数理解析研究所講究録
第2009巻 2016年 105-123

KPZ equation with fractional derivatives of white noise

Masato Hoshino
Graduate School of Mathematical Sciences
The University of Tokyo

Abstract

In this paper, we consider the KPZ equation driven by space-time white noise replaced with its fractional derivatives of order $\gamma>0$ in spatial variable. A well-posedness theory for the KPZ equation is established by Hairer [3] as an application of the theory of regularity structures. Our aim is to see to what extent his theory works if noises become rougher. We can expect that his theory works if and only if $\gamma<1/2$. However, we show that the renormalization like "$(\partial_{x}h)^{2}-\infty$" is well-posed only if $\gamma<1/4$.

1 Introduction

In this paper, we discuss the stochastic partial differential equation

$$\partial_{t}h(t, x)=\partial_{x}^{2}h(t, x)+(\partial_{x}h(t, x))^{2}+\partial_{x}^{\gamma}\xi(t, x) \tag{1.1}$$

for $(t, x)\in[0, \infty)\times \mathbb{T}$ with $\mathbb{T}=\mathbb{R}/\mathbb{Z}$, which is equivalent to $[0,1]$ with periodic boundary conditions, and $\gamma\geq 0$. Here $h(t, x)$ is a continuous stochastic process and ξ is a space-time white noise. $\partial_{x}^{\gamma}=-(-\partial_{x}^{2})^{\frac{\gamma}{2}}$ is the fractional derivative. If $\gamma=0$, (1.1) is the KPZ equation, which is proposed in [8] as a model of surface growth.

The equation (1.1) is ill-posed. Formally speaking, h has the same regularity as the solution of the linear equation

$$\partial_{t}h(t, x)=\partial_{x}^{2}h(t, x)+\partial_{x}^{\gamma}\xi(t, x). \tag{1.2}$$

Then $h(t, \cdot)$ belongs to Hölder space $C^{\alpha}(\mathbb{T})$ with $\alpha<\frac{1}{2}-\gamma$ for each fixed t. However this implies that the nonlinear term $(\partial_{x}h)^{2}$ is the square of the distribution, which generally does not make sense.

Hairer discussed the solution of the KPZ equation in [2] and [4]. It is natural to replace ξ by a smooth approximation ξ_{ϵ}, which is obtained by a convolution with a smooth mollifier, and consider the classical solution of the KPZ equation with ξ_{ϵ}. He showed that there exists a sequence of constants $C_{\epsilon}\sim\frac{1}{\epsilon}$ such that, the sequence of solutions h_{ϵ} of

$$\partial_{t}h_{\epsilon}(t, x)=\partial_{x}^{2}h_{\epsilon}(t, x)+(\partial_{x}h_{\epsilon}(t, x))^{2}-C_{\epsilon}+\xi_{\epsilon}(t, x) \tag{1.3}$$

has a unique limit h in probability, which is independent of the choice of a mollifier.

Our goal is to make the noise rougher and see to what extent this theory works. Because of the "local subcriticality" (Assumption 8.3 of [3]), we can expect that similar results hold if $\gamma<\frac{1}{2}$. If we write $h^{\delta}(t, x)=\delta^{-\frac{1}{2}+\gamma}h(\delta^{2}t, \delta x)$ and $\xi^{\delta}(t, x)=\delta^{\frac{3}{2}}\xi(\delta^{2}t, \delta x)$ for $\delta>0$, then ξ^{δ} is equal to ξ in distribution and h^{δ} satisfies

$$\partial_{t}h^{\delta}(t, x)=\partial_{x}^{2}h^{\delta}(t, x)+\delta^{\frac{1}{2}-\gamma}(\partial_{x}h^{\delta}(t, x))^{2}+\partial_{x}^{\gamma}\xi^{\delta}(t, x).$$

As $\delta\rightarrow 0$, we can see that the nonlinear term vanishes. Formally speaking, this means that h behaves like the solution of (1.2) at small scales. His theory implies that it is possible to devise a suitable renormalization in this case. However, we prove that the renormalization like (1.3) is possible only if $\gamma<\frac{1}{4}$ in this paper. In the case $\gamma\geq\frac{1}{4}$, see Subsection 4.4.

Theorem 1.1. *Let ρ be a function on $\mathbb{R}^{2}$ which is smooth, compactly supported, symmetric in x, non-negative, and satisfies $\int_{\mathbb{R}^{2}}\rho(t, x)dtdx=1$. Set $\rho_{\epsilon}(t, x)=\epsilon^{-3}\rho(\epsilon^{-2}t, \epsilon^{-1}x)$ for $\epsilon>0$, and $\xi_{\epsilon}=\xi*\rho_{\epsilon}$*

(space-time convolution). Let $0 \le \gamma < \frac{1}{4}$ and $0 < \eta < \frac{1}{2} - \gamma$. Then there exists a sequence of constants $C_\epsilon \sim_{\gamma,\rho} \epsilon^{-1-2\gamma}$ such that, for every initial condition $h_0 \in \mathcal{C}^\alpha(\mathbb{T})$ the sequence of solutions h_ϵ of

$$\partial_t h_\epsilon(t,x) = \partial_x^2 h_\epsilon(t,x) + (\partial_x h_\epsilon(t,x))^2 - C_\epsilon + \partial_x^\gamma \xi_\epsilon(t,x)$$

converges to a unique stochastic process h, which is independent of the choice of ρ. Precisely, $h_\epsilon(t,\cdot)$ exists until the survival time $T_\epsilon \in (0,\infty]$ which satisfies $\liminf_{\epsilon\downarrow 0} T_\epsilon > 0$, and h_ϵ convergences to h in probability in the uniform norm on $[0,T]\times\mathbb{T}$ and η-Hölder norm on all compact sets in $(0,T]\times\mathbb{T}$ for every $T < \liminf_\epsilon T_\epsilon$.

This theorem is obtained from Theorem 3.7, Proposition 4.4 and Theorem 4.5 for $0 \le \gamma < \frac{1}{6}$, and from Theorem 3.7, Proposition 4.7 and Theorem 4.8 for $\frac{1}{6} \le \gamma < \frac{1}{4}$. The estimate of C_ϵ is in Propositions 5.2 and 5.5.

We note that above result is related to [6], where the equation

$$\partial_t u(t,x) = \partial_x^2 u(t,x) + u\partial_t\partial_x W(t,x)$$

is studied. Here W is a standard Brownian motion in t and a fractional Brownian motion with Hurst parameter $H \in (\frac{1}{4},\frac{1}{2})$ in x. The relation between both equations is $H = \frac{1}{2} - \gamma$, so that the same boundary $\gamma = H = \frac{1}{4}$ appears. Although both equations are same after performing the Cole-Hopf transformation $u = e^h$, their results do not imply Theorem 1.1 since Itô calculus is only stable under spatial regularizations.

The organization of this paper is as follows. In Section 2, we introduce some notations and fractional calculus. In Section 3, we briefly recall the theory of regularity structures and prepare some tools for the proof of Theorem 1.1. We discuss the renormalization of models in Section 4. Detailed estimates are shown in Section 5. See [5] for complete proof.

2 Notations and fractional calculus

We introduce some notations and definitions.

2.1 Notations

We denote by $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ the 1-dimensional torus. For a function f on $\mathbb{R}$, we write $f \in L^2(\mathbb{T})$ if and only if $f(\cdot + n) = f$ for every $n \in \mathbb{Z}$ and $\|f\|_{L^2(\mathbb{T})} := (\int_0^1 |f(x)|^2 dx)^{\frac{1}{2}} < \infty$. For $f, g \in L^2(\mathbb{T})$, we define

$$(f,g)_{L^2(\mathbb{T})} = \int_0^1 f(x)\overline{g(x)}dx.$$

For a function f on $\mathbb{R}^2$, we write $f \in L^2(\mathbb{R}\times\mathbb{T})$ if and only if $f(t,\cdot) \in L^2(\mathbb{T})$ for every $t \in \mathbb{R}$ and $\|f\|_{L^2(\mathbb{R}\times\mathbb{T})} := (\int_{(t,x)\in\mathbb{R}\times[0,1)} |f(t,x)|^2 dtdx) < \infty$. For $f,g \in L^2(\mathbb{R}\times\mathbb{T})$, we define

$$(f,g)_{L^2(\mathbb{R}\times\mathbb{T})} = \int_{(t,x)\in\mathbb{R}\times[0,1)} f(t,x)\overline{g(t,x)}dtdx.$$

We denote by $\mathcal{F}f = \mathcal{F}_{\mathbb{T}}f$ the Fourier transform of $f \in L^2(\mathbb{T})$, which is defined by

$$\mathcal{F}f(n) = \int_0^1 f(x)e^{-2\pi i n x}dx \quad (n \in \mathbb{Z}).$$

For $f \in L^2(\mathbb{R}\times\mathbb{T})$, we define $\mathcal{F}f(t,n) = \mathcal{F}(f(t,\cdot))(n)$.

We denote by $\mathcal{F}f = \mathcal{F}_{\mathbb{R}}f$ the Fourier transform of $f \in L^1(\mathbb{R})$, which is defined by

$$\mathcal{F}f(\xi) = \int_{\mathbb{R}} f(x)e^{-2\pi i \xi x}dx \quad (\xi \in \mathbb{R}).$$

For a function f on $\mathbb{R}^2$ such that $f(t,\cdot) \in L^1(\mathbb{R})$ for every $t \in \mathbb{R}$, we define $\mathcal{F}f(t,\xi) = \mathcal{F}(f(t,\cdot))(\xi)$.

For a function f on $\mathbb{R}$ decreasing sufficiently fast as $|x| \to \infty$, we define a function πf by

$$(\pi f)(x) = \sum_{n\in\mathbb{Z}} f(x+n) \quad (x \in \mathbb{R}).$$

We note that $\mathcal{F}_{\mathbb{T}}(\pi f)(n) = \mathcal{F}_{\mathbb{R}} f(n)$ for every $n \in \mathbb{Z}$. For a suitable function f on $\mathbb{R}^2$, we write $(\pi f)(t,x) = \pi(f(t,\cdot))(x)$.

For $r \in \mathbb{Z}_+ := \mathbb{Z} \cap [0,\infty)$, we denote by $C^r(\mathbb{R}^2)$ the space of r times continuously differentiable functions on $\mathbb{R}^2$. We also denote by $C_0^r(\mathbb{R}^2)$ the space of compactly supported functions $f \in C^r(\mathbb{R}^2)$. We write $\mathcal{B}_0^r$ for the set of all functions $f \in C_0^r(\mathbb{R}^2)$ such that $\|f\|_{C^r} := \sum_{k_0+k_1\le r} \|\partial_t^{k_0}\partial_x^{k_1} f\|_{L^\infty} \le 1$ and $\operatorname{supp} f \subset \{(t,x) \in \mathbb{R}^2 \,;\, |t|^{\frac{1}{2}} + |x| \le 1\}$.

We denote by $\mathcal{S}(\mathbb{R}^2)$ the space of Schwartz functions on $\mathbb{R}^2$. We denote by $\mathcal{S}'(\mathbb{R}^2)$ its dual. We also denote by $(C_0^r)'(\mathbb{R}^2)$ the dual of $C_0^r(\mathbb{R}^2)$. We write $\xi(f)$ for a pairing of $f \in C_0^r(\mathbb{R}^2)$ and $\xi \in (C_0^r)'(\mathbb{R}^2)$.

We usually use a variable $z = (t,x)$ as a point in $\mathbb{R}^2$ and write $dz = dtdx$. For $z = (t,x) \in \mathbb{R}^2$, we write $\|z\|_{\mathfrak{s}} = \sqrt{|t|} + |x|$. We also define $B_{\mathfrak{s}}(z,r) = \{\bar{z} \in \mathbb{R}^2 \,;\, \|z - \bar{z}\|_{\mathfrak{s}} \le r\}$. For $z = (t,x) \in \mathbb{R}^2$, $\delta > 0$ and a function ρ on $\mathbb{R}^2$, we define ρ_z^δ by

$$\rho_z^\delta(\bar{z}) = \delta^{-3}\rho(\delta^{-2}(\bar{t}-t), \delta^{-1}(\bar{x}-x)) \quad (\bar{z} = (\bar{t},\bar{x}) \in \mathbb{R}^2).$$

For a multiindex $k = (k_0,k_1) \in \mathbb{Z}_+^2$, we write $|k|_{\mathfrak{s}} = 2k_0 + k_1$ and $\partial^k = \partial_t^{k_0}\partial_x^{k_1}$. Especially, we write $' = \partial^{(0,1)}$. For $z = (t,x) \in \mathbb{R}^2$ and $k = (k_0,k_1) \in \mathbb{Z}_+^2$, we write $z^k = t^{k_0}x^{k_1}$.

Let $\alpha \in (0,1)$. We denote by $\mathcal{C}_{\mathfrak{s}}^\alpha(\mathbb{R}^2)$ the space of functions f on $\mathbb{R}^2$ such that for every compact set $\mathfrak{K} \subset \mathbb{R}^2$ we have

$$\|f\|_{\alpha;\mathfrak{K}} := \sup_{z,\bar{z}\in\mathfrak{K}} \frac{|f(z) - f(\bar{z})|}{\|z - \bar{z}\|_{\mathfrak{s}}^\alpha} < \infty.$$

We also denote by $\mathcal{C}_{\mathfrak{s}}^\alpha((0,\infty)\times\mathbb{R})$ the space of functions f on $(0,\infty)\times\mathbb{R}$ such that $\|f\|_{\alpha;\mathfrak{K}} < \infty$ for every compact set $\mathfrak{K} \subset (0,\infty)\times\mathbb{R}$.

