla\psi_{1,r}^{(1)}\|_{L_2(\Omega)}^2 + \left\|\frac{1}{r}\psi_{,r}^{(1)}\right\|_{L_2(\Omega)}^2 + \int_{-a}^{a} |\psi_{1,r}^{(1)}|^2|_{r=0} dz \le c\|\omega_1\|_{L_2(\Omega)}^2. \tag{3.24}$$

Expressing (3.19) in the form

$$-\psi_{1,zz}^{(1)} = \psi_{1,rr}^{(1)} + \frac{3}{r}\psi_{1,r}^{(1)} - \left(2\psi_{1,r}\dot{\varphi}^{(1)} + \psi_1\ddot{\varphi}^{(1)} + \frac{2}{r}\psi_1\dot{\varphi}^{(1)}\right) + \omega_1^{(1)}$$

we have

$$\|\psi_{1,zz}^{(1)}\|_{L_2(\Omega)}^2 \le c\left(\|\psi_{1,rr}^{(1)}\|_{L_2(\Omega)}^2 + \left\|\frac{1}{r}\psi_{1,r}^{(1)}\right\|_{L_2(\Omega)}^2 + \|\psi_1\|_{H^1(\Omega)}^2 + \|\omega_1^{(1)}\|_{L_2(\Omega)}^2\right).$$

In view of (3.18) and (3.24) it follows that

$$\|\dot{\psi}_{1,zz}^{(1)}\|_{L_2(\Omega)}^2 \le c\|\omega_1\|_{L_2(\Omega)}^2 \tag{3.25}$$

Estimates (3.18), (3.24) and (3.25) imply

$$\|\psi_1^{(1)}\|_{H^2(\Omega)}^2 + \left\|\frac{1}{r}\psi_{1,r}^{(1)}\right\|_{L_2(\Omega)}^2 \le c\|\omega_1\|_{L_2(\Omega)}^2. \tag{3.26}$$

Next we examine solutions to (3.14) in a neighborhood located in a positive distance from the axis of symmetry. For this purpose we multiply (3.14) by $\varphi^{(2)}$. Then we get

$$\begin{aligned} &-\Delta\psi_1^{(2)} - \frac{2}{r}\psi_{1,r}^{(2)} + 2\psi_{1,r}\dot{\varphi}^{(2)} + \psi_1\ddot{\varphi}^{(2)} + \frac{2}{r}\psi_1\dot{\varphi}^{(2)} = \omega_1^{(2)}, \\ &\psi_1^{(2)}|_S = 0, \quad \psi_1^{(2)}|_{r=r_0} = 0. \end{aligned} \tag{3.27}$$

From (3.18) and (3.27) we have

$$\|\Delta\psi_1^{(2)}\|_{L_2(\Omega)} \le c(\|\psi_1\|_{H^1(\Omega)} + \|\omega_1^{(2)}\|_{L_2(\Omega)}) \le c\|\omega_1\|_{L_2(\Omega)}. \tag{3.28}$$

From (3.28) and boundary conditions in (3.27) we derive

$$\|\psi_1^{(2)}\|_{H^2(\Omega)} \le c\|\omega_1\|_{L_2(\Omega)}. \tag{3.29}$$

To prove (3.29) we need local considerations. Expecially, to perform the estimate near the angle between S_1 and S_2 we have to use reflections with respect to S_1 and S_2, respectively. Hence, (3.26) and (3.29) imply (3.16). This concludes the proof. □

Since we are not able to derive an estimate for the third derivatives with respect to r to solutions to problem (3.14) we obtain such estimate for solutions to problem (3.12). Hence, we have

Lemma 3.3. *Assume that $\omega \in H^1(\Omega)$. Assume that solutions to problem (3.12) vanish sufficiently fast near the axis of symmetry.*
Then the following a priori estimate holds

$$
\begin{aligned}
&\|\psi\|_{H^3(\Omega)} + \left\|\frac{1}{r}\psi_{,rr}\right\|_{L_2(\Omega)} + \left\|\frac{1}{r}\psi_{,zz}\right\|_{L_2(\Omega)} + \left\|\frac{1}{r}\psi_{1,r}\right\|_{L_2(\Omega)} \\
&+ \left(\int_{-a}^{a} \frac{1}{r^2}\psi_{,r}^2|_{r=0}dz\right)^{1/2} \le c\|\omega\|_{H^1(\Omega)}.
\end{aligned} \tag{3.30}
$$

Proof. First we examine solutions to problem (3.12) in a neighborhood of the axis of symmetry. Multiplying (3.12) by $\varphi^{(1)}$ we get

$$
\begin{aligned}
-\Delta\psi^{(1)} + \frac{\psi^{(1)}}{r^2} &= -(2\psi_{,r}\dot{\varphi}^{(1)} + \psi\ddot{\varphi}^{(1)}) + \omega^{(1)}, \\
\psi^{(1)}|_{S_2} &= 0.
\end{aligned} \tag{3.31}
$$

