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Partial regularity and extension of solutions to the Navier-Stokes equations

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Abstract

It is quite well-known that we cannot assure the existence of global-in-time solutions to the Navier-Stokes equations for large initial data, but we have local-in-time solutions at least. The purpose of this talk is to get another time extension criterion for that local-in-time solution. Specifically, We work on smooth classical solutions which satisfy so called Leray-Hopf class on $\mathbb{R}^n \times (0, T)$, and then establish an time-extension criterion beyond T by estimating a sort of Morrey type functional of solutions. A key idea here is to utilize the ϵ-regularity argument which has been the critical part of the theory of suitable weak solutions. We note that this article is based on the author's recent work [23] and also contains similar results for bounded domains.

1 Introduction

1.1 Motivation and Purpose

This note is a brief survey of my recent work [23], so I would like to introduce the content of that paper here briefly. We consider the initial value problem for the incompressible Navier-Stokes equation in $\mathbb{R}^n$ in a very simple setting as follows:

$$\begin{aligned} u_t - \Delta u + (u, \nabla)u + \nabla p = 0 \quad & \text{in } \mathbb{R}^n \times (0, T) \\ \operatorname{div} u = 0 \quad & \text{in } \mathbb{R}^n \times (0, T) \\ u(x, 0) = u_0(x) \quad & \text{on } \mathbb{R}^n \end{aligned} \tag{1.1}$$

Here the dimension n is 3 or 4, and $u = u(x, t) = (u_1(x, t), u_2(x, t), ..., u_n(x, t))$ and $p = p(x, t)$ stand for the unknown velocity and the unknown pressure of the fluid at the point $(x, t) \in \mathbb{R}^n \times (0, T)$ respectively. Furthermore, $u_0(x)$ is the given initial velocity of the fluid.

As is mentioned above, we have a unique local-in-time solution for any initial data due to the pioneer works (see Fujita-Kato[7], Giga[8], Kato[9]). They proved that for sufficiently small initial data we can construct a unique global-in-time mild solution which is also smooth, but for large initial data we can only say that we can construct a unique local-in-time mild solution which is smooth. No one knows if this constructed local-in-time solution blows up at some point or turns out to be a global-in-time smooth solution. Actually this point is one of the well-known Millennium problems. In order to tackle this thorny subject, it is surely an important step to investigate whether that local-in-time solution u on $\mathbb{R}^n \times (0, T)$ would blow up at $t = T$ or can be continued beyond T. We are going to establish a new time extension criterion to the local-in-time solution u here.

Let me refer to some known results discussing the time extension criteria for strong solutions to Euler and Navier-Stokes equations. Beale-Kato-Majda [1] and Kato-Ponce [10] worked on the 3D Euler equations and proved that the maximum norm of the vorticity controls the breakdown of smooth solutions. In other words, a smooth solution persists and can be continued if the vorticity remains bounded. This blow-up criterion is also applicable to the Navier-Stokes equations. Chemin [3] and Kozono-Ogawa-Taniuchi [12] improved their results by replacing the L^{∞}-norm by some Besov norm. Kozono-Yatsu [13] and Kozono-Shimada [14] proved blow-up criteria which are similar to the results above by using some BMO-norm and Triebel-Lizorkin-norm. There are blow-up criteria for bounded domains as well. Actually the same result as Beale-Kato-Majda [1] has been proven (see Shirota-Yanagisawa [22], Ferrari [6], and Zajackowski [25]). Ogawa-Taniuchi [17] got similar criterion by using *bmo*-norm, and recently Taniuchi has improved that result by using more general functional space and confirmed that the result by Ogawa-Taniuchi [17] which was proved only for the Euler equations are actually applicable to the Navier-Stokes equations. He gave a talk on that topic at this very same conference so please refer to his paper for more details.

1.2 Key Ideas

Some of the improvements above were obtained by improving upon some essential inequalities(see Kozono-Ogawa-Taniuchi[12] for example). We take a slightly different viewpoint and focus on the partial regularity of the solution. Actually the solution which we are working on is already smooth, so what we want to do here is to investigate the behavior of the solution near the final time T. For that purpose, we first of all establish so-called ϵ-regularity theorem. Generally, the statement of ϵ-regularity theorem is that the solution u is locally bounded if some term is sufficiently small, smaller than some constant ϵ.