Let $\alpha < 0$ and $r = \lceil -\alpha \rceil$ (r is the smallest integer such that $-\alpha < r$). We say that $\xi \in \mathcal{S}'(\mathbb{R}^2)$ belongs to $\mathcal{C}_{\mathfrak{s}}^\alpha(\mathbb{R}^2)$ if and only if it belongs to $(C_0^r)'(\mathbb{R}^2)$ and for every compact set $\mathfrak{K} \subset \mathbb{R}^2$ we have

$$\|\xi\|_{\alpha;\mathfrak{K}} := \sup_{z\in\mathfrak{K}} \sup_{\rho\in\mathcal{B}_0^r} \sup_{\delta\in(0,1]} \delta^{-\alpha}|\xi(\rho_z^\delta)| < \infty.$$

The operator $*_{t,x}$ denotes the convolution in (t,x). $*_t$ and $*_x$ denote the convolutions in t and x, respectively. The operator $*$ always denotes $*_{t,x}$.

For a function f on $\mathbb{R}^2$, we define $\overleftarrow{f}(t,x) = f(-t,-x)$.

For linear spaces A and B, we denote by $\mathcal{L}(A,B)$ the space of linear maps form A to B. We write $\mathcal{L}(A) = \mathcal{L}(A,A)$ and $A^* = \mathcal{L}(A,\mathbb{R})$. $\mathrm{id} = \mathrm{id}_A \in \mathcal{L}(A)$ denotes the identity map on A. For a subset $S \subset A$, we denote by $\langle S \rangle$ the linear subspace of A spanned by S.

2.2 Fractional derivatives of white noise

For $\gamma > 0$, we write $H^\gamma(\mathbb{T}) := \{f \in L^2(\mathbb{T}) \,;\, \|f\|_{H^\gamma(\mathbb{T})} := (\sum_{n\in\mathbb{Z}}(2\pi|n|)^{2\gamma}|\mathcal{F}f(n)|^2)^{\frac{1}{2}} < \infty\}$. For $f,g \in H^\gamma(\mathbb{T})$, we define

$$(f,g)_{H^\gamma(\mathbb{T})} = \sum_{n\in\mathbb{Z}} (2\pi|n|)^{2\gamma}\mathcal{F}f(n)\overline{\mathcal{F}g(n)}.$$

We define the operator $(-\Delta)^{\frac{\gamma}{2}} : H^\gamma(\mathbb{T}) \to L^2(\mathbb{T})$ by

$$(-\Delta)^{\frac{\gamma}{2}} f(x) = \sum_{n\in\mathbb{Z}} (2\pi|n|)^\gamma \mathcal{F}f(n)e^{2\pi inx} \quad (x \in \mathbb{R}).$$

For a function f on $\mathbb{R}\times\mathbb{T}$ such that $f(t,\cdot) \in H^\gamma(\mathbb{T})$ for every $t \in \mathbb{R}$, we write $\partial_x^\gamma f(t,x) = ((-\Delta)^{\frac{\gamma}{2}} f(t,\cdot))(x)$.

Let $\{W_h\}_{h\in L^2(\mathbb{R}\times\mathbb{T})}$ be a space-time white noise on $\mathbb{R}\times\mathbb{T}$ defined on a probability space $(\Omega,\mathcal{F},\mathbb{P})$. That is, $\{W_h\}$ is a collection of centered Gaussian random variables such that $\mathbb{E}(W_hW_g) = (h,g)_{L^2(\mathbb{R}\times\mathbb{T})}$. For a function h on $\mathbb{R}\times\mathbb{T}$ such that $\int_{\mathbb{R}} \|h(t,\cdot)\|_{H^\gamma(\mathbb{T})}^2 dt < \infty$, we define $\partial_x^\gamma W_h := W_{\partial_x^\gamma h}$. Then we have

$$\mathbb{E}(\partial_x^\gamma W_h \partial_x^\gamma W_g) = \int_{\mathbb{R}} (h(t,\cdot), g(t,\cdot))_{H^\gamma(\mathbb{T})} dt.$$

We define the periodic extension of $\partial_x^\gamma W$.

Lemma 2.1. *Let $0 < \gamma < \frac{3}{2}$. Then for every $h \in \mathcal{S}(\mathbb{R}^2) \cup C_0^2(\mathbb{R}^2)$, we have $\int_{\mathbb{R}} \|\pi h(t,\cdot)\|_{H^\gamma(\mathbb{T})}^2 dt < \infty$.*

proof. Since $\mathcal{F}_{\mathbb{T}}(\pi h)(t,n) = \mathcal{F}_{\mathbb{R}} h(t,n)$ for every $n \in \mathbb{Z}$, we have

$$\begin{aligned}\int_{\mathbb{R}} \|\pi h(t,\cdot)\|_{H^\gamma(\mathbb{T})}^2 dt &= \int_{\mathbb{R}} \sum_n (2\pi|n|)^{2\gamma} |\mathcal{F}h(t,n)|^2 dt \\ &= \int_{\mathbb{R}} \sum_n (2\pi|n|)^{2\gamma} (4\pi^2 n^2)^{-2} |\mathcal{F}(\partial_x^2 h)(t,n)|^2 dt \\ &\le \sum_{n\neq 0} (2\pi|n|)^{2\gamma-4} \int_{\mathbb{R}} \|\partial_x^2 h(t,\cdot)\|_{L^1(\mathbb{R})}^2 dt.\end{aligned}$$

The last term is finite because $\sum_{n\neq 0} |n|^{2\gamma-4} < \infty$. □

For $h \in \mathcal{S}(\mathbb{R}^2) \cup C_0^2(\mathbb{R}^2)$, we define $\partial_x^\gamma \xi(h) := \partial_x^\gamma W_{\pi h}$.

Lemma 2.2. *Let $0 < \gamma < \frac{1}{2}$ and $-2 < \alpha < -\frac{3}{2} - \gamma$. Then for every $p \ge 1$ and compact set $\mathfrak{K} \subset \mathbb{R}^2$, we have*

$$\mathbb{E}\|\partial_x^\gamma \xi\|_{\alpha;\mathfrak{K}}^p < \infty.$$

Furthermore, let $\rho \in C_0^\infty(\mathbb{R}^2)$ be a function such that $\int \rho(z)dz = 1$, and for $\epsilon > 0$ we define a function $\partial_x^\gamma \xi_\epsilon$ on $\mathbb{R}^2$ by $\partial_x^\gamma \xi_\epsilon(z) := \partial_x^\gamma \xi(\rho_z^\epsilon)$. Then for every $\kappa \in (0, -\frac{3}{2} - \gamma - \alpha)$, $p \ge 1$ and compact set $\mathfrak{K} \subset \mathbb{R}^2$, we have

$$\mathbb{E}\|\partial_x^\gamma \xi - \partial_x^\gamma \xi_\epsilon\|_{\alpha;\mathfrak{K}}^p \lesssim \epsilon^{\kappa p}.$$

proof. See Lemma 2.2 of [5]. □

3 Regularity structures

We recall some concepts in the theory of regularity structures from [3].

3.1 Definitions

Definition 3.1. *We say that a triplet $\mathcal{T} = (A, T, G)$ is a* regularity structure *with* index set A, model space T *and* structure group G, *if and only if*

1. *A is a countable subset of $\mathbb{R}$, $0 \in A$, bounded from below, and has no accumulation points.*
2. *$T = \bigoplus_{\alpha\in A} T_\alpha$ is a direct sum of normed spaces. Furthermore, $\dim T_0 = 1$ and its unit vector is denoted by $\mathbf{1}$.*
3. *G is a subgroup of $\mathcal{L}(T)$ such that, for every $\Gamma \in G$, $\alpha \in A$, and $\tau \in T_\alpha$ we have*

$$\Gamma\tau - \tau \in \bigoplus_{\beta<\alpha} T_\beta.$$

Furthermore, $\Gamma\mathbf{1} = \mathbf{1}$ for every $\Gamma \in G$.

The norm of T_α is denoted by $\|\cdot\|_\alpha$. For an element $\tau \in T$, we write $\|\tau\|_\alpha = \|\tau_\alpha\|_\alpha$, where τ_α is the component of τ in T_α. For $\beta > 0$, we write $T_\beta^- = \bigoplus_{\alpha<\beta} T_\alpha$.

Definition 3.2. *Let $\mathcal{T} = (A, T, G)$ be a regularity structure. Let B be a subset of A such that $0 \in B$, and $V_\beta \neq \{0\}$ be a linear subspace of T_β for each $\beta \in B$. We say that $V = \bigoplus_{\beta\in B} V_\beta$ is a* sector *of $\mathcal{T}$, if and only if $\Gamma V \subset V$ for every $\Gamma \in G$.*

We say that $\beta = \min B$ is the *regularity* of V. One important example of a regularity structure is the structure generated by polynomials. Let X_0, X_1 be dummy variables and $T = \mathbb{R}[X_0, X_1]$ be the linear space of polynomials in X_0, X_1. For a multiindex $k = (k_0, k_1) \in \mathbb{Z}_+^2$, we write $X^k = X_0^{k_0} X_1^{k_1}$. We denote by $\mathbf{1} = X^{(0,0)}$.

Definition 3.3. *The polynomial regularity structure* (A, T, G) *consists of the following elements.*

1. $A = \mathbb{Z}_+$.
2. $T = \bigoplus_{n\in\mathbb{Z}_+} T_n$, *where* $T_n = \langle X^k ; |k|_\mathfrak{s} = n\rangle$.
3. $G = \{\Gamma_h ; h \in \mathbb{R}^2\}$, *where* Γ_h *is defined by* $\Gamma_h X^k = (X + h\mathbf{1})^k$.

We define models for a regularity structure.

Definition 3.4. *Let* $\mathcal{T} = (A, T, G)$ *be a regularity structure with regularity* $\alpha_0 \le 0$, *and* $r = \lceil -\alpha_0 \rceil$ *We say that* $Z = (\Pi, \Gamma)$ *is a* model *for* $\mathcal{T}$, *if and only if*

1. Γ *is a map* $\mathbb{R}^2 \times \mathbb{R}^2 \to G$ *such that* $\Gamma_{z,z} = \mathrm{id}_T$ *and* $\Gamma_{z,\bar z}\Gamma_{\bar z,\bar{\bar z}} = \Gamma_{z,\bar{\bar z}}$ *for every* $z, \bar z, \bar{\bar z} \in \mathbb{R}^2$. *Furthermore, for every* $\gamma > 0$ *and compact set* $\mathfrak{K} \subset \mathbb{R}^2$, *we have*
$$\|\Gamma\|_{\gamma;\mathfrak{K}} := \sup\left\{\frac{\|\Gamma_{z,\bar z}\tau\|_\beta}{\|\tau\|_\alpha \|z - \bar z\|_\mathfrak{s}^{\alpha-\beta}} ; \beta < \alpha < \gamma,\ \tau \in T_\alpha,\ (z, \bar z) \in \mathfrak{K}^2\right\} < \infty.$$
2. Π *is a map* $\mathbb{R}^2 \to \mathcal{L}(\mathcal{T}, \mathcal{S}'(\mathbb{R}^2))$ *such that* $\Pi_z\Gamma_{z,\bar z} = \Pi_{\bar z}$ *for every* $z, \bar z \in \mathbb{R}^2$. *Furthermore, for every* $\gamma > 0$ *and compact set* $\mathfrak{K} \subset \mathbb{R}^2$, *we have*
$$\|\Pi\|_{\gamma;\mathfrak{K}} := \sup\left\{\frac{|(\Pi_z\tau)(\rho_z^\delta)|}{\|\tau\|_\alpha \delta^\alpha} ; \alpha < \gamma,\ \tau \in T_\alpha,\ z \in \mathfrak{K},\ \rho \in \mathcal{B}_0^r,\ \delta \in (0, 1]\right\} < \infty.$$

For models $Z = (\Pi, \Gamma)$ and $\bar Z = (\bar\Pi, \bar\Gamma)$ on $\mathcal{T}$, we write
$$|\!|\!| Z |\!|\!|_{\gamma;\mathfrak{K}} = \|\Gamma\|_{\gamma;\mathfrak{K}} + \|\Pi\|_{\gamma;\mathfrak{K}}, \quad |\!|\!| Z - \bar Z |\!|\!|_{\gamma;\mathfrak{K}} = \|\Gamma - \bar\Gamma\|_{\gamma;\mathfrak{K}} + \|\Pi - \bar\Pi\|_{\gamma;\mathfrak{K}}.$$

3.2 Regularity structures for (1.1)

We construct regularity structures for (1.1), following Section 8.1 of [3]. We use dummy variables Ξ (noise), $\mathbf{1}$ (constant), X_1 (time variable), X_2 (spatial variable), and abstract operators $\{\mathcal{I}_k\}_{k\in\mathbb{Z}_+^2}$ (convolution with k-th derivative of heat kernel). Especially we write $\mathcal{I} = \mathcal{I}_{(0,0)}$ and $\mathcal{I}' = \mathcal{I}_{(0,1)}$. We define $\widetilde{\mathcal{F}}$ as the minimal set of variables such that
$$(1)\ \Xi, \mathbf{1}, X_0, X_1 \in \widetilde{\mathcal{F}}, \quad (2)\ \tau, \bar\tau \in \widetilde{\mathcal{F}} \Rightarrow \tau\bar\tau \in \widetilde{\mathcal{F}},$$
$$(3)\ \tau \in \widetilde{\mathcal{F}} \setminus \{X^l ; l \in \mathbb{Z}_+^2\}, k \in \mathbb{Z}_+^2 \Rightarrow \mathcal{I}_k\tau \in \widetilde{\mathcal{F}}.$$

In (2), we postulate that $\tau\bar\tau = \bar\tau\tau$. For a fixed number $\alpha_0 \in \mathbb{R}$, we can define the homogeneity (Besov index in parabolic scaling) of each variable by
$$|\Xi|_\mathfrak{s} = \alpha_0, \quad |\mathbf{1}|_\mathfrak{s} = 0, \quad |X_0|_\mathfrak{s} = 2, \quad |X_1|_\mathfrak{s} = 1$$
$$|\tau\bar\tau|_\mathfrak{s} = |\tau|_\mathfrak{s} + |\bar\tau|_\mathfrak{s} \quad |\mathcal{I}_k\tau|_\mathfrak{s} = |\tau|_\mathfrak{s} + 2 - |k|_\mathfrak{s}.$$