It is convenient to express (3.31) in the form

$$
\begin{aligned}
&-\left(\psi_{,rr}^{(1)} + \psi_{,zz}^{(1)} + \frac{1}{r}\psi_{,r}^{(1)}\right) + \frac{\psi^{(1)}}{r^2} = -(2\psi_{,r}\dot{\varphi}^{(1)} + \psi\ddot{\varphi}^{(1)}) + \omega^{(1)} \\
&\psi^{(1)}|_S = 0.
\end{aligned} \tag{3.32}
$$

Differentiating (3.32) twice with respect to r, multiplying the result by $\psi_{,rr}^{(1)}$ and integrating over Ω, gives

$$
\begin{aligned}
&-\int_\Omega \left(\psi_{,rrrr}^{(1)} + \psi_{,rrzz}^{(1)} + \frac{1}{r}\psi_{,rrr}^{(1)} - \frac{2}{r^2}\psi_{,rr}^{(1)} + \frac{2}{r^3}\psi_{,r}^{(1)}\right)\psi_{,rr}^{(1)}dx \\
&+\int_\Omega \left(\frac{\psi^{(1)}}{r^2}\right)_{,rr}\psi_{,rr}^{(1)}dx = -\int_\Omega (2\psi_{,r}\dot{\varphi}^{(1)} + \psi\ddot{\varphi}^{(1)})_{,rr}\psi_{,rr}^{(1)}dx \\
&+\int_\Omega \omega_{,rr}^{(1)}\psi_{,rr}^{(1)}dx.
\end{aligned} \tag{3.33}
$$

Using that $\psi_{,rr}^{(1)}|_{S_2} = 0$, we obtain from (3.33) the inequality

$$
\begin{aligned}
&\int_\Omega |\nabla\psi_{,rr}^{(1)}|^2dx + 2\int_\Omega \frac{1}{r^2}|\psi_{,rr}^{(1)}|^2dx - 2\int_\Omega \frac{1}{r^3}\psi_{,r}^{(1)}\psi_{,rr}^{(1)}dx \\
&+\int_\Omega \left(\frac{\psi^{(1)}}{r^2}\right)_{,rr}\psi_{,rr}^{(1)}dx \le \varepsilon\|\psi_{,rrr}^{(1)}\|_{L_2(\Omega)}^2 + c/\varepsilon\|\psi_1\|_{H^2(\Omega)}^2 \\
&+\int_\Omega (\omega_{,r}^{(1)}\psi_{,rr}^{(1)}r)_{,r}drdz - \int_\Omega \omega_{,r}^{(1)}\psi_{,rrr}^{(1)}dx - \int_\Omega \omega_{,r}^{(1)}\psi_{,rr}^{(1)}drdz.
\end{aligned} \tag{3.34}
$$

Exploiting that $\int_\Omega (\omega^{(1)}_{,r}\psi^{(1)}_{,rr}r)_{,z}drdz = 0$ the third term on the r.h.s. of (3.34) equals

$$\int_\Omega [(\omega^{(1)}_{,r}\psi^{(1)}_{,zr}r)_{,r} + (\omega^{(1)}_{,r}\psi^{(1)}_{,rr}r)_{,z}]drdz = \int_\Omega \operatorname{div}\,(\overline{\omega^{(1)}_{,r}\psi^{(1)}_{,rr}})dx = 0,$$

where $\overline{\overset{1}{\omega}_{,r}\psi^{(1)}_{,rr}} = (\overset{1}{\omega}_{,r}\psi^{(1)}_{,rr}, \overset{1}{\omega}_{,r}\psi^{(1)}_{,rr})$. The fourth term on the r.h.s. of (3.34) is estimated by

$$\varepsilon\|\psi^{(1)}_{,rrr}\|^2_{L_2(\Omega)} + c/\varepsilon\|\omega^{(1)}_{,r}\|^2_{L_2(\Omega)}.$$

Finally, the last term on the r.h.s. of (3.34) is bounded by

$$\begin{aligned}&\left|\int_\Omega \omega^{(1)}_{,r}(r\psi^{(1)}_1)_{,rr}drdz\right| = \left|\int_\Omega \omega^{(1)}_{,r}(2\psi^{(1)}_{1,r} + r\psi^{(1)}_{1,rr})drdz\right| \\ &\le \varepsilon\left(\int_\Omega \frac{|\psi^{(1)}_{1,r}|^2}{r^2}dx + \int_\Omega \psi^2_{1,rr}dx\right) + c/\varepsilon\|\omega^{(1)}_{,r}\|^2_{L_2(\Omega)}.\end{aligned}$$