There are two steps to get to the time-extension criterion we want. First we construct Theorem 1.1 which says that if a sort of Morrey type functional is sufficiently small, then the solution u is bounded near final time T. Next, we can construct another mild solution starting at t_0 which is near T enough, and in fact the classical solution and that mild solution coincide. The estimate of lifetime of that mild solution confirms that the classical solution is continued beyond T.

Let me show you some known results about partial regularity of suitable weak solutions and ϵ-regularity theorem for them. For three dimensional case, many authors have investigated the partial regularity of suitable weak solutions of the Navier-Stokes equations. Scheffer [18, 19] studied the partial regularity and proved the existence of weak solutions such that the two dimensional Hausdorff measure of the singular set is finite. Caffarelli-Kohn-Nirenberg [2] introduced suitable weak solutions which satisfy so called generalized energy inequality, then proved that the one dimensional Hausdorff measure of the singular set is equal to zero. They constructed an ϵ-regularity theorem to prove that. Seregin-Šverák [21] proved an ϵ-regularity criterion estimating a Morrey type functional of the solution. We will state the relation between their paper and our results. For four dimensional case, Dong-Du [4] has investigated partial regularity in four dimensional case. They worked on the classical solutions and established some ϵ-regularity theorems and estimate about the singular set near the final time T (see also Dong-Gu[5], Ladyzhenskaya-Seregin[15], Lin[16], Scheffer[20], Wang-Wu[24]).

1.3 Results

Let us state our results. Let u be the classical solution to (1.1) which satisfies so called Leray-Hopf class ($u \in L^\infty(0,T;L^2(\mathbb{R}^n)) \cap L^2(0,T;H^1(\mathbb{R}^n))$), and $u_0 \in L^2_\sigma$. We start with Theorem 1.1. discussing the partial regularity of the solution near the final time T.

Theorem 1.1 ([21], [23]). *There is a positive number ϵ satisfying the following property. Assume that, for some positive R and a point $z_0 = (x_0, T) \in \mathbb{R}^n \times T$ the inequality*

$$\sup_{0<r<R} \sup_{T-r^2 \le t < T} \frac{1}{r^{n-2}} \int_{B(x_0,r)} |u(x,t)|^2 dx < \epsilon$$

holds. Then z_0 is a regular point. More precisely,

$$\sup_{z \in Q(z_0,r)} |u(z)| \le \frac{C}{r}$$

for some r, where C is positive constant which is independent of x_0.

Remark 1.1. *Although we only discuss the Cauchy problem here, this theorem is also valid if we consider this problem on any domain just because the arguments here are all local ones. So we don't care about the boundary or boundary value at all. Seregin-Šverák [21] also established the ϵ-regularity theorem on any domain.*

This statement was essentially proved by Seregin-Šverák [21] when $n = 3$, but the local estimate of u on a parabolic cylinder was not apparently discussed there. So we prove that estimate by using the scaling method, and that estimate plays an important role later in establishing the time-extension criterion. Now we are in position to state our main result.

Theorem 1.2 ([23]). *If there exists $R > 0$ such that*

$$\sup_{0<r<R, x_0 \in \mathbb{R}^4} \sup_{T-r^2 \le t < T} \frac{1}{r^{n-2}} \int_{B(x_0,r)} |u(x,t)|^2 dx < \epsilon$$

holds. Then there exists $T' > T$, and u can be extended to the classical solution on $\mathbb{R}^n \times (0, T')$. Moreover, this solution satisfies $u \in L^\infty(0,T';L^2(\mathbb{R}^n)) \cap L^2(0,T';H^1(\mathbb{R}^n))$

Remark 1.2. *This Theorem1.2. says that the local-in-time classical solution u on $\mathbb{R}^n \times (0,T)$ can be continued beyond T when a sort of Morrey type functional of u is sufficiently small near the final time T. We can say that this assumption is weaker than that Morrey norm is sufficiently small, because the parameter r does not go to the infinity. This is a time-extension criterion of a new type compared to the known results which are written above.*

Remark 1.3. *We can prove by a similar argument that this criterion is also valid for bounded domains. Let me show you how to prove it later very briefly. But we are not sure about other cases such as exterior domains, because we have to construct a mild solution which has a good estimate about lifetime.*

2 Preliminaries

Before proving our results, some preliminaries are needed. First of all we introduce the notation of balls and parabolic cylinders. $B_n(x_0, r) = \{x \in \mathbb{R}^n | |x - x_0| < r\}$, $Q_n(x_0, r) = B_n(x_0, r) \times (t_0 - r^2, t_0)$. Then, we can introduce the notation of mean value of functions. We define $[u]_{x_0,r,n}(t) = \frac{1}{|B_n(r)|}\int_{B_n(x_0,r)} u(x,t)dx$. Let me omit the indices which would make the following argument unnecessarily complicated.