We define the abstract operator ∂ acting on $\{\mathcal{I}\tau\} \cup \{X^k\}$ by
$$\partial\mathcal{I}\tau = \mathcal{I}'\tau, \quad \partial(X_0^{k_0}X_1^{k_1}) = \mathbf{1}_{k_1\neq 0}k_1X_0^{k_0}X_1^{k_1-1}.$$

We define the sets $\mathcal{U}_n$ and $\mathcal{V}_n$ for $n \in \mathbb{Z}_+$ recursively by
$$\mathcal{U}_0 = \mathcal{V}_0 = \{X^k ; k \in \mathbb{Z}_+^2\},$$
$$\mathcal{V}_n = \{\Xi\} \cup \{\partial\tau_1\partial\tau_2 ; \tau_1, \tau_2 \in \mathcal{U}_{n-1}\}, \quad \mathcal{U}_n = \mathcal{U}_{n-1} \cup \{\mathcal{I}\tau ; \tau \in \mathcal{V}_n\} \quad (n \ge 1).$$

We set $\mathcal{U} = \bigcup_{n\ge 0}\mathcal{U}_n$, $\mathcal{V} = \bigcup_{n\ge 0}\mathcal{V}_n$ and $\mathcal{F} = \mathcal{U} \cup \mathcal{V}$. We define
$$T_\alpha = \langle \tau \in \mathcal{F} ; |\tau|_\mathfrak{s} = \alpha\rangle, \quad T = \langle\mathcal{F}\rangle, \quad U = \langle\mathcal{U}\rangle, \quad V = \langle\mathcal{V}\rangle.$$

In order to define T as a model space of a regularity structure, the set $\{|\tau|_\mathfrak{s} ; \tau \in \mathcal{F}\}$ must be bounded from below. A nonlinear SPDE is called *subcritical*, if nonlinear terms formally disappear in some scaling which keeps the linear part and the noise term invariant. This is equivalent to the property that all variables except Ξ defined as above have homogeneities strictly greater than $|\Xi|_\mathfrak{s}$ (Assumption 8.3 of [3]). In the present case, this is equivalent to $|\mathcal{I}'(\Xi)|_\mathfrak{s} = 2(1 + \alpha_0) > \alpha_0 \Leftrightarrow \alpha_0 > -2$. Additionally, we should assume $-2 < \alpha_0 < -\frac{3}{2} - \gamma$ from Lemma 2.2, which implies $0 \le \gamma < \frac{1}{2}$.

Lemma 3.1 (Lemma 8.10 of [3]). *Let $0 \le \gamma < \frac{1}{2}$ and $-2 < \alpha_0 < -\frac{3}{2} - \gamma$. Then the set $\{\tau \in \mathcal{F}\,;|\tau|_\mathfrak{s} < r\}$ is finite for every $r > 0$.*

3.3 Structure group

We define the structure group on T following Section 8.1 of [3]. Instead of $\{\mathcal{I}_k\}_{k\in\mathbb{Z}_+^2}$, we use operators $\{\mathcal{J}_k\}_{k\in\mathbb{Z}_+^2}$. We write $\mathcal{J} = \mathcal{J}_{(0,0)}$ and $\mathcal{J}' = \mathcal{J}_{(0,1)}$. We define $\mathcal{F}^+$ as the minimal set of variables such that

$$(1)\ \mathbf{1}, X_0, X_1 \in \mathcal{F}^+, \quad (2)\ \tau, \bar\tau \in \mathcal{F}^+ \Rightarrow \tau\bar\tau \in \mathcal{F}^+,$$
$$(3)\ \tau \in \mathcal{F} \setminus \{X^l\,; l \in \mathbb{Z}_+^2\}, k \in \mathbb{Z}_+^2, |\tau|_\mathfrak{s} + 2 - |k|_\mathfrak{s} > 0 \Rightarrow \mathcal{J}_k\tau \in \mathcal{F}^+.$$

We can define the homogeneity of each variable by

$$|\mathbf{1}|_\mathfrak{s} = 0, \quad |X_0|_\mathfrak{s} = 2, \quad |X_1|_\mathfrak{s} = 1, \quad |\tau\bar\tau|_\mathfrak{s} = |\tau|_\mathfrak{s} + |\bar\tau|_\mathfrak{s}, \quad |\mathcal{J}_k\tau|_\mathfrak{s} = |\tau|_\mathfrak{s} + 2 - |k|_\mathfrak{s}.$$

We write $\mathcal{H} = \langle\mathcal{F}\rangle$ and $\mathcal{H}^+ = \langle\mathcal{F}^+\rangle$. We have a natural linear map $\hat{\mathcal{J}}_k : \mathcal{H} \to \mathcal{H}^+$ defined by the linear extension of

$$\hat{\mathcal{J}}_k(\tau) = \begin{cases} \mathcal{J}_k\tau & |\tau|_\mathfrak{s} + 2 - |k|_\mathfrak{s} > 0 \\ 0 & |\tau|_\mathfrak{s} + 2 - |k|_\mathfrak{s} \le 0 \end{cases}, \quad \tau \in \mathcal{F}.$$

We simply write again $\mathcal{J}_k$ instead of $\hat{\mathcal{J}}_k$.

We define the linear map $\Delta : \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}^+$ by

$$\Delta\mathbf{1} = \mathbf{1}\otimes\mathbf{1}, \quad \Delta X_i = X_i \otimes \mathbf{1} + \mathbf{1} \otimes X_i\ (i = 0,1), \quad \Delta\Xi = \Xi \otimes \mathbf{1},$$
$$\Delta(\tau\bar\tau) = (\Delta\tau)(\Delta\bar\tau), \quad \Delta(\mathcal{I}_k\tau) = (\mathcal{I}_k \otimes \mathrm{id})\Delta\tau + \sum_{l,m} \frac{X^l}{l!} \otimes \frac{X^m}{m!}\mathcal{J}_{k+l+m}\tau.$$

We also define the linear map $\Delta^+ : \mathcal{H}^+ \to \mathcal{H}^+ \otimes \mathcal{H}^+$ by

$$\Delta^+\mathbf{1} = \mathbf{1}\otimes\mathbf{1}, \quad \Delta^+X_i = X_i \otimes \mathbf{1} + \mathbf{1} \otimes X_i\ (i = 0,1),$$
$$\Delta^+(\tau\bar\tau) = (\Delta^+\tau)(\Delta^+\bar\tau), \quad \Delta^+(\mathcal{J}_k\tau) = \sum_l \left(\mathcal{J}_{k+l} \otimes \frac{(-X)^l}{l!}\right)\Delta\tau + \mathbf{1}\otimes\mathcal{J}_k\tau.$$

Furthermore, we define the linear map $\mathcal{A} : \mathcal{H}^+ \to \mathcal{H}^+$ by

$$\mathcal{A}\mathbf{1} = \mathbf{1}, \quad \mathcal{A}X_i = -X_i\ (i = 0,1), \quad \mathcal{A}(\tau\bar\tau) = (\mathcal{A}\tau)(\mathcal{A}\bar\tau),$$
$$\mathcal{A}\mathcal{J}_k\tau = -\sum_l \mathcal{M}\left(\mathcal{J}_{k+l} \otimes \frac{X^l}{l!}\mathcal{A}\right)\Delta\tau.$$

Here $\mathcal{M} : \mathcal{H}^+ \times \mathcal{H}^+ \to \mathcal{H}^+$ is the multiplication operator defined by $\mathcal{M}(\tau\otimes\bar\tau) = \tau\bar\tau$. We define the product $\circ$ on $(\mathcal{H}^+)^*$ by

$$(g \circ \bar g)(\tau) = (g \otimes \bar g)(\Delta^+\tau) \quad (g, \bar g \in (\mathcal{H}^+)^*, \tau \in \mathcal{H}^+).$$

Lemma 3.2 (Theorem 8.16 of [3]). *$\mathcal{H}^+$ is a Hopf algebra with the antipode $\mathcal{A}$. $\mathcal{H}$ is a comodule over $\mathcal{H}^+$.*

We denote by G the set of algebra homomorphisms $g : \mathcal{H}^+ \to \mathbb{R}$. Then G is a group with the product $\circ$. The inverse of $g \in G$ is given by $g^{-1} = g\mathcal{A}$. For $g \in G$, we define the operator $\Gamma_g \in \mathcal{L}(T)$ by

$$\Gamma_g\tau = (\mathrm{id}\otimes g)\Delta\tau \quad (\tau \in T),$$

where we identify $T \otimes \mathbb{R}$ with T by $\tau \otimes a \mapsto a\tau$. Since $g \mapsto \Gamma_g$ is a group homomorphism (Proposition 8.19 of [3]), we can identify G with $\{\Gamma_g\,; g \in G\}$.

Lemma 3.3 (Theorem 8.24 of [3]). *Let $0 \le \gamma < \frac{1}{2}$, $-2 < \alpha_0 < -\frac{3}{2} - \gamma$ and $A = \{|\tau|_\mathfrak{s}\,; \tau \in \mathcal{F}\}$. Then (A, T, G) is a regularity structure.*

Given $r > 0$, obviously

$$\mathcal{T}^{(r)} = (A \cap (-\infty, r), T_r^-, G|_{T_r^-})$$

is also a regularity structure.

3.4 Admissible models

Fix $C, r > 0$. Let $K : \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}$ be a function such that

1. K is smooth except at 0, and supported in $B_{\mathfrak{s}}(0, C)$.
2. $K(t, \cdot) = 0$ for $t \le 0$, and $K(t, x) = K(t, -x)$ for every $(t, x) \in \mathbb{R}^2$.
3. For every $(t, x) \in B_{\mathfrak{s}}(0, \frac{C}{2})$, we have
$$K(t, x) = \mathbf{1}_{t>0} \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}}.$$
4. For every multiindex $k = (k_1, k_2) \in \mathbb{Z}_+^2$ with $|k|_{\mathfrak{s}} \le r$, we have
$$\iint K(t, x) t^{k_1} x^{k_2} dt dx = 0.$$

See Lemma 5.5 of [3]. It is sufficient to take small C, see [5].

Definition 3.5. *We say that a model (Π, Γ) on $\mathcal{T}^{(r)}$ is* admissible*, if and only if for every $z = (t, x), \bar{z} = (\bar{t}, \bar{x}) \in \mathbb{R}^2$, multiindex $k \in \mathbb{Z}_+^2$ and $\tau \in \mathcal{F}$, Π satisfies*

$$(\Pi_z \mathcal{I}_k \tau)(\bar{z}) = \partial^k K * (\Pi_z \tau)(\bar{z}) + \sum_l \frac{(\bar{z} - z)^l}{l!} f_z(\mathcal{J}_{k+l} \tau), \quad (\Pi_z X^k)(\bar{z}) = (\bar{z} - z)^k,$$

and Γ satisfies $\Gamma_{z\bar{z}} = (\Gamma_{f_z})^{-1} \Gamma_{f_{\bar{z}}}$. Here $\{f_z \in G\,; z \in \mathbb{R}^2\}$ is a family defined by

$$\begin{aligned} &f_z(X_0) = -t, \ f_z(X_1) = -x, \\ &f_z(\mathcal{J}_k \tau) = -\partial^k K * (\Pi_z \tau)(z) \quad (|\tau|_{\mathfrak{s}} + 2 - |k|_{\mathfrak{s}} > 0). \end{aligned}$$

From the definition of admissible models, the map $\Pi_z (\Gamma_{f_z})^{-1} : \mathcal{H} \to \mathcal{S}'(\mathbb{R}^2)$ is independent to z. Hence we can write $\mathbf{\Pi} = \Pi_z (\Gamma_{f_z})^{-1}$. If (Π, Γ) is admissible, for every $z = (t, x) \in \mathbb{R}^2$, $k \in \mathbb{Z}_+^2$ and $\tau \in \mathcal{F}$ we have

$$(\mathbf{\Pi} \mathbf{1})(z) = 1, \quad (\mathbf{\Pi} X_0)(z) = t, \quad (\mathbf{\Pi} X_1)(z) = x, \quad (\mathbf{\Pi} \mathcal{I}_k \tau)(z) = \partial^k K * (\mathbf{\Pi} \tau)(z).$$

Conversely, if a linear map $\mathbf{\Pi} : \mathcal{H} \to \mathcal{S}'(\mathbb{R}^2)$ satisfies these conditions, and a family $\{f_z\,; z \in G\}$ satisfies

$$\begin{aligned} &f_z(X_0) = -t, \quad f_z(X_1) = -x, \\ &f_z(\mathcal{J}_k \tau) = -\partial^k K * ((\mathbf{\Pi} \otimes f_z) \Delta \tau)(z) \quad (|\tau|_{\mathfrak{s}} + 2 - |k|_{\mathfrak{s}} > 0), \end{aligned}$$

then an admissible model (Π, Γ) is uniquely determined by

$$\Pi_z = (\mathbf{\Pi} \otimes f_z) \Delta, \quad \Gamma_{z\bar{z}} = (\Gamma_{f_z})^{-1} \Gamma_{f_{\bar{z}}}.$$

Definition 3.6. *We say that a model (Π, Γ) on $\mathcal{T}^{(r)}$ is* periodic (in x)*, if and only if*

$$(\Pi_{(t, x+n)} \tau)(\varphi_n) = (\Pi_{(t,x)} \tau)(\varphi), \quad \Gamma_{(t,x+n),(\bar{t},\bar{x}+n)} = \Gamma_{(t,x),(\bar{t},\bar{x})},$$

for every $(t, x), (\bar{t}, \bar{x}) \in \mathbb{R}^2$, $n, \bar{n} \in \mathbb{Z}$ and $\varphi \in \mathcal{S}(\mathbb{R}^2)$. Here $\varphi_n(t, x) = \varphi(t, x - n)$.