Employing the above considerations in (3.34) and assuming that ε is sufficiently small we derive the inequality

$$\begin{aligned}&\int_\Omega |\nabla\psi^{(1)}_{,rr}|^2|dx + 2\int_\Omega \frac{1}{r^2}|\psi^{(1)}_{,rr}|^2dx - 2\int_\Omega \frac{1}{r^3}\psi^{(1)}_{,r}\psi^{(1)}_{,rr}dx \\ &+ \int_\Omega \left(\frac{\psi^{(1)}}{r^2}\right)_{,rr}\psi^{(1)}_{,rr}dx \le \varepsilon\int_\Omega \frac{|\psi^{(1)}_{1,r}|^2}{r^2}dx + \varepsilon\int_\Omega \psi^2_{1,rr}dx \\ &+ c/\varepsilon\|\omega^{(1)}_{,r}\|^2_{L_2(\Omega)} + c\|\psi_1\|^2_{H^2(\Omega)}.\end{aligned} \tag{3.35}$$

From (3.16), (3.35) and sufficiently small ε we have

$$\begin{aligned}&\|\nabla\psi^{(1)}_{,rr}\|^2_{L_2(\Omega)} + \|\psi_1\|^2_{H^2(\Omega)} + 2\int_\Omega \frac{1}{r^2}|\psi^{(1)}_{,rr}|^2dx + \int_\Omega \frac{1}{r^2}|\psi^{(1)}_{1,r}|^2dx \\ &- 2\int_\Omega \frac{1}{r^3}\psi^{(1)}_{,r}\psi^{(1)}_{,rr}dx + \int_\Omega \left(\frac{\psi^{(1)}}{r^2}\right)_{,rr}\psi^{(1)}_{,rr}dx \le c\|\omega^{(1)}_{,r}\|^2_{L_2(\Omega)} \\ &+ c\|\omega_1\|^2_{L_2(\Omega)}.\end{aligned} \tag{3.36}$$

The fifth term on the l.h.s. of (3.36) equals

$$\begin{aligned}&-\int_\Omega \frac{1}{r^3}(|\psi^{(1)}_{,r}|^2)_{,r}dx = -\int_\Omega \frac{1}{r^2}(|\psi^{(1)}_{,r}|^2)_{,r}drdz = -\int_\Omega \partial_r\left(\frac{1}{r^2}|\psi^{(1)}_{,r}|^2\right)drdz \\ &- 2\int_\Omega \frac{1}{r^4}|\psi^{(1)}_{,r}|^2dx = \int_{-a}^{a} \frac{1}{r^2}|\psi^{(1)}_{,r}|^2|_{r=0}dz - 2\int_\Omega \frac{1}{r^4}|\psi^{(1)}_{,r}|^2dx.\end{aligned} \tag{3.37}$$

The last integral in (3.37) is examined in the following way

$$\int_\Omega \frac{1}{r^4}|\psi^{(1)}_{,r}|^2dx = \int_\Omega \frac{1}{r^4}|(r\psi^{(1)}_1)_{,r}|^2dx \le \int_\Omega \frac{1}{r^4}(|r\psi^{(1)}_{1,r}|^2+|\psi^{(1)}_1|^2)dx$$
$$\le \int_\Omega \frac{1}{r^2}|\psi^{(1)}_{1,r}|^2dx + \int_\Omega \frac{1}{r^4}|\psi^{(1)}_1|62dx \le c\int_\Omega \frac{1}{r^2}|\psi^{(1)}_{1,r}|^2dx,$$

where the last inequality follows from the Hardy inequality. The last term on the l.h.s. of (3.36) equals

$$I_1 \equiv \int_\Omega \left(\left(\frac{\psi^{(1)}}{r^2}\right)_{,r}\psi^{(1)}_{,rr}r\right)_{,r}drdz - \int_\Omega \left(\frac{\psi^{(1)}}{r^2}\right)_{,r}\psi^{(1)}_{,rrr}dx - \int_\Omega \left(\frac{\psi^{(1)}}{r^2}\right)_{,r}\psi^{(1)}_{,rr}drdz,$$

where the first integral in I_1 vanishes because $\psi^{(1)}_{,rr}|_{S_2}=0$ and the same idea implying vanishing of the first term in (3.22) is used and the second and the third terms are bounded by

$$\varepsilon\int_\Omega |\psi^{(1)}_{,rrr}|^2dx + \varepsilon\int_\Omega \frac{1}{r^2}|\psi^{(1)}_{,rr}|^2dx + c/\varepsilon\int_\Omega \left(\frac{\psi^{(1)}_1}{r}\right)^2_{,r}dx,$$