3.5 Modelled distributions

Following Section 6 of [3], we define modelled distributions with singularity at $t = 0$. We write $P = \{(t, x) \in \mathbb{R}^2\,; t = 0\}$ and

$$\|(t, x)\|_P = 1 \wedge \sqrt{|t|}, \quad \|z, \bar{z}\|_P = \|z\|_P \wedge \|\bar{z}\|_P.$$

For a subset $\mathfrak{K} \subset \mathbb{R}^2$, we write

$$\mathfrak{K}_P := \{(z, \bar{z}) \in (\mathfrak{K} \setminus P)^2\,; z \neq \bar{z}, \|z - \bar{z}\|_{\mathfrak{s}} \le \|z, \bar{z}\|_P\}.$$

Definition 3.7. *Let $Z = (\Pi, \Gamma)$ be a model on $\mathcal{T}^{(r)}$, $\theta > 0$ and $\eta \in \mathbb{R}$. For a T_θ^--valued function f on $\mathbb{R}^2$, and a compact set $\mathfrak{K} \subset \mathbb{R}^2$, we define*

$$\|f\|_{\theta,\eta;\mathfrak{K}} = \sup_{z\in\mathfrak{K}\setminus P} \sup_{l<\theta} \frac{\|f(z)\|_l}{\|z\|_P^{(\eta-l)\wedge 0}},$$

$$|||f|||_{\theta,\eta;\mathfrak{K}} = \|f\|_{\theta,\eta;\mathfrak{K}} + \sup_{(z,\bar{z})\in\mathfrak{K}_P} \sup_{l<\theta} \frac{\|f(z) - \Gamma_{z\bar{z}}f(\bar{z})\|_l}{\|z-\bar{z}\|_{\mathfrak{s}}^{\theta-l}\|z,\bar{z}\|_P^{\eta-\theta}}.$$

We write $f \in \mathcal{D}_P^{\theta,\eta} = \mathcal{D}_P^{\theta,\eta}(Z)$ if and only if $|||f|||_{\theta,\eta;\mathfrak{K}} < \infty$ for every compact subset $\mathfrak{K} \subset \mathbb{R}^2$.

If f takes value in a sector W of $\mathcal{T}^{(r)}$, we write $f \in \mathcal{D}_P^{\theta,\eta}(W; Z)$. For models Z, $\bar{Z}$ and $f \in \mathcal{D}_P^{\theta,\eta}(W; Z)$, $\bar{f} \in \mathcal{D}_P^{\theta,\eta}(W; \bar{Z})$, we define

$$|||f;\bar{f}|||_{\theta,\eta;\mathfrak{K}} = \|f-\bar{f}\|_{\theta,\eta;\mathfrak{K}} + \sup_{(z,\bar{z})\in\mathfrak{K}_P} \sup_{l<\theta} \frac{\|f(z) - \bar{f}(z) - \Gamma_{z\bar{z}}f(\bar{z}) + \bar{\Gamma}_{z\bar{z}}\bar{f}(\bar{z})\|_l}{\|z-\bar{z}\|_{\mathfrak{s}}^{\theta-l}\|z,\bar{z}\|_P^{\eta-\theta}}.$$

We denote by $\mathcal{M} \ltimes \mathcal{D}_P^{\theta,\eta}$ the set of all pairs (Z, f) of a model Z and $f \in \mathcal{D}_P^{\theta,\eta}(Z)$. The topology on $\mathcal{M} \ltimes \mathcal{D}_P^{\theta,\eta}$ is defined by the family of pseudo-metrics $\{|||\cdot;\cdot|||_{\theta,\eta;\mathfrak{K}}\}$.

Theorem 3.4 (Theorem 3.10, Lemma 6.7 and Proposition 6.9 of [3])**.** *Let $Z = (\Pi, \Gamma)$ be a model on $\mathcal{T}^{(r)}$. Let W be a sector of $\mathcal{T}^{(r)}$ with regularity $\alpha \le 0$. If $\theta > 0$, $\eta \le \theta$, and $\alpha \wedge \eta > -2$, then there exists a unique continuous linear map $\mathcal{R} : \mathcal{D}_P^{\theta,\eta}(W; Z) \to \mathcal{C}_{\mathfrak{s}}^{\alpha\wedge\eta}$ such that for every compact set $\mathfrak{K} \in \mathbb{R}^2$, we have*

$$|(\mathcal{R}f - \Pi_z f(z))(\rho_z^\delta)| \lesssim \lambda^{\eta-\theta}\delta^\theta \|\Pi\|_{\gamma,\bar{\mathfrak{K}}} |||f|||_{\theta,\eta;\bar{\mathfrak{K}}},$$

uniformly over $f \in \mathcal{D}_P^{\theta,\eta}$, $\rho \in \mathcal{B}_0^2$, $\delta \in (0,1]$, $z \in \mathfrak{K}$, and $\lambda \in (0,1]$ such that $\inf\{|t|\,;(t,x) \in B_{\mathfrak{s}}(z, 2\delta)\} \ge \lambda$. *Here $\bar{\mathfrak{K}}$ is the 1-fattening of $\mathfrak{K}$. Furthermore, the map $\mathcal{M} \ltimes \mathcal{D}_P^{\theta,\eta}(W) \ni (Z, f) \to \mathcal{R}^Z f \in \mathcal{C}_{\mathfrak{s}}^{\alpha\wedge\eta}$ is locally uniformly continuous.*

Let W be a sector of $\mathcal{T}^{(r)}$ such that there exists $\beta \in (0,1)$ such that

$$\langle X^k\,; |k|_{\mathfrak{s}} < r\rangle \subset W, \quad W = \langle \mathbf{1}\rangle \oplus \bigoplus_{l\ge\beta} W_l.$$

Then for every $f \in \mathcal{D}_P^{\theta,\eta}(W)$, $\mathcal{R}f$ coincides with the component of f in T_0. Furthermore we have $\mathcal{R}f \in \mathcal{C}_{\mathfrak{s}}^\beta((0,\infty)\times\mathbb{R})$, and the map $\mathcal{M} \ltimes \mathcal{D}_P^{\theta,\eta}(W) \ni (Z, f) \to \mathcal{R}^Z f \in \mathcal{C}_{\mathfrak{s}}^\beta((0,\infty)\times\mathbb{R})$ is locally uniformly continuous. In fact, for every model Z, $\bar{Z}$, modelled distribution $f \in \mathcal{D}_P(Z)$, $\bar{f} \in \mathcal{D}(\bar{Z})$ and compact set $\mathfrak{K}$ away from P, we have

$$\|\mathcal{R}f - \bar{\mathcal{R}}\bar{f}\|_{\beta,\mathfrak{K}} \lesssim (\inf\{|s|\,; \exists x \in \mathbb{R}\ (s,x) \in \mathfrak{K}\})^{\eta-\theta}(|||f;\bar{f}|||_{\theta,\eta;\bar{\mathfrak{K}}} + |||Z;\bar{Z}|||_{\theta,\bar{\mathfrak{K}}})$$

(Proposition 3.28 of [3] and the definition of modelled distributions).

3.6 Solution map

From now on we fix $r > 2$ in Subsection 3.4. We write (1.1) by the mild form:

$$h = G *_{t,x} \{\mathbf{1}_{t>0}((\partial_x h)^2 + \partial_x^\gamma \xi)\} + G *_x h_0,$$

where, G is the heat kernel on $(0,\infty)\times\mathbb{R}$ and h_0 is an initial condition. We reformulate it as an equation of $H \in \mathcal{D}_P^{\theta,\eta}(U)$:

$$H = \mathcal{G}\mathbf{1}_{t>0}((\partial H)^2 + \Xi) + Gh_0. \tag{3.1}$$

Here $Gh_0 = G *_x h_0$ is lifted to an element of $\mathcal{D}_P^{\theta,\eta}(\langle X^k\rangle)$ $(\theta > \eta\vee 0)$ by Lemma 7.5 of [3]. $\mathcal{G}$ is an operator in the following lemma.

Lemma 3.5 (Proposition 6.16, Theorem 7.1 and Lemma 7.3 of [3]). *Let $\theta \in (0, r-2)$ and $\eta \in (-2, \theta)$. Assume that $\theta + 2, \eta + 2 \notin \mathbb{N}$. Then, for each admissible and periodic model Z on $\mathcal{T}^{(r)}$, there exists a continuous linear map $\mathcal{G} : \mathcal{D}_P^{\theta,\eta}(V) \to \mathcal{D}_P^{\bar{\theta},\bar{\eta}}(U)$ ($\bar{\theta} = \theta + 2$, $\bar{\eta} = \eta \wedge \alpha_0 + 2$) such that for every $f \in \mathcal{D}_P^{\theta,\eta}(V)$ we have*

1. *$\mathcal{G}f - \mathcal{I}f$ takes values in $\langle X^k \rangle$.*
2. *$\mathcal{R}\mathcal{G}f = G * \mathcal{R}f$.*

The nonlinearity is naturally extended to $\mathcal{D}_P^{\theta,\eta}$.

Lemma 3.6 (Propositions 6.12 and 6.15 of [3]). *For every $f \in \mathcal{D}_P^{\theta,\eta}(U)$ with $\theta > -\alpha_0$ and $\eta < \alpha_0 + 2$, we have $(\partial f)^2 + \Xi \in \mathcal{D}_P^{\theta+\alpha_0,\eta+\alpha_0}(V)$. Furthermore, the map $f \mapsto (\partial f)^2 + \Xi$ is locally Lipschitz continuous.*

proof. From Proposition 6.15 of [3], $\partial f \in \mathcal{D}_P^{\theta-1,\eta-1}$. Since the regularity of the sector where ∂f takes value is $|\mathcal{I}'(\Xi)|_\mathfrak{s} = \alpha_0 + 1$, we have $(\partial f)^2 + \Xi \in \mathcal{D}_P^{\bar{\theta},\bar{\eta}}$ with $\bar{\theta} = (\theta - 1) + (\alpha_0 + 1) = \theta + \alpha_0$ and $\bar{\eta} = (\eta + \alpha_0) \wedge (2\alpha_0 + 2) = \eta + \alpha_0$, from Proposition 6.15 of [3]. Lipschitz continuity is also obtained from these propositions. □

Now we have the continuity of the solution map.

Theorem 3.7 (Theorem 7.8 of [3]). *Let $0 \le \gamma < \frac{1}{2}$, and $-2 < \alpha_0 < -\frac{3}{2} - \gamma$. Let $\theta \in (-\alpha_0, r - \alpha_0 - 2)$, and $\eta \in (0, \alpha_0 + 2)$. Then, for every periodic $h_0 \in \mathcal{C}^\eta(\mathbb{R})$ and admissible and periodic model Z, there exists a time $T = T(h_0, Z) > 0$ such that, for every $t < T$ there exists a unique solution $H \in \mathcal{D}_P^{\theta,\eta}(U)$ to (3.1) on $[0, t]$, and it holds that $T = \infty$ or $\lim_{t\to\infty} \|\mathcal{R}H(t,\cdot)\|_\eta = \infty$. Furthermore, the map $(h_0, Z) \mapsto T \in (0, \infty]$ is lower semi-continuous and the solution map $\mathcal{S} : (h_0, Z) \mapsto H$ is jointly uniformly continuous in a neighborhood around (h_0, Z).*

4 Renormalization

For each $\epsilon > 0$, we can lift the noise $\partial_x^\gamma \xi_\epsilon$ to a canonical model $Z^{(\epsilon)} = (\Pi^{(\epsilon)}, \Gamma^{(\epsilon)})$ on $\mathcal{T}^{(r)}$. We define the linear map $\mathbf{\Pi}^{(\epsilon)} : T_r^- \to C(\mathbb{R}^2)$ by

$$(\mathbf{\Pi}^{(\epsilon)}\mathbf{1})(z) = 1, \quad (\mathbf{\Pi}^{(\epsilon)}X_0)(z) = t, \quad (\mathbf{\Pi}^{(\epsilon)}X_1)(z) = x, \quad (\mathbf{\Pi}^{(\epsilon)}\Xi)(z) = \partial_x^\gamma \xi_\epsilon(z),$$
$$(\mathbf{\Pi}^{(\epsilon)}\tau\bar{\tau})(z) = (\mathbf{\Pi}^{(\epsilon)}\tau)(z)(\mathbf{\Pi}^{(\epsilon)}\bar{\tau})(z), \quad (\mathbf{\Pi}^{(\epsilon)}\mathcal{I}_k\tau)(z) = \partial^k K * (\mathbf{\Pi}^{(\epsilon)}\tau)(z).$$

We define the family $\{f_z^{(\epsilon)} \in G\,;\, z \in \mathbb{R}^2\}$ by

$$f_z^{(\epsilon)}(\mathbf{1}) = 1, \quad f_z^{(\epsilon)}(X_0) = -t, \quad f_z^{(\epsilon)}(X_1) = -x,$$
$$f_z^{(\epsilon)}(\mathcal{J}_k\tau) = -\partial^k K * (\mathbf{\Pi}^{(\epsilon)}\Gamma_{f_z^{(\epsilon)}}\tau)(z) \quad (|\tau|_\mathfrak{s} + 2 - |k|_\mathfrak{s} > 0).$$

Lemma 4.1. *Let $\Pi_z^{(\epsilon)} = \mathbf{\Pi}^{(\epsilon)}\Gamma_{f_z^{(\epsilon)}}$ and $\Gamma_{z\bar{z}}^{(\epsilon)} = (\Gamma_{f_z^{(\epsilon)}})^{-1}\Gamma_{f_{\bar{z}}^{(\epsilon)}}$. Then $Z^{(\epsilon)} = (\Pi^{(\epsilon)}, \Gamma^{(\epsilon)})$ is an admissible and periodic model.*

4.1 Renormalization procedure

We introduce a renormalization of $Z^{(\epsilon)}$ following Section 8.3 of [3]. Let $\mathcal{F}_0 \subset \mathcal{F}$ be a subset such that

1. $\{\tau \in \mathcal{F}\,;\, |\tau|_\mathfrak{s} \le 0\} \subset \mathcal{F}_0$.
2. There exists a subset $\mathcal{F}_* \subset \mathcal{F}_0$ such that $\Delta\mathcal{F}_0 \subset \langle \mathcal{F}_0 \rangle \otimes \langle \mathcal{F}_0^+ \rangle$, where $\mathcal{F}_0^+$ is the minimal subset of $\mathcal{F}^+$ such that

$$(1)\ \mathbf{1}, X_0, X_1 \in \mathcal{F}_0^+, \quad (2)\ \tau, \bar{\tau} \in \mathcal{F}_0^+ \Rightarrow \tau\bar{\tau} \in \mathcal{F}_0^+,$$
$$(3)\ \tau \in \mathcal{F}_* \setminus \{X^l\,;\, l \in \mathbb{Z}_+^2\}, k \in \mathbb{Z}_+^2, |\tau|_\mathfrak{s} + 2 - |k|_\mathfrak{s} > 0 \Rightarrow \mathcal{J}_k\tau \in \mathcal{F}_0^+.$$

We write $\mathcal{H}_0 = \langle \mathcal{F}_0 \rangle$ and $\mathcal{H}_0^+ = \langle \mathcal{F}_0^+ \rangle$.