where the last integral is bounded by

$$c\int_\Omega \frac{1}{r^2}|\psi^{(1)}_{1,r}|^2dx + c\int_\Omega \frac{1}{r^4}|\psi^{(1)}_1|^2dx \le c\int_\Omega \frac{1}{r^2}|\psi^{(1)}_{1,r}|^2dx,$$

where the Hardy inequality is used. Employing the above estimates in (3.36) and using again (3.16) we have

$$\begin{aligned}&\|\nabla\psi^{(1)}_{,rr}\|^2_{L_2(\Omega)} + \|\psi_1\|^2_{H^2(\Omega)} + \int_\Omega \frac{1}{r^2}|\psi^{(1)}_{,rr}|^2dx + \int_\Omega \frac{1}{r^2}|\psi^{(1)}_{1,r}|^2dx\\ &+\int_{-a}^{a} \frac{1}{r^2}|\psi^{(1)}_{,r}|^2|_{r=0}dz \le c\|\omega^{(1)}_{,r}\|^2_{L_2(\Omega)} + c\|\omega_1\|^2_{L_2(\Omega)}.\end{aligned} \tag{3.38}$$

Differentiating (3.12) twice with respect to z, multiplying by $\psi_{,zz}$, integrating over Ω and by parts we get

$$-\int_{S_2} \bar{n}\cdot\nabla\psi_{,zz}\cdot\psi_{,zz}dS_2 + \int_\Omega |\nabla\psi_{,zz}|^2dx + \int_\Omega \frac{1}{r^2}|\psi_{,zz}|^2dx = \int_\Omega \omega_{,zz}\psi_{,zz}dx. \tag{3.39}$$

Using that $\bar{n}$ is the unit outward vector normal to S_2, the first term on the l.h.s. of (3.39) equals

$$I_1 = -\int_0^R \psi_{,zzz}\psi_{,zz}|^{z=a}_{z=-a}rdr$$

Since

$$\psi_{,zz} = \omega = 0 \quad \text{on} \quad S_2, \tag{3.40}$$

I_1 vanishes.
Similarly, the term on the r.h.s. of (3.39) equals

$$I_2 = \int\limits_\Omega (\omega_{,z}\psi_{,zz})_{,z}dx - \int\limits_\Omega \omega_{,z}\psi_{,zzz}dx,$$

where the first integral in I_2 vanishes. Hence

$$|I_2| \le \varepsilon\|\psi_{,zz}\|^2_{H^1(\Omega)} + c/\varepsilon\|\omega\|^2_{H^1(\Omega)}.$$

The second integral in I_2 is bounded by A. Summarizing and assuming that ε is sufficiently small we obtain from (3.39) the inequality

$$\int\limits_\Omega |\nabla\psi_{,zz}|^2 dx + \int\limits_\Omega \frac{1}{r^2}|\psi_{,zz}|^2 dx \le c\|\omega\|^2_{H^1(\Omega)}. \tag{3.41}$$

Finally, we have to obtain an estimate for $\|\nabla\psi^{(2)}_{,rr}\|_{L_2(\Omega)}$. For this purpose we multiply (3.12) by $\varphi^{(2)}$. Then we get

$$\begin{aligned} &-\Delta\psi^{(2)} + \frac{\psi^{(2)}}{r^2} = -(2\psi_{,r}\dot\varphi^{(2)} + \psi\ddot\varphi^{(2)})^{\cdot} + \omega^{(2)}, \\ &\psi^{(2)}|_S = 0. \end{aligned} \tag{3.42}$$

It is more convenient to write (3.42) in the form

$$-\psi^{(2)}_{,rr} - \psi^{(2)}_{,zz} - \frac{1}{r}\psi^{(2)}_{,r} + \frac{\psi^{(2)}}{r^2} = -(2\psi_{,r}\dot\varphi^{(2)} + \psi\ddot\varphi^{(2)}) + \omega^{(2)}. \tag{3.43}$$

Differentiating (3.43) with respect to r and taking L_2 norm we get

$$\|\psi^{(2)}_{,rrr}\|^2_{L_2(\Omega)} \le \|\psi^{(2)}_{,rzz}\|^2_{L_2(\Omega)} + c\|\psi^{(2)}\|^2_{H^2(\Omega)} + c\|\psi\|^2_{H^2(\Omega)} + c\|\omega^{(2)}\|^2_{H^1(\Omega)}. \tag{3.44}$$

Since the first norm on the r.h.s. of (3.44) is estimated in view of (3.41) we obtain from (3.38), (3.41), (3.44) and the estimate

$$\|\psi\|_{H^2(\Omega)} \le \|r\psi_1\|_{H^2(\Omega)} \le c\|\omega_1\|_{L_2(\Omega)}$$

which holds in view of (3.16), the estimate

$$\begin{aligned} &\|\psi\|_{H^3(\Omega)} + \left\|\frac{1}{r}\psi_{,zz}\right\|_{L_2(\Omega)} + \left\|\frac{1}{r}\psi_{,rr}\right\|_{L_2(\Omega)} + \left\|\frac{1}{r}\psi_{1,r}\right\|_{L_2(\Omega)} \\ &+ \left(\int\limits_{-a}^{a} \frac{1}{r^2}\psi^2_{,r}|_{r=0}dz\right)^{1/2} \le c(\|\omega\|_{H^1(\Omega)} + \|\omega_1\|_{L_2(\Omega)}). \end{aligned}$$

This concludes the proof of Lemma 3.3. □

From Lemmas 3.1–3.3 and [3] we have

Lemma 3.4. *Assume that* $\overset{1}{\omega}_1(0) \in L_2(\Omega)$, $\overset{1}{F}_{\varphi 1} \in L_2(kT,(k+1)T; L_{6/5}(\Omega))$, $k \in \mathbb{N}_0$. *Then there exists a solution to problem (1.13)–(1.17) such that*

$$\begin{aligned} &\overset{1}{\omega}_1, \overset{1}{\omega} \in L_\infty(\mathbb{R}_+; L_2(\Omega)) \cap L_2(kT,(k+1)T; H^1(\Omega)),\\ &\overset{1}{\psi} \in L_\infty(\mathbb{R}_+; H^2(\Omega)) \cap L_2(kT,(k+1)T; H^3(\Omega)). \end{aligned} \tag{3.45}$$

Proof. The estimates follow from Lemmas 3.1–3.3. Existence of solutions to (1.13)–(1.17) such that $\overset{1}{\omega} \in L_\infty(\mathbb{R}_+; L_2(\Omega))$ and $\overset{1}{\psi} \in L_\infty(\mathbb{R}_+; H^2(\Omega))$ is proved by Ladyzhenskaya in [3]. Further regularity follows from the classical technique of increasing regularity. Moreover, since the estimates in Lemmas 3.1–3.3 are shown by the energy method the existence of solutions (3.45) can be proved by the Faedo-Galerkin method. This concludes the proof. □

4 Stability

From Lemmas 3.1–3.3 we have that

$$\begin{aligned} &\|\overset{1}{\psi}_1\|^2_{L_\infty(kT,t;H^2(\Omega))} + \int_{kT}^{t} \bigg(\|\overset{1}{\psi}(t')\|^2_{H^3(\Omega)} + \left\| \frac{1}{r}\psi_{,rr} \right\|^2_{L_2(\Omega)} \\ &\quad + \left\| \frac{1}{r}\psi_{,zz} \right\|^2_{L_2(\Omega)} + \left\| \frac{1}{r}\psi_{1,r} \right\|^2_{L_2(\Omega)} \bigg) dt' \le c(A_1^2 + A_2^2), \end{aligned} \tag{4.1}$$

where $t \in (kT,(k+1)T]$, $k \in \mathbb{N}_0$ and A_1, A_2 are introduced in (3.1). Then (1.15) implies

$$\overset{1}{v}_r, \overset{1}{v}_z \in L_\infty(kT,t; H^1(\Omega)) \cap L_2(kT,t; H^2(\Omega)), \tag{4.2}$$

where $t \in (kT,(k+1)T]$, $k \in \mathbb{N}_0$. To show stability we need

Lemma 4.1. *Assume that*

$$
(4.3)\qquad
\begin{aligned}
&1.\quad \overset{1}{\omega} \in L_2(kT,(k+1)T;H^1(\Omega)),\quad k\in\mathbb{N}_0,\\
&\quad\ F_{\varphi_1}\in C(\mathbb{R}_+;L_{6/5}(\Omega)),\quad f_{\varphi_1}\in C(\mathbb{R}_+;L_{4/3}(\Omega))\\
&2.\quad \textit{let } \gamma\in(0,\gamma_*],\ \textit{where } \gamma_* \textit{ is so small that}\\
&\quad\ \nu_1-\frac{c_0}{\nu_1}\gamma_*\ge\frac{c_*}{2},\ \textit{where } c_*\in(0,\nu_1] \textit{ and } \nu_1=\frac{\nu c_1}{2}, c_1=\min\{1,c_p\},\\
&\quad\ c_p \textit{ is the Poincaré constant,}\\
&\quad\ \textit{and } c_0 \textit{ is introduced in (4.11).}\\
&3.\quad \|\omega_1(0)\|^2_{L_2(\Omega)}+\|u_1(0)\|^4_{L_4(\Omega)}\le\gamma\\
&4.\quad \textit{let } c_0(\|F_{\varphi_1}(t)\|^2_{L_{6/5}(\Omega)}+\|f_{\varphi_1}(t)\|^4_{L_{4/3}(\Omega)})\le\frac{c_*}{4}\gamma \quad \textit{for } t\in\mathbb{R}_+,\\
&\quad\ c_0\int_{kT}^{(k+1)T}(\|F_{\varphi_1}(t)\|^2_{L_{6/5}(\Omega)}+\|f_{\varphi_1}(t)\|^4_{L_{4/3}(\Omega)})dt\le\alpha\gamma,\quad k\in\mathbb{N}_0.\\
&5.\quad \textit{let } \bar{A}_1^2(k)=\frac{c_0}{\nu_1^2}\int_{kT}^{(k+1)T}\|\overset{1}{F}_{\varphi 1}(t)\|^2_{L_{6/5}(\Omega)}dt+\frac{1}{\nu_1}(\|\overset{1}{\omega}_1(kT)\|^2_{L_2(\Omega)}\\
&\quad\ -\|\overset{1}{\omega}_1((k+1)T)\|^2_{L_2(\Omega)}).\\
&6.\quad \bar{A}_1^2\equiv\sup_{k\in\mathbb{N}_0}\bar{A}_1^2(k)\le\frac{c_*}{4}T,\ \alpha\exp(\bar{A}_1^2)+\exp\Big(-\frac{c_*}{4}T\Big)\le 1.
\end{aligned}
$$