Let $M : \mathcal{H}_0 \to \mathcal{H}_0$ be a linear map such that $M\mathcal{I}_k\tau = \mathcal{I}_k M\tau$ for every $\tau \in \mathcal{F}_0$ such that $\mathcal{I}_k\tau \in \mathcal{F}_0$, and $MX^k = X^k$. Then two linear maps $\hat{M} : \mathcal{H}_0^+ \to \mathcal{H}_0^+$ and $\Delta^M : \mathcal{H}_0 \to \mathcal{H}_0 \otimes \mathcal{H}_0^+$ are uniquely determined by the following properties.

1. $\hat{M}\mathcal{J}_k\tau = \mathcal{M}(\mathcal{J}_k \otimes \mathrm{id})\Delta^M\tau$, $\hat{M}X^k = X^k$,
2. $\hat{M}(\tau\bar{\tau}) = (\hat{M}\tau)(\hat{M}\bar{\tau})$,
3. $(\mathrm{id} \otimes \mathcal{M})(\Delta \otimes \mathrm{id})\Delta^M\tau = (M \otimes \hat{M})\Delta\tau$

(Proposition 8.36 of [3]). Furthermore, the linear map $\hat{\Delta}^M : \mathcal{H}_0^+ \to \mathcal{H}_0^+ \otimes \mathcal{H}_0^+$ is defined by

$$(\mathcal{A}\hat{M}\mathcal{A} \otimes \hat{M})\Delta^+ = (\mathrm{id} \otimes \mathcal{M})(\Delta^+ \otimes \mathrm{id})\hat{\Delta}^M,$$

since $(\mathrm{id} \otimes \mathcal{M})(\Delta^+ \otimes \mathrm{id})$ is invertible on $\mathcal{H}_0^+ \otimes \mathcal{H}_0^+$ from the definition of Δ^+.

Lemma 4.2 (Theorem 8.44 of [3])**.** *Let $\mathcal{F}_0$ and M as above. Assume that for every $\tau \in \mathcal{F}_0$ and $\hat{\tau} \in \mathcal{F}_0^+$ we can write*

$$\Delta^M\tau = \tau \otimes \mathbf{1} + \sum_{|\tau^{(1)}|_s > |\tau|_s} \tau^{(1)} \otimes \tau^{(2)}, \quad \hat{\Delta}^M\hat{\tau} = \hat{\tau} \otimes \mathbf{1} + \sum_{|\hat{\tau}^{(1)}|_s > |\hat{\tau}|_s} \hat{\tau}^{(1)} \otimes \hat{\tau}^{(2)}.$$

Then for every admissible model $(\mathbf{\Pi}, f)$ on $\mathcal{T}^{(r)}$, the maps $\mathbf{\Pi}^M : \mathcal{H}_0 \to \mathcal{S}'(\mathbb{R}^2)$ and $f_z^M : \mathcal{H}_0^+ \to \mathbb{R}$ defined by

$$\mathbf{\Pi}^M = \mathbf{\Pi}M, \quad f_z^M = f_z\hat{M}$$

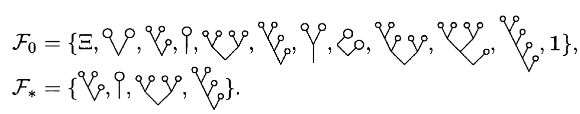
are uniquely extended to an admissible model $Z^M = (\Pi^M, \Gamma^M)$ on $\mathcal{T}^{(r)}$.

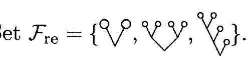
4.2 Renormalization map

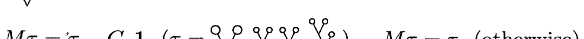
We construct renormalization maps when $0 \le \gamma < \frac{1}{4}$. We write each element of $\mathcal{F}_0$ as a graph with one root. We draw a circle to represent Ξ. In order to represent $\mathcal{I}'(\tau)$, we draw a downward line starting at the root of τ. Then the root of $\mathcal{I}'(\tau)$ is another vertex of this downward line, which is not the root of τ. In order to represent $\tau\bar{\tau}$, we joint the trees τ and $\bar{\tau}$ at their roots. For example,

4.2.1 $0 \le \gamma < \frac{1}{6}$

Provided that $\alpha_0 \in (-\frac{3}{2} - \frac{1}{6}, -\frac{3}{2} - \gamma)$, it is sufficient to set



Set  For constants C_τ $(\tau \in \mathcal{F}_{\mathrm{re}})$, we define a linear map $M : \mathcal{H}_0 \to \mathcal{H}_0$ by

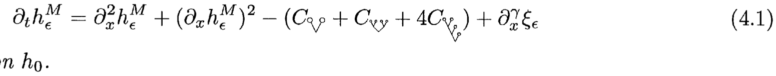


Lemma 4.3. *M satisfies the condition of Lemma 4.2. For every $\tau \in \mathcal{F}_0$, the renormalized model $Z^{(\epsilon),M} = (Z^{(\epsilon)})^M$ satisfies*

$$\Pi_z^{(\epsilon),M}\tau = \Pi_z^{(\epsilon)}M\tau.$$

proof. See Lemma 4.3 of [5]. □

Proposition 4.4. *Let $\mathcal{S} : (h_0, Z) \mapsto H$ be the solution map defined in Theorem 3.7. Then $h_\epsilon^M = \mathcal{RS}(h_0, Z^{(\epsilon),M})$ solves the equation*



with the initial condition h_0.

proof. By the definition of $\mathcal{T}^{(r)}$, every solution $H \in \mathcal{D}_P^{\theta,\eta}(U)$ to (3.1) with $\theta > -\alpha_0$ has the following form.

$$H = h\mathbf{1} + \mathcal{I}(\Xi) + \mathcal{I}([\text{tree}]) + h'X_1 + 2\mathcal{I}([\text{tree}]) + 2h'\mathcal{I}([\text{tree}]) + \mathcal{I}([\text{tree}]) + 4\mathcal{I}([\text{tree}]).$$

Here h and h' are functions on $\mathbb{R}^2$. Then ∂H and $(\partial H)^2 + \Xi$ are given by

$$\partial H = [\text{tree}] + [\text{tree}] + h'\mathbf{1} + 2\mathcal{I}'([\text{tree}]) + 2h'\mathcal{I}'([\text{tree}]) + \mathcal{I}'([\text{tree}]) + 4\mathcal{I}'([\text{tree}]),$$
$$(\partial H)^2 + \Xi = \Xi + [\text{tree}] + 2[\text{tree}] + 2h'[\text{tree}] + [\text{tree}] + 4[\text{tree}] + 2h'[\text{tree}] + 4h'[\text{tree}] + 4[\text{tree}] + 2[\text{tree}] + 8[\text{tree}].$$

From these representations, we obtain

$$M(\partial H) = \partial H, \quad M((\partial H)^2 + \Xi) = (\partial H)^2 + \Xi - (C_{[\text{tree}]} + C_{[\text{tree}]} + 4C_{[\text{tree}]}).$$

Hence we have

$$M((\partial H)^2 + \Xi) = (M\partial H)^2 + \Xi - (C_{[\text{tree}]} + C_{[\text{tree}]} + 4C_{[\text{tree}]}).$$

From Remark 3.15 of [3] and Lemma 4.3, for the reconstruction operator $\mathcal{R}$ of $\hat{Z}^{(\epsilon),M}$ we have

$$\mathcal{R}((\partial H)^2 + \Xi) = (\partial_x \mathcal{R}H)^2 + \partial_x^\gamma \xi_\epsilon - (C_{[\text{tree}]} + C_{[\text{tree}]} + 4C_{[\text{tree}]}).$$

Let $H = \mathcal{S}(h_0, Z^{(\epsilon),M})$. Applying $\mathcal{R}^{(\epsilon),M}$ to both sides of (3.1), we have

$$\mathcal{R}H = G * \{\mathbf{1}_{t>0}((\partial_x \mathcal{R}H)^2 + \partial_x^\gamma \xi_\epsilon - (C_{[\text{tree}]} + C_{[\text{tree}]} + 4C_{[\text{tree}]}))\} + Gh_0.$$

Hence $h_\epsilon^M = \mathcal{R}H$ solves (4.1). □

The goal of this paper is the following theorem.

Theorem 4.5. *Let C_τ $(\tau \in \mathcal{F}_{re})$ be constants defined in Section 5 of [5]. Let $\hat{Z}^{(\epsilon)} = Z^{(\epsilon),M^{(\epsilon)}}$ be the corresponding renormalized model. Then there exists an admissible and periodic random model $\hat{Z}$ and $\kappa > 0$, such that for every $\zeta > 0$, compact set $\mathfrak{K}$ and $p \geq 1$, we have the bounds*

$$\mathbb{E}\|\hat{Z}\|_{\zeta;\mathfrak{K}}^p \lesssim 1, \quad \mathbb{E}\|\hat{Z}^{(\epsilon)}; \hat{Z}\|_{\zeta;\mathfrak{K}}^p \lesssim \epsilon^{\kappa p}.$$

4.2.2 $\frac{1}{6} \leq \gamma < \frac{1}{4}$

Provided $\alpha_0 \in (-\frac{3}{2} - \frac{1}{4}, -\frac{3}{2} - \gamma)$, it is sufficient to set

$$\mathcal{F}_0 = \{\Xi, [\text{tree}], [\text{tree}], \ldots, [\text{tree}], \mathbf{1}\},$$
$$\mathcal{F}_* = \{[\text{tree}], [\text{tree}], \ldots, [\text{tree}]\}.$$

Set $\mathcal{F}_{\text{re}} = \{[\text{tree}], \ldots, [\text{tree}]\}$. For constants C_τ $(\tau \in \mathcal{F}_{\text{re}})$, we define a linear map $M : \mathcal{H}_0 \to \mathcal{H}_0$ by

$$M\tau = \tau - C_\tau \quad (\tau \in \mathcal{F}_{\text{re}}), \quad M\tau = \tau \quad (\text{otherwise}).$$

The following results are obtained similarly to Lemma 4.3 and Proposition 4.4.

Lemma 4.6. *M satisfies the condition of Lemma 4.2. For every $\tau \in \mathcal{F}_0$, the renormalized model $Z^{(\epsilon),M}$ satisfies*

$$\Pi_z^{(\epsilon),M}\tau = \Pi_z^{(\epsilon)} M\tau.$$

Proposition 4.7. *$h_\epsilon^M = \mathcal{RS}(h_0, Z^{(\epsilon),M})$ solves the equation*

$$\partial_t h_\epsilon^M = \partial_x^2 h_\epsilon^M + (\partial_x h_\epsilon^M)^2 - (C_{[\text{tree}]} + C_{[\text{tree}]} + 4C_{[\text{tree}]} + 4C_{[\text{tree}]} + 2C_{[\text{tree}]} + 8C_{[\text{tree}]} + 8C_{[\text{tree}]} + 4C_{[\text{tree}]} + 16C_{[\text{tree}]}) + \partial_x^\gamma \xi_\epsilon$$

with the initial condition h_0.

The goal of this paper is the following theorem.

Theorem 4.8. *Let $C_\tau^{(\epsilon)}$ ($\tau \in \mathcal{F}_{\rm re}$) be constants defined in Section 7 of [5]. Let $\hat{Z}^{(\epsilon)} = Z^{(\epsilon),M^{(\epsilon)}}$ be the corresponding renormalized model. Then there exists an admissible and periodic random model $\hat{Z}$ and $\kappa > 0$, such that for every $\zeta > 0$, compact set $\mathfrak{K}$ and $p \geq 1$, we have the bounds*

$$\mathbb{E}\|\hat{Z}\|_{\zeta;\mathfrak{K}}^p \lesssim 1, \quad \mathbb{E}\|\hat{Z}^{(\epsilon)}; \hat{Z}\|_{\zeta;\mathfrak{K}}^p \lesssim \epsilon^{\kappa p}.$$

4.3 Wiener chaos expansion

The processes $\Pi_z\tau$ belong to Wiener chaos constructed from the space-time white noise $\{W_h\}$ on $\mathbb{R} \times \mathbb{T}$. We denote by $I_1 : H = L^2(\mathbb{R} \times \mathbb{T}) \to L^2(\Omega, \mathbb{P})$ a linear isometry defined by $I_1(h) = W_h$. We also denote by $I_k : H^{\otimes k}(\simeq L^2((\mathbb{R} \times \mathbb{T})^k)) \to L^2(\Omega, \mathbb{P})$ the k-th stochastic integral defined in Theorem 7.25 of [7].