Then

$$
(4.4)\qquad \|\omega_1(t)\|^2_{L_2(\Omega)}+\|u_1(t)\|^4_{L_4(\Omega)}\le\gamma \ \textit{for } t\in\mathbb{R}_+.
$$

Proof. Multiplying $(1.30)_2$ by ω_1, integrating over Ω and using boundary conditions $(1.30)_6$ yield

$$
(4.5)\qquad
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\omega_1\|^2_{L_2(\Omega)}+\int_\Omega v\cdot\nabla\overset{1}{\omega}_1\omega_1dx+\nu\int_\Omega|\nabla\omega_1|^2dx+\nu\int_{-a}^{a}\omega_1^2|_{r=0}dx\\
&=\int_\Omega\partial_z u_1^2\omega_1dx+\int_\Omega F_{\varphi_1}\omega_1dx.
\end{aligned}
$$

The second term on the l.h.s. of (4.5) is bounded by

$$
\varepsilon\|\omega_1\|^2_{L_6(\Omega)}+c/\varepsilon\|v\|^2_{L_3(\Omega)}\|\nabla\overset{1}{\omega}_1\|^2_{L_2(\Omega)}.
$$

The first term on the r.h.s. of (4.5) is treated as folllows

$$
\Big|-\int_\Omega u_1^2\omega_{1,z}dx\Big|\le\varepsilon\|\omega_{1,z}\|^2_{L_2(\Omega)}+c/\varepsilon\|u_1\|^4_{L_4(\Omega)}.
$$

Finally, the last term on the r.h.s. of (4.5) is estimated by

$$
\varepsilon\|\omega_1\|^2_{L_6(\Omega)}+c/\varepsilon\|F_{\varphi_1}\|^2_{L_{6/5}(\Omega)}.
$$

Employing the above estimates in (4.5) and assuming that ε is sufficiently small we get

$$\frac{d}{dt}\|\omega_1\|^2_{L_2(\Omega)} + \nu\|\nabla\omega_1\|^2_{L_2(\Omega)} \le c\|u_1\|^4_{L_4(\Omega)} + c\|\nabla\overset{1}{\omega}\|^2_{L_2(\Omega)}\|v\|^2_{L_3(\Omega)} + c\|F_{\varphi_1}\|^2_{L_{6/5}(\Omega)}. \tag{4.6}$$

Multiplying $(1.30)_1$ by $u_1|u_1|^2$ and integrating over Ω gives

$$\frac{1}{4}\frac{d}{dt}\|u_1\|^4_{L_4(\Omega)} + \nu\int\limits_\Omega |\nabla u_1^2|^2dx + \frac{\nu}{2}\int\limits_{-a}^{a} u_1^4|_{r=0}dz = 2\int\limits_\Omega u_1^4\psi_{1,z}dx + \int\limits_\Omega u_1^4\overset{1}{\psi}_{1,z}dx + \int\limits_\Omega f_{\varphi_1}u_1|u_1|^2dx. \tag{4.7}$$

The first term on the r.h.s. of (4.7) equals

$$-4\int\limits_\Omega u_1^2\partial_z u_1^2\psi_1 dx$$

so it is bounded by

$$\varepsilon\int\limits_\Omega |\partial_z u_1^2|^2dx + c/\varepsilon\sup_\Omega|\psi_1|^2\int\limits_\Omega u_1^4dx.$$

The last term on the r.h.s. of (4.7) is bounded by

$$\|f_{\varphi_1}\|_{L_{4/3}(\Omega)}\|u_1\|^3_{L_{12}(\Omega)} \le \varepsilon\|u_1\|^4_{L_{12}(\Omega)} + c/\varepsilon\|f_{\varphi_1}\|^4_{L_{4/3}(\Omega)}.$$

The Poincaré inequality implies

$$\|u_1^2\|^2_{H^1(\Omega)} \le c\|\nabla u_1^2\|^2_{L_2(\Omega)}. \tag{4.8}$$

Then the Sobolev imbeddings yield

$$\|u_1\|^4_{L_{12}(\Omega)} \le c\|u_1^2\|^2_{H^1(\Omega)}. \tag{4.9}$$

In view of the above estimates, (4.8), (4.9) and ε sufficiently small we derive the inequality

$$\frac{d}{dt}\|u_1\|^4_{L_4(\Omega)} + \nu\|\nabla u_1^2\|^2_{L_2(\Omega)} \le c\|\omega_1\|^2_{L_2(\Omega)}\|u_1\|^4_{L_4(\Omega)} + c\|\overset{1}{\omega}_1\|^2_{L_2(\Omega)}\|u_1\|^4_{L_4(\Omega)} + c\|f_{\varphi_1}\|^4_{L_{4/3}(\Omega)}. \tag{4.10}$$

Adding appropriately (4.6) and (4.10) gives

$$\begin{aligned}&\frac{d}{dt}(\|\omega_1\|^2_{L_2(\Omega)} + \|u_1\|^4_{L_4(\Omega)}) + \nu(\|\nabla\omega_1\|^2_{L_2(\Omega)} + \|\nabla u_1^2\|^2_{L_2(\Omega)})\\ &\le c_0\|\overset{1}{\omega}\|^2_{H^1(\Omega)}\|\omega_1\|^2_{L_2(\Omega)} + c_0\|\omega_1\|^2_{L_2(\Omega)}\|u_1\|^4_{L_4(\Omega)} + c_0\|\overset{1}{\omega}_1\|^2_{L_2(\Omega)}\|u_1\|^4_{L_4(\Omega)}\\ &\quad + c_0(\|f_{\varphi_1}\|^4_{L_{4/3}(\Omega)} + \|F_{\varphi_1}\|^2_{L_{6/5}(\Omega)}),\end{aligned} \tag{4.11}$$

where we used that

$$\|v\|_{L_3(\Omega)} \le c\|\omega_1\|_{L_2(\Omega)}. \tag{4.12}$$

Let us introduce the notation

$$\begin{aligned} X^2(t) &= \|\omega_1(t)\|^2_{L_2(\Omega)} + \|u_1(t)\|^4_{L_4(\Omega)}, \\ A^2(t) &= c_0\|\overset{1}{\omega}(t)\|^2_{H^1(\Omega)}, \\ G^2(t) &= c_0(\|F_{\varphi 1}(t)\|^2_{L_{6/5}(\Omega)} + \|f_{\varphi 1}(t)\|^4_{L_{4/3}(\Omega)}). \end{aligned} \tag{4.13}$$

Then (4.11) takes the form

$$\frac{d}{dt}X^2 \le -X^2\Big(\nu - \frac{c_0}{\nu}X^2\Big) + A^2X^2 + G^2. \tag{4.14}$$

Let $\gamma \in (0, \gamma_*]$, where γ_* is so small that

$$\nu - \frac{c_0}{\nu}\gamma_* \ge \frac{c_*}{2}, \quad 0 < c_* \le \nu. \tag{4.15}$$

Assume that

$$X^2(kT) \le \gamma, \quad G^2(t) \le c_*\frac{\gamma}{4}, \quad t \in [kT, (k+1)T]. \tag{4.16}$$

Let us introduce the quantity

$$Z^2(t) = \exp\Big(-\int_{kT}^{t} A^2(t')dt'\Big)X^2(t), \quad t \in [kT, (k+1)T].$$

Then (4.14) takes the form

$$\frac{d}{dt}Z^2 \le -\Big(\nu - \frac{c_0}{\nu}X^2\Big)Z^2 + \bar{G}^2. \tag{4.17}$$

where $\bar{G}(t) = G^2(t)\exp\big(-\int_{kT}^t A^2(t')dt'\big)$. Suppose, that

$$\begin{aligned} t_* &= \inf\{t \in (kT, (k+1)T] :\ X^2(t) > \gamma\} \\ &= \inf\Big\{t \in (kT, (k+1)T] :\ Z^2(t) > \gamma\exp\Big(-\int_{kT}^{t} A^2(t')dt'\Big)\Big\} > kT. \end{aligned}$$