For each $\tau \in \mathcal{F}$, we can define the number $\|\tau\|$ by

$$\|\mathbf{1}\| = \|X_0\| = \|X_1\| = 0, \quad \|\Xi\| = 1, \quad \|\tau\bar{\tau}\| = \|\tau\| + \|\bar{\tau}\|, \quad \|\mathcal{I}_k\tau\| = \|\tau\|.$$

Lemma 4.9. *Let $C_\tau^{(\epsilon)}$ ($\tau \in \mathcal{F}_{\rm re}$) be constants defined in Sections 5 and 7 of [5]. For each $\tau \in \mathcal{F}_- := \{\tau \in \mathcal{F}\,; |\tau|_\mathfrak{s} \leq 0\}$ and $z \in \mathbb{R}^2$, there exists a family of functions $\{\hat{\mathcal{W}}^{(\epsilon,k)}(\tau)(z) = \hat{\mathcal{W}}^{(\epsilon,k)}(\tau)(z; w_1, \ldots, w_k) \in H^{\otimes k}\}_{k \in \|\tau\| - 2\mathbb{Z}_+, k \geq 0}$ such that*

$$(\hat{\Pi}_0^{(\epsilon)}\tau)(z) = \sum_k I_k(\hat{\mathcal{W}}^{(\epsilon,k)}(\tau)(z)). \tag{4.2}$$

proof. For Ξ, by the definition of $\partial_x^\gamma \xi_\epsilon$ we have

$$\begin{aligned}(\hat{\Pi}_0^{(\epsilon)}\Xi)(z) &= \partial_x^\gamma \xi_\epsilon(z) = \partial_x^\gamma \xi(\rho_\epsilon(z - \cdot)) = \partial_x^\gamma W(\pi\rho_\epsilon(z - \cdot)) \\ &= W(\partial_x^\gamma(\mathbb{T})\pi\rho_\epsilon(z - \cdot)) = W(\pi\partial_x^\gamma(\mathbb{R})\rho_\epsilon(z - \cdot)).\end{aligned}$$

Here $\partial_x^\gamma(\mathbb{T})$ and $\partial_x^\gamma(\mathbb{R})$ denote the fractional derivatives defined in $\mathbb{T}$ and $\mathbb{R}$, respectively. Hence

$$(\hat{\Pi}_0^{(\epsilon)}\Xi)(z) = I_1(\hat{\mathcal{W}}^{(\epsilon,1)}(\Xi)(z)), \quad \hat{\mathcal{W}}^{(\epsilon,1)}(\Xi)(z; w) = \pi\partial_x^\gamma \rho_\epsilon(z - w).$$

If τ satisfies (4.2), it is clear that $\mathcal{I}\tau$ and $\mathcal{I}'\tau$ also. Regarding the product of two variables, we can use Theorem 7.33 of [7]. □

In order to prove Theorems 4.5 and 4.8, it is sufficient to obtain estimates in the following theorem.

Theorem 4.10 (Theorem 10.7 of [3]). *For the renormalized model $\hat{Z}^{(\epsilon)}$ in Theorems 4.5 and 4.8, assume that there exist $\kappa > 0$ and $\iota > 0$, such that for every $\varphi \in \mathcal{B}_0^2$ and $\tau \in \mathcal{F}_-$ there exists a random variable $(\hat{\Pi}_0\tau)(\varphi)$ belonging to the inhomogeneous Wiener chaos of order $\|\tau\|$ such that*

$$\mathbb{E}|(\hat{\Pi}_0\tau)(\varphi_0^\lambda)|^2 \lesssim \lambda^{2|\tau|_\mathfrak{s}+\iota}, \quad \mathbb{E}|(\hat{\Pi}_0\tau - \hat{\Pi}_0^{(\epsilon)}\tau)(\varphi_0^\lambda)|^2 \lesssim \epsilon^{2\kappa}\lambda^{2|\tau|_\mathfrak{s}+\iota}.$$

uniformly over $0 < \lambda < C$. Then there exists a unique admissible and periodic random model $\hat{Z}$, such that for every $\zeta > 0$, compact set $\mathfrak{K} \subset \mathbb{R}^2$ and $p \geq 1$ we have

$$\mathbb{E}\|\hat{Z}\|_{\zeta;\mathfrak{K}}^p \lesssim 1, \quad \mathbb{E}\|\hat{Z}; \hat{Z}^{(\epsilon)}\|_{\zeta;\mathfrak{K}}^p \lesssim \epsilon^{\kappa p}.$$

Theorem 4.11. *Let $0 \leq \gamma < \frac{1}{4}$. For every $\tau \in \mathcal{F}_-$, k, l, and $\varphi \in \mathcal{B}_0^2$, the function*

$$\int_{\mathbb{R}^2} \varphi(z)\hat{\mathcal{W}}_l^{(k)}(\tau)(z;\cdot)dz$$

belongs to $H^{\otimes k}$. If we define random variables by

$$(\hat{\Pi}_0\tau)(\varphi) = \sum_k \mathcal{I}_k \left(\sum_l \int \varphi(u)\hat{\mathcal{W}}_l^{(k)}(\tau)(u)du \right),$$

then there exist $\iota, \kappa > 0$ such that we have

$$\mathbb{E}|(\hat{\Pi}_0\tau)(\varphi_0^\lambda)| \lesssim \lambda^{2|\tau|_\mathfrak{s}+\iota}, \quad \mathbb{E}|(\hat{\Pi}_0\tau - \hat{\Pi}_0^{(\epsilon)}\tau)(\varphi_0^\lambda)| \lesssim \epsilon^{2\kappa}\lambda^{2|\tau|_\mathfrak{s}+\iota}$$

uniformly over $0 < \lambda < C$.

proof. For $\tau = \Xi, \mathbf{1}$, by definitions we have

$$(\hat{\Pi}_0\Xi)(\varphi) = \partial_x^\gamma \xi(\varphi), \quad (\hat{\Pi}_0^{(\epsilon)}\Xi)(\varphi) = \int_{\mathbb{R}^2} \partial_x^\gamma \xi_\epsilon(z)\varphi(z)dz,$$
$$(\hat{\Pi}_0\mathbf{1})(\varphi) = (\hat{\Pi}_0^{(\epsilon)}\mathbf{1})(\varphi) = \int_{\mathbb{R}^2} \varphi(z)dz.$$

Then the required bounds are trivial for $\mathbf{1}$, and obtained as a result of Lemma 2.2 for Ξ.
For $\tau \neq \Xi, \mathbf{1}$, from Theorem 7.26 of [7] we have

$$\begin{aligned}\mathbb{E}|(\hat{\Pi}_0\tau)(\varphi_0^\lambda)|^2 &\lesssim \sum_{k,l} \left\| \int_{\mathbb{R}^2} \varphi_0^\lambda(z)\hat{\mathcal{W}}_l^{(k)}(\tau)(z)dz \right\|_{H^{\otimes k}}^2 \\ &= \sum_{k,l} \iint_{\mathbb{R}^2\times\mathbb{R}^2} \varphi_0^\lambda(z)\varphi_0^\lambda(\bar{z})(\hat{\mathcal{W}}_l^{(k)}(\tau)(z), \hat{\mathcal{W}}_l^{(k)}(\tau)(\bar{z}))_{H^{\otimes k}} dz d\bar{z},\end{aligned}$$

and

$$\begin{aligned}&\mathbb{E}|(\hat{\Pi}_0\tau - \hat{\Pi}_0^{(\epsilon)}\tau)(\varphi_0^\lambda)|^2 \\ &\lesssim \sum_{k,l} \iint_{\mathbb{R}^2\times\mathbb{R}^2} \varphi_0^\lambda(z)\varphi_0^\lambda(\bar{z})(\delta\hat{\mathcal{W}}_l^{(\epsilon,k)}(\tau)(z), \delta\hat{\mathcal{W}}_l^{(\epsilon,k)}(\tau)(\bar{z}))_{H^{\otimes k}} dz d\bar{z},\end{aligned}$$

where $\delta\hat{\mathcal{W}}_l^{(\epsilon,k)}(\tau) = \hat{\mathcal{W}}_l^{(k)}(\tau) - \hat{\mathcal{W}}_l^{(\epsilon,k)}(\tau)$. We write

$$\mathcal{P}_l^{(k)}(\tau)(z;\bar{z}) = (\hat{\mathcal{W}}_l^{(k)}(\tau)(z), \hat{\mathcal{W}}_l^{(k)}(\tau)(\bar{z}))_{H^{\otimes k}},$$
$$\mathcal{P}_l^{(\epsilon,k)}(\tau)(z;\bar{z}) = (\delta\hat{\mathcal{W}}_l^{(\epsilon,k)}(\tau)(z), \delta\hat{\mathcal{W}}_l^{(\epsilon,k)}(\tau)(\bar{z}))_{H^{\otimes k}}.$$

In Sections 5-8 of [5], we prove estimates

$$|\mathcal{P}_l^{(k)}(\tau)(z;\bar{z})| \lesssim \begin{cases} \|z-\bar{z}\|_\mathfrak{s}^{2|\tau|_\mathfrak{s}+\iota} & \text{or} \\ \sum_{\zeta_1,\zeta_2>0} \|z\|_\mathfrak{s}^{\zeta_1}\|\bar{z}\|_\mathfrak{s}^{\zeta_2}\|z-\bar{z}\|_\mathfrak{s}^{2|\tau|_\mathfrak{s}-\zeta_1-\zeta_2+\iota} & \text{or} \\ \|z\|_\mathfrak{s}^{|\tau|_\mathfrak{s}+\iota/2}\|\bar{z}\|_\mathfrak{s}^{|\tau|_\mathfrak{s}+\iota/2} & \end{cases}$$

and

$$|\mathcal{P}_l^{(\epsilon,k)}(\tau)(z;\bar{z})| \lesssim \begin{cases} \epsilon^\kappa\|z-\bar{z}\|_\mathfrak{s}^{2|\tau|_\mathfrak{s}+\iota} & \text{or} \\ \epsilon^\kappa\sum_{\zeta_1,\zeta_2>0} \|z\|_\mathfrak{s}^{\zeta_1}\|\bar{z}\|_\mathfrak{s}^{\zeta_2}\|z-\bar{z}\|_\mathfrak{s}^{2|\tau|_\mathfrak{s}-\zeta_1-\zeta_2+\iota} & \text{or} \\ \epsilon^\kappa\|z\|_\mathfrak{s}^{|\tau|_\mathfrak{s}+\iota/2}\|\bar{z}\|_\mathfrak{s}^{|\tau|_\mathfrak{s}+\iota/2} & \end{cases}$$

uniformly over $\epsilon > 0$ and $z, \bar{z} \in B_{\mathfrak{s}}(0, C)$ with some $\kappa, \iota > 0$. Here the sums run over finitely many ζ_1 and ζ_2. Since it turns out that all of negative indexes which appear in the above bounds are greater than -3, they are integrable around $0 \in \mathbb{R}^2$ and we have

$$\iint_{\mathbb{R}^2\times\mathbb{R}^2} \varphi_0^\lambda(z)\varphi_0^\lambda(\bar{z})\mathcal{P}_l^{(k)}(\tau)(z;\bar{z})dzd\bar{z} \lesssim \lambda^{2|\tau|_{\mathfrak{s}}+\iota},$$
$$\iint_{\mathbb{R}^2\times\mathbb{R}^2} \varphi_0^\lambda(z)\varphi_0^\lambda(\bar{z})\mathcal{P}_l^{(\epsilon,k)}(\tau)(z;\bar{z})dzd\bar{z} \lesssim \epsilon^{\kappa}\lambda^{2|\tau|_{\mathfrak{s}}+\iota}.$$

□

4.4 Notes on the case $\gamma \geq \frac{1}{4}$

In order to obtain the required estimates as in Theorem 4.11, all of labels in the graphs must be greater than -3. If $\gamma \geq \frac{1}{4}$, we cannot define the limit of $\hat{\Pi}_0^{(\epsilon)}{}^{\circ}\!\vee^{\circ}$ by the renormalization described above. From the definition of ${}^{\circ}\!\vee^{\circ}$, we have

$$(\Pi_0^{(\epsilon)}{}^{\circ}\!\vee^{\circ})(z) = I_2(\mathcal{W}^{(\epsilon,2)}({}^{\circ}\!\vee^{\circ})(z)) + \mathcal{W}^{(\epsilon,0)}({}^{\circ}\!\vee^{\circ})(z),$$

where

$$\mathcal{W}^{(\epsilon,2)}({}^{\circ}\!\vee^{\circ})(z;w_1,w_2) = \pi\partial_x^\gamma K_\epsilon'(z-w_1)\pi\partial_x^\gamma K_\epsilon'(z-w_2),$$
$$\mathcal{W}^{(\epsilon,0)}({}^{\circ}\!\vee^{\circ})(z) = \int_{w\in\mathbb{R}\times[0,1)} (\pi\partial_x^\gamma K_\epsilon'(z-w))^2 dw.$$

Since the constant term diverges, we define $C_{{}^{\circ}\!\vee^{\circ}}^{(\epsilon)} = \mathcal{W}^{(\epsilon,0)}({}^{\circ}\!\vee^{\circ})(z)$ and

$$(\hat{\Pi}_0^{(\epsilon)}{}^{\circ}\!\vee^{\circ})(z) = I_2(\hat{\mathcal{W}}^{(\epsilon,2)}({}^{\circ}\!\vee^{\circ})(z)), \quad \hat{\mathcal{W}}^{(\epsilon,2)}({}^{\circ}\!\vee^{\circ})(z) = \mathcal{W}^{(\epsilon,2)}({}^{\circ}\!\vee^{\circ})(z).$$

If $\gamma < \frac{1}{4}$, we can define $\hat{\Pi}_0{}^{\circ}\!\vee^{\circ}(z)$ by

$$(\hat{\Pi}_0{}^{\circ}\!\vee^{\circ})(z) = I_2(\hat{\mathcal{W}}^{(2)}({}^{\circ}\!\vee^{\circ})(z;\cdot,\cdot)),$$
$$\hat{\mathcal{W}}^{(2)}({}^{\circ}\!\vee^{\circ})(z;w_1,w_2) = \pi\partial_x^\gamma K'(z-w_1)\pi\partial_x^\gamma K'(z-w_2),$$

because we have the bound

$$\mathcal{P}^{(2)}({}^{\circ}\!\vee^{\circ})(z;\bar{z}) \lesssim \|z-\bar{z}\|_{\mathfrak{s}}^{-2-4\gamma}$$

and $-2-4\gamma > -3$. See Section 5.

On the other hand, if $\gamma \geq \frac{1}{4}$ we cannot use this estimate. In fact, for every test function φ we obtain the following estimate.

$$(\hat{\Pi}_0^{(\epsilon,2)}{}^{\circ}\!\vee^{\circ})(\varphi) \sim \begin{cases} |\log\epsilon| & \gamma = \frac{1}{4} \\ \epsilon^{1-4\gamma} & \gamma > \frac{1}{4}. \end{cases}$$

Now we prove it. Let $G_\epsilon = G * \rho_\epsilon$. In $z, \bar{z} \in B_{\mathfrak{s}}(0, \frac{1}{2})$, $\mathcal{P}^{(\epsilon,2)}({}^{\circ}\!\vee^{\circ})(z;\bar{z})$ is given by

$$\begin{aligned}\mathcal{P}^{(\epsilon,2)}({}^{\circ}\!\vee^{\circ})(z;\bar{z}) &\sim \int_{w_1,w_2\in\mathbb{R}\times[0,1)} \pi\partial_x^\gamma G_\epsilon'(z-w_1)\pi\partial_x^\gamma G_\epsilon'(z-w_2) \\ &\qquad\qquad \times \pi\partial_x^\gamma G_\epsilon'(\bar{z}-w_1)\pi\partial_x^\gamma G_\epsilon'(\bar{z}-w_2)dw_1dw_2 \\ &\sim (\partial_x^{2\gamma}G * \rho_\epsilon * \overleftarrow{\rho_\epsilon})(|t-\bar{t}|, x-\bar{x})^2 =: I_\epsilon(z-\bar{z}),\end{aligned}$$

since $G' * \overleftarrow{G'}(t,x) = -\frac{1}{2}G(|t|,x)$. From the scaling property of G, we have $I_\epsilon(t,x) = \epsilon^{-2-4\gamma}I_1(\epsilon^{-2}t, \epsilon^{-1}x)$. Then we have

$$\begin{aligned}(\hat{\Pi}_0^{(\epsilon,2)}{}^{\circ}\!\vee^{\circ})(\varphi) &\sim \int_{\mathbb{R}^2\times\mathbb{R}^2} \varphi(z)\varphi(\bar{z})I_\epsilon(z-\bar{z})dzd\bar{z} \\ &= \epsilon^{1-4\gamma}\int_{\mathbb{R}^2} \epsilon^3\varphi(\epsilon^2 t, \epsilon x)\left(\int_{\mathbb{R}^2} \varphi(\epsilon^2\bar{t}, \epsilon\bar{x})I_1(t-\bar{t}, x-\bar{x})d\bar{t}d\bar{x}\right)dtdx\end{aligned}$$

$$\sim \epsilon^{1-4\gamma} \int_{\|z\|_{\mathfrak{s}} \lesssim \epsilon^{-1}} I_1(z)dz.$$

Since there exists $\psi \in \mathcal{S}(\mathbb{R})$ such that

$$I_1(t,x) \sim \frac{1}{t^{1+2\gamma}} \left(\partial_x^{2\gamma} \psi \left(\frac{x}{\sqrt{t}} \right) \right)^2$$

for $\|z\|_{\mathfrak{s}} > 1$, we have

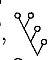
$$\int_{\|z\|_{\mathfrak{s}} \lesssim \epsilon^{-1}} I_1(z)dz \sim \int_{|t| \lesssim \epsilon^{-2}} 1 \wedge t^{-\frac{1}{2}-2\gamma} dt \sim \begin{cases} |\log \epsilon| & \epsilon = \frac{1}{4} \\ 1 & \epsilon > \frac{1}{4}. \end{cases}$$

5 Renormalization in $0 \le \gamma < \frac{1}{4}$

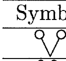
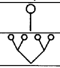
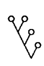
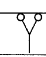
In this section, we show the bounds of $\mathcal{P}(\tau)(z;\bar{z})$ and $C_\tau^{(\epsilon)}$. We omit the proof because it is very long. See Sections 5-8 of [5].

5.1 Renormalization in $0 \le \gamma < \frac{1}{10}$

Let $0 \le \gamma < \frac{1}{10}$. We choose sufficiently small $\kappa > 0$ and set $\alpha_0 = -\frac{3}{2} - \gamma - \kappa$. Then all elements in $\mathcal{F}_-$ are as follows.

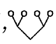
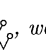
Homogeneity	Symbol
$-\frac{3}{2} - \gamma - \kappa$	Ξ
$-1 - 2\gamma - 2\kappa$	
$-\frac{1}{2} - 3\gamma - 3\kappa$	
$-\frac{1}{2} - \gamma - \kappa$	
$-4\gamma - 4\kappa$	, 
$-2\gamma - 2\kappa$	, 
0	$\mathbf{1}$

Proposition 5.1. *For each $\tau \in \mathcal{F}_-$, we have the bounds of $\mathcal{P}(z;\bar{z})$ and $\mathcal{P}^{(\epsilon)}(z;\bar{z})$ as follows.*

Symbol	$\mathcal{P}(z;\bar{z})$	$\mathcal{P}^{(\epsilon)}(z;\bar{z})$
	$\|z-\bar{z}\|_{\mathfrak{s}}^{-2-4\gamma}$	$\epsilon^\theta \|z-\bar{z}\|_{\mathfrak{s}}^{-2-4\gamma-\theta}$
	$\|z-\bar{z}\|_{\mathfrak{s}}^{-1-6\gamma}$	$\epsilon^\theta \|z-\bar{z}\|_{\mathfrak{s}}^{-1-6\gamma-\theta}$
	$\|z-\bar{z}\|_{\mathfrak{s}}^{-1-2\gamma}$	$\epsilon^\theta \|z-\bar{z}\|_{\mathfrak{s}}^{-1-2\gamma-\theta}$
	$\|z-\bar{z}\|_{\mathfrak{s}}^{-8\gamma}$	$\epsilon^\theta \|z-\bar{z}\|_{\mathfrak{s}}^{-8\gamma-\theta}$
	$\|z\|_{\mathfrak{s}}^{\frac{1}{2}-3\gamma-\theta} \|\bar{z}\|_{\mathfrak{s}}^{\frac{1}{2}-3\gamma-\theta}$ $\times \|z-\bar{z}\|_{\mathfrak{s}}^{-1-2\gamma}$, $\|z-\bar{z}\|_{\mathfrak{s}}^{-8\gamma}$, $\|z\|_{\mathfrak{s}}^{-4\gamma-\theta} \|\bar{z}\|_{\mathfrak{s}}^{-4\gamma-\theta}$	$\epsilon^\theta \|z\|_{\mathfrak{s}}^{\frac{1}{2}-3\gamma-\theta_1} \|\bar{z}\|_{\mathfrak{s}}^{\frac{1}{2}-3\gamma-\theta_1}$ $\times \|z-\bar{z}\|_{\mathfrak{s}}^{-1-2\gamma-\theta_2}$, $\epsilon^\theta \|z-\bar{z}\|_{\mathfrak{s}}^{-8\gamma-\theta}$, $\epsilon^\theta \|z\|_{\mathfrak{s}}^{-4\gamma-\theta_1} \|\bar{z}\|_{\mathfrak{s}}^{-4\gamma-\theta_1}$
	$\|z-\bar{z}\|_{\mathfrak{s}}^{-4\gamma}$	$\epsilon^\theta \|z-\bar{z}\|_{\mathfrak{s}}^{-4\gamma-\theta}$

	$\|z\|_{\mathfrak{s}}^{\frac{1}{2}-\gamma-\theta}\|\bar{z}\|_{\mathfrak{s}}^{\frac{1}{2}-\gamma-\theta}$ $\times\ \|z-\bar{z}\|_{\mathfrak{s}}^{-1-2\gamma}$, $\|z\|_{\mathfrak{s}}^{-2\gamma}\|\bar{z}\|_{\mathfrak{s}}^{-2\gamma}$	$\epsilon^{\theta}\|z\|_{\mathfrak{s}}^{\frac{1}{2}-\gamma-\theta_1}\|\bar{z}\|_{\mathfrak{s}}^{\frac{1}{2}-\gamma-\theta_1}$ $\times\ \|z-\bar{z}\|_{\mathfrak{s}}^{-1-2\gamma-\theta_2}$, $\epsilon^{\theta}\|z\|_{\mathfrak{s}}^{-2\gamma-\theta}\|\bar{z}\|_{\mathfrak{s}}^{-2\gamma-\theta}$

Here $\theta, \theta_1, \theta_2 > 0$ are any sufficiently small numbers.

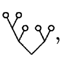
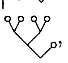
Proposition 5.2. *For* ,  *and* , *we have*

$$C^{(\epsilon)}_{\text{[tree]}} \sim \epsilon^{-1-2\gamma}, \quad |C^{(\epsilon)}_{\text{[tree]}}| \lesssim \epsilon^{-4\gamma}, \quad |C^{(\epsilon)}_{\text{[tree]}}| \lesssim \epsilon^{-4\gamma}.$$

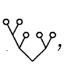
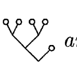
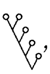
  

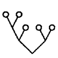
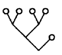
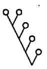
5.2 Renormalization in $\frac{1}{10} \le \gamma < \frac{1}{6}$

Let $\frac{1}{10} \le \gamma < \frac{1}{6}$. Then all elements in $\mathcal{F}_-$ are as follows.

Homogeneity	Symbol
$-\frac{3}{2}-\gamma-\kappa$	Ξ
$-1-2\gamma-2\kappa$	
$-\frac{1}{2}-3\gamma-3\kappa$	
$-\frac{1}{2}-\gamma-\kappa$	
$-4\gamma-4\kappa$	, 
$-2\gamma-2\kappa$	, 
$\frac{1}{2}-5\gamma-5\kappa$	, , 
0	**1**

It remains to obtain the bounds of elements with homogeneity $\frac{1}{2}-5\gamma-5\kappa$. The other bounds are same as before.

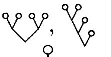
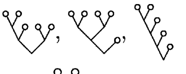
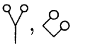
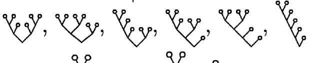
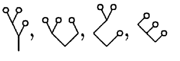
Proposition 5.3. *For* ,  *and* , *we have the bounds of $\mathcal{P}(z;\bar{z})$ as follows.*

Symbol	$\mathcal{P}(z;\bar{z})$
	$\|z\|_{\mathfrak{s}}^{\frac{1}{2}-3\gamma-\theta}\|\bar{z}\|_{\mathfrak{s}}^{\frac{1}{2}-3\gamma-\theta}\|z-\bar{z}\|_{\mathfrak{s}}^{-4\gamma}$, $\|z-\bar{z}\|_{\mathfrak{s}}^{1-10\gamma-\theta}$, $\|z\|_{\mathfrak{s}}^{\frac{1}{2}-5\gamma-\theta}\|\bar{z}\|_{\mathfrak{s}}^{\frac{1}{2}-5\gamma-\theta}$
	$\|z\|_{\mathfrak{s}}^{1-4\gamma-\theta}\|\bar{z}\|_{\mathfrak{s}}^{1-4\gamma-\theta}\|z-\bar{z}\|_{\mathfrak{s}}^{-1-2\gamma}$, $\|z-\bar{z}\|_{\mathfrak{s}}^{1-10\gamma-\theta}$, $\|z\|_{\mathfrak{s}}^{\frac{1}{2}-5\gamma-\theta}\|\bar{z}\|_{\mathfrak{s}}^{\frac{1}{2}-5\gamma-\theta}$
	$(\|z\|_{\mathfrak{s}}^{3-14\gamma-\theta}+\|\bar{z}\|_{\mathfrak{s}}^{3-14\gamma-\theta})\|z-\bar{z}\|_{\mathfrak{s}}^{-2+4\gamma}$, $\|z-\bar{z}\|_{\mathfrak{s}}^{1-10\gamma-\theta}$, $\|z\|_{\mathfrak{s}}^{\frac{1}{2}-5\gamma-\theta}\|\bar{z}\|_{\mathfrak{s}}^{\frac{1}{2}-5\gamma-\theta}$

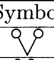
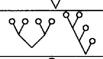
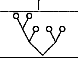
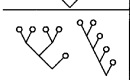
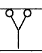
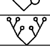
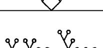
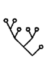
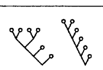
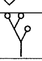
Here θ is any sufficiently small number. The bounds of $\mathcal{P}^{(\epsilon)}(z;\bar{z})$ are obtained by multiplying ϵ^{θ} to above bounds whose indexes are slightly subtracted as in Proposition 5.1.

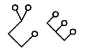
5.3 Renormalization in $\frac{1}{6} \leq \gamma < \frac{3}{14}$

Let $\frac{1}{6} \leq \gamma < \frac{3}{14}$. Then all elements in $\mathcal{F}_-$ are as follows.