By (4.15) for $t \in (kT, t_*]$ inequality (4.17) takes the form

$$\frac{d}{dt}Z^2 \le -\frac{c_*}{2}Z^2 + \bar{G}^2(t). \tag{4.18}$$

Clearly, we have

$$\begin{aligned} &Z^2(t_*) = \gamma\exp\Big(-\int_{kT}^{t_*} A^2(t)dt\Big) \quad \text{and} \\ &Z^2(t) > \gamma\exp\Big(-\int_{kT}^{t} A^2(t')dt'\Big) \quad \text{for } t > t_*. \end{aligned} \tag{4.19}$$

But (4.16) and (4.18) yield

$$\frac{d}{dt}Z^2|_{t=t_*} \le c_*\Big(-\frac{\gamma}{2}+\frac{\gamma}{4}\Big)\exp\Big(-\int_{kT}^{t} A^2(t')dt'\Big) < 0$$

contrary to (4.19). Hence $Z^2(t) \le \gamma \exp\big(-\int_{kT}^{t_*} A^2(t)dt\big)$ for $t > t_*$. Definition of $Z^2(t)$ implies

$$X^2(t) \le \gamma \exp\Big(\int_{t_*}^{t} A^2 dt'\Big) \quad \text{for } t > t_*.$$

In view of (3.11) we have

$$\int_{kT}^{(k+1)T} A^2 dt \le \frac{c_s}{\nu_1^2}\int_{kT}^{(k+1)T} \|\overset{1}{F}_{\varphi 1}(t)\|^2_{L_{6/5}(\Omega)}dt + \frac{1}{\nu_1}(\|\overset{1}{\omega}_1(kT)\|^2_{L_2(\Omega)} - \|\overset{1}{\omega}_1((k+1)T)\|^2_{L_2(\Omega)}) \equiv \bar{A}_1^2(k).$$

For sufficiently small γ inequality (4.14) takes the form

$$\frac{d}{dt}X^2 + \frac{c_*}{2}X^2 \le A^2X^2 + G^2. \tag{4.20}$$

Integrating (4.20) with respect to time from $t = kT$ to $t = (k+1)T$ gives

$$\begin{aligned} X^2((k+1)T) &\le \exp\Big(\int_{kT}^{(k+1)T} A^2(t)dt\Big)\int_{kT}^{(k+1)T} G^2(t)dt \\ &+ \exp\Big(-\frac{c_*}{2}T + \int_{kT}^{(k+1)T} A^2(t)dt\Big)X^2(kT). \end{aligned} \tag{4.21}$$

In view of the assumptions

$$\frac{c_*}{4}T \ge \int_{kT}^{(k+1)T} A^2(t)dt, \quad \int_{kT}^{(k+1)T} G^2(t)dt \le \alpha\gamma, \tag{4.22}$$

where α is so small and T so large that

$$\alpha \exp\Big(\int_{kT}^{(k+1)T} A^2 dt\Big) + \exp\Big(-\frac{c_*}{4}T\Big) \le 1, \tag{4.23}$$

$X^2((k+1)T) \le \gamma$. Then induction proves the lemma. □

Next, we have

Lemma 4.2. *Let the assumptions of Lemma 4.1 be satisfied. Let $k \in \mathbb{N}_0$. Assume that $f_{\varphi 1} \in L_4(kT,(k+1)T;L_{4/3}(\Omega))$, $F_{\varphi 1} \in L_2(kT,(k+1)T;L_{6/5}(\Omega))$. Then there exists a solution to problem (1.30) such that*

$$\begin{aligned}&\|\omega_1(t)\|^2_{L_2(\Omega)} + \|u_1(t)\|^4_{L_4(\Omega)} + \nu\int_{kT}^{t}(\|\omega_1(t')\|^2_{H^1(\Omega)} + \|u_1^2\|^2_{H^1(\Omega)})dt' \\ &\le c(\gamma, A_1, A_2) + \int_{kT}^{t}(\|f_{\varphi 1}(t')\|^4_{L_{4/3}(\Omega)} + \|F_{\varphi 1}(t')\|^2_{L_{6/5}(\Omega)})dt'\end{aligned} \tag{4.24}$$

where $t \in (kT,(k+1)T]$.

Proof. Estimate (4.24) follows from integration (4.11) with respect to time from $t = kT$ to $t \in (kT,(k+1)T]$ and application of estimates (3.2) and (4.4). The existence follows from the Faedo-Galerkin method used in each time step $[kT,(k+1)T]$, $k \in \mathbb{N}_0$, separately. This concludes the proof. □

Acknowledgements

The research leading to these results has received funding from the People Programme (Marie Curie Actions) of the European Union's Seventh Framework Programme FP7/2007-2013/ under REA grant agreement n° 319012 and from the Funds for International Co-operation under Polish Ministry of Science and Higher Education grant agreement n° 2853/7.PR/2013/2.

The author thanks to professor Yoshihiro Shibata for very important comments concerning the proof of Lemma 4.1.

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