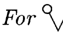
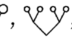
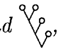
Homogeneity	Symbol
$-\frac{3}{2} - \gamma - \kappa$	Ξ
$-1 - 2\gamma - 2\kappa$	
$-\frac{1}{2} - 3\gamma - 3\kappa$	
$-4\gamma - 4\kappa$	
$-\frac{1}{2} - \gamma - \kappa$	
$\frac{1}{2} - 5\gamma - 5\kappa$	
$-2\gamma - 2\kappa$	
$1 - 6\gamma - 6\kappa$	
$\frac{1}{2} - 3\gamma - 3\kappa$	
0	$\mathbf{1}$

Proposition 5.4. *For each $\tau \in \mathcal{F}_-$, we have the bounds of $\mathcal{P}(z;\bar{z})$ as follows.*

Symbol	$\mathcal{P}(z;\bar{z})$
	$\|z-\bar{z}\|_{\mathfrak{s}}^{-2-4\gamma}$
	$\|z-\bar{z}\|_{\mathfrak{s}}^{-1-6\gamma}$
	$\|z-\bar{z}\|_{\mathfrak{s}}^{-8\gamma}$
	$\|z-\bar{z}\|_{\mathfrak{s}}^{-1-2\gamma}$
	$\|z-\bar{z}\|_{\mathfrak{s}}^{1-10\gamma-\theta}$
	$\|z\|_{\mathfrak{s}}^{1-4\gamma-\theta}\|\bar{z}\|_{\mathfrak{s}}^{1-4\gamma-\theta}\|z-\bar{z}\|_{\mathfrak{s}}^{-1-2\gamma}$, $\|z-\bar{z}\|_{\mathfrak{s}}^{1-10\gamma-\theta}$, $\|z\|_{\mathfrak{s}}^{\frac{1}{2}-5\gamma-\theta}\|\bar{z}\|_{\mathfrak{s}}^{\frac{1}{2}-5\gamma-\theta}$
	$\|z-\bar{z}\|_{\mathfrak{s}}^{-4\gamma}$
	$\|z\|_{\mathfrak{s}}^{\frac{1}{2}-\gamma-\theta}\|\bar{z}\|_{\mathfrak{s}}^{\frac{1}{2}-\gamma-\theta}\|z-\bar{z}\|_{\mathfrak{s}}^{-1-2\gamma}$, $\|z\|_{\mathfrak{s}}^{-2\gamma}\|\bar{z}\|_{\mathfrak{s}}^{-2\gamma}$
	$\|z-\bar{z}\|_{\mathfrak{s}}^{2-12\gamma-\theta}$
	$\|z\|_{\mathfrak{s}}^{1-4\gamma-\theta}\|\bar{z}\|_{\mathfrak{s}}^{1-4\gamma-\theta}\|z-\bar{z}\|_{\mathfrak{s}}^{-4\gamma}$, $\|z-\bar{z}\|_{\mathfrak{s}}^{2-12\gamma-\theta}$, $\|z\|_{\mathfrak{s}}^{1-6\gamma-\theta}\|\bar{z}\|_{\mathfrak{s}}^{1-6\gamma-\theta}$
	$\|z\|_{\mathfrak{s}}^{\frac{3}{2}-5\gamma-\theta}\|\bar{z}\|_{\mathfrak{s}}^{\frac{3}{2}-5\gamma-\theta}\|z-\bar{z}\|_{\mathfrak{s}}^{-1-2\gamma}$, $\|z-\bar{z}\|_{\mathfrak{s}}^{2-12\gamma-\theta}$, $\|z\|_{\mathfrak{s}}^{1-6\gamma-\theta}\|\bar{z}\|_{\mathfrak{s}}^{1-6\gamma-\theta}$
	$(\|z\|_{\mathfrak{s}}^{5-18\gamma-\theta}+\|\bar{z}\|_{\mathfrak{s}}^{5-18\gamma-\theta})\|z-\bar{z}\|_{\mathfrak{s}}^{-3+6\gamma}$, $\|z-\bar{z}\|_{\mathfrak{s}}^{2-12\gamma-\theta}$, $\|z\|_{\mathfrak{s}}^{1-6\gamma-\theta}\|\bar{z}\|_{\mathfrak{s}}^{1-6\gamma-\theta}$
	$\|z-\bar{z}\|_{\mathfrak{s}}^{1-6\gamma}$
	$\|z\|_{\mathfrak{s}}^{\frac{1}{2}-\gamma-\theta}\|\bar{z}\|_{\mathfrak{s}}^{\frac{1}{2}-\gamma-\theta}\|z-\bar{z}\|_{\mathfrak{s}}^{-4\gamma}$, $\|z-\bar{z}\|_{\mathfrak{s}}^{1-6\gamma}$, $\|z\|_{\mathfrak{s}}^{\frac{1}{2}-3\gamma-\theta}\|\bar{z}\|_{\mathfrak{s}}^{\frac{1}{2}-3\gamma-\theta}$

	$\|z\|_{\mathfrak{s}}^{1-2\gamma-\theta}\|\bar{z}\|_{\mathfrak{s}}^{1-2\gamma-\theta}\|z-\bar{z}\|_{\mathfrak{s}}^{-1-2\gamma}$, $\|z-\bar{z}\|_{\mathfrak{s}}^{1-6\gamma}$, $\|z\|_{\mathfrak{s}}^{\frac{1}{2}-3\gamma-\theta}\|\bar{z}\|_{\mathfrak{s}}^{\frac{1}{2}-3\gamma-\theta}$

Here θ is any sufficiently small number. The bounds of $\mathcal{P}^{(\epsilon)}(z;\bar{z})$ are obtained by multiplying ϵ^{θ} to above bounds whose indexes are slightly subtracted as in Proposition 5.1.

Proposition 5.5. *For* , , *and* , *we have*

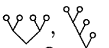
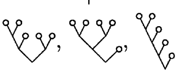
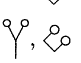
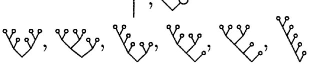
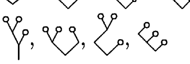
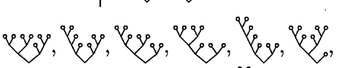
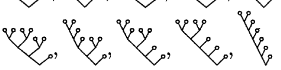
$C^{(\epsilon)}$ $\sim \epsilon^{-1-2\gamma}$, $|C^{(\epsilon)}$$| \lesssim \epsilon^{-4\gamma}$, $|C^{(\epsilon)}$$| \lesssim \epsilon^{-4\gamma}$.

For $\tau \in \mathcal{F}_-$ with $\|\tau\| = 6$ and $\kappa > 0$, we have

$$|C_{\tau}^{(\epsilon)}| \lesssim \epsilon^{1-6\gamma-\kappa}.$$

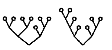
5.4 Renormalization in $\frac{3}{14} \leq \gamma < \frac{1}{4}$

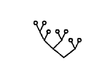
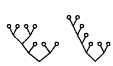
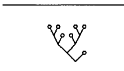
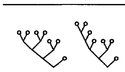
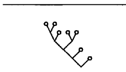
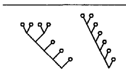
Let $\frac{3}{14} \leq \gamma < \frac{1}{4}$. Then all elements in $\mathcal{F}_-$ are as follows.

Homogeneity	Symbol
$-\frac{3}{2}-\gamma-\kappa$	Ξ
$-1-2\gamma-2\kappa$	
$-\frac{1}{2}-3\gamma-3\kappa$	
$-4\gamma-4\kappa$	
$-\frac{1}{2}-\gamma-\kappa$	
$\frac{1}{2}-5\gamma-5\kappa$	
$-2\gamma-2\kappa$	
$1-6\gamma-6\kappa$	
$\frac{1}{2}-3\gamma-3\kappa$	
$\frac{3}{2}-7\gamma-7\kappa$	 
0	**1**

It remains to obtain the bounds for elements with homogeneity $\frac{3}{2}-7\gamma-7\kappa$. The other bounds are same as before.

Proposition 5.6. *For every τ with $\|\tau\| = \frac{3}{2}-7\gamma-7\kappa$, we have the bounds of $\mathcal{P}(z;\bar{z})$ as follows.*

Symbol	$\mathcal{P}(z;\bar{z})$
	$\|z\|_{\mathfrak{s}}^{1-4\gamma-\theta}\|\bar{z}\|_{\mathfrak{s}}^{1-4\gamma-\theta}\|z-\bar{z}\|_{\mathfrak{s}}^{1-6\gamma}$, $\|z-\bar{z}\|_{\mathfrak{s}}^{3-14\gamma-\theta}$, $\|z\|_{\mathfrak{s}}^{\frac{3}{2}-7\gamma-\theta}\|\bar{z}\|_{\mathfrak{s}}^{\frac{3}{2}-7\gamma-\theta}$

	$\Vert z\Vert_{\mathfrak{s}}^{\frac{3}{2}-5\gamma-\theta}\Vert \bar{z}\Vert_{\mathfrak{s}}^{\frac{3}{2}-5\gamma-\theta}\Vert z-\bar{z}\Vert_{\mathfrak{s}}^{-4\gamma}$, $\Vert z-\bar{z}\Vert_{\mathfrak{s}}^{3-14\gamma-\theta}$, $\Vert z\Vert_{\mathfrak{s}}^{\frac{3}{2}-7\gamma-\theta}\Vert \bar{z}\Vert_{\mathfrak{s}}^{\frac{3}{2}-7\gamma-\theta}$
	$(\Vert z\Vert_{\mathfrak{s}}^{5-18\gamma-\theta}+\Vert \bar{z}\Vert_{\mathfrak{s}}^{5-18\gamma-\theta})\Vert z-\bar{z}\Vert_{\mathfrak{s}}^{-2+4\gamma}$, $\Vert z-\bar{z}\Vert_{\mathfrak{s}}^{3-14\gamma-\theta}$, $\Vert z\Vert_{\mathfrak{s}}^{\frac{3}{2}-7\gamma-\theta}\Vert \bar{z}\Vert_{\mathfrak{s}}^{\frac{3}{2}-7\gamma-\theta}$
	$\Vert z\Vert_{\mathfrak{s}}^{2-6\gamma-\theta}\Vert \bar{z}\Vert_{\mathfrak{s}}^{2-6\gamma-\theta}\Vert z-\bar{z}\Vert_{\mathfrak{s}}^{-1-2\gamma}$, $\Vert z-\bar{z}\Vert_{\mathfrak{s}}^{3-14\gamma-\theta}$, $\Vert z\Vert_{\mathfrak{s}}^{\frac{3}{2}-7\gamma-\theta}\Vert \bar{z}\Vert_{\mathfrak{s}}^{\frac{3}{2}-7\gamma-\theta}$
	$(\Vert z\Vert_{\mathfrak{s}}^{6-20\gamma-\theta}+\Vert \bar{z}\Vert_{\mathfrak{s}}^{6-20\gamma-\theta})\Vert z-\bar{z}\Vert_{\mathfrak{s}}^{-3+6\gamma}$, $\Vert z-\bar{z}\Vert_{\mathfrak{s}}^{3-14\gamma-\theta}$, $\Vert z\Vert_{\mathfrak{s}}^{\frac{3}{2}-7\gamma-\theta}\Vert \bar{z}\Vert_{\mathfrak{s}}^{\frac{3}{2}-7\gamma-\theta}$
	$(\Vert z\Vert_{\mathfrak{s}}^{7-22\gamma-\theta}+\Vert \bar{z}\Vert_{\mathfrak{s}}^{7-22\gamma-\theta})\Vert z-\bar{z}\Vert_{\mathfrak{s}}^{-4+8\gamma}$, $\Vert z-\bar{z}\Vert_{\mathfrak{s}}^{3-14\gamma-\theta}$, $\Vert z\Vert_{\mathfrak{s}}^{\frac{3}{2}-7\gamma-\theta}\Vert \bar{z}\Vert_{\mathfrak{s}}^{\frac{3}{2}-7\gamma-\theta}$
	$(\Vert z\Vert_{\mathfrak{s}}^{9-30\gamma-\theta}+\Vert \bar{z}\Vert_{\mathfrak{s}}^{9-30\gamma-\theta})\Vert z-\bar{z}\Vert_{\mathfrak{s}}^{-6+16\gamma}$, $\Vert z-\bar{z}\Vert_{\mathfrak{s}}^{3-14\gamma-\theta}$, $\Vert z\Vert_{\mathfrak{s}}^{\frac{3}{2}-7\gamma-\theta}\Vert \bar{z}\Vert_{\mathfrak{s}}^{\frac{3}{2}-7\gamma-\theta}$

Here θ is any sufficiently small number. The bounds of $\mathcal{P}^{(\epsilon)}(z;\bar{z})$ are obtained by multiplying ϵ^{θ} to above bounds whose indexes are slightly subtracted as in Proposition 5.1.

Acknowledgements

The author would like to thank Professor T. Funaki for leading him to the problem discussed in the present paper and asking Professor M. Hairer about this problem, who kindly pointed out that $\gamma = \frac{1}{4}$ is the border.

References

[1] E. Di Nezza, G. Palatucci, and E. Valdinoci, *Hitchhikers guide to the fractional Sobolev spaces*, Bull. Sci. Math. **136** (2012) 521-573.

[2] M. Hairer, *Solving the KPZ equation*, Ann. Math. (2) **178** (2013) 559–664.

[3] M. Hairer, *A theory of regularity structures*, Invent. Math. **198** (2014) 269–504.

[4] M. Hairer, *Singular stochastic PDEs*, arXiv:1403.6353 (2014).

[5] M. Hoshino, *KPZ equation with fractional derivatives of white noise*, arXiv:1602.04570 (2016).

[6] Y. Hu, J. Huang, K. Lê, D. Nualart, and S. Tindel, *Stochastic heat equation with rough dependence in space*, arXiv:1505.04924 (2015).

[7] S. Janson, *Gaussian Hilbert Spaces*, Cambridge Univ. Press, 1997.

[8] M. Kardar, G. Parisi, and Y.-C Zhang, *Dynamic scaling of growing interfaces*, Phys. Rev. Lett. **56** (1986) 889–892.

Graduate School of Mathematical Sciences
The University of Tokyo
3-8-1 Komaba, Meguro-ku, Tokyo 153-8914, Japan
E-mail address: hoshino@ms.u-tokyo.ac.jp