2.1 Setting for 3D case

To establish an ϵ-regularity theorem and an estimate of solution which are crucial to prove the time-extension criterion we wan to obtain, we have to introduce following quantities. They are all scaling invariant and well-known terms in the theory of partial regularity. If you want to know more about why they are important and how they were introduced, please refer to Caffarelli-Kohn-Nirenberg [2], Ladyzhenskaya-Seregin[15], Lin[16], and Seregin-Šverák [21] for further details.

$$
\begin{aligned}
A_3(r) = A_3(r, z_0) &= \sup_{t_0 - r^2 \le t < t_0} \frac{1}{r} \int_{B(x_0,r)} |u(x,t)|^2 dx \\
C_3(r) = C_3(r, z_0) &= \frac{1}{r^2} \int_{Q(z_0,r)} |u(x,t)|^3 dz \\
D_3(r) = D_3(r, z_0) &= \frac{1}{r^2} \int_{Q(z_0,r)} |p(x,t)|^{\frac{3}{2}} dz \\
E_3(r) = E_3(r, z_0) &= \frac{1}{r} \int_{Q(z_0,r)} |\nabla u(x,t)|^2 dz
\end{aligned}
$$

2.2 Setting for 4D case

Now we want to introduce similar scaling-invariant terms in 4D case as well. For that purpose, we first of all introduce the decomposition of pressure function $p(x,t)$, which splits the pressure function into the harmonic part and the non-harmonic part. That kind of technique has been used by lots of papers, and following setting is based on Dong-Du [4]. Please refer to their paper for more precise and further details about the decomposition of pressure.

We define $\eta(x)$ as a cut-off function on $\mathbb{R}^4$ supported in $B(1)$, with $0 \le \eta \le 1$, $\eta = 1$ on $\overline{B}(\frac{2}{3})$. When $z_0 \in \mathbb{R}^4 \times (0, T]$ and $r > 0$ satisfy $Q(z_0, r) \subset \mathbb{R}^4 \times (0, T)$, for a.e. $t \in (t_0 - r^2, t_0)$, we can get by a straightforward calculation that

$$
\Delta p = \frac{\partial^2}{\partial x_i \partial x_j}\{(u_i - [u_i]_{x_0,r})(u_j - [u_j]_{x_0,r})\} \quad in \;\; B(x_0, r)
$$

Furthermore, Let $\tilde{p}_{x_0,r}$ be the solution to following Poisson equation.

$$
\Delta \tilde{p}_{x_0,r} = (u_i - [u_i]_{x_0,r})(u_j - [u_j]_{x_0,r})\eta(\frac{x}{r})
$$

Now we are in position to introduce the decomposition of pressure function $p(x,t)$

$$
p(x,t) = \tilde{p}_{x_0,r}(x,t) + h_{x_0,r}(x,t)
$$

As is written above, this decomposition splits $p(x,t)$ into the harmonic part and the nonharmonic part. Actually $h_{x_0,r}(x,t)$ is harmonic in $B(x_0,\frac{r}{2})$ by the definition. Let me omit the indices of $\tilde{p}$ and h if there is no confusion, because they would make the following arguments look so complicated. Now we can finally introduce the scaling invariant terms we want, which are crucial to establish an ϵ-regularity theorem.

$$
\begin{aligned}
A_4(r) = A_4(r,z_0) &= \sup_{t_0-r^2\le t<t_0} \frac{1}{r^2}\int_{B(x_0,r)} |u(x,t)|^2 dx \\
C_4(r) = C_4(r,z_0) &= \frac{1}{r^3}\int_{Q(z_0,r)} |u(x,t)|^3 dz \\
D_4(r) = D_4(r,z_0) &= \frac{1}{r^3}\int_{Q(z_0,r)} |p(x,t)-[h]_{x_0,r}(t)|^{\frac{3}{2}} dz \\
E_4(r) = E_4(r,z_0) &= \frac{1}{r^2}\int_{Q(z_0,r)} |\nabla u(x,t)|^2 dz \\
F_4(r) = F_4(r,z_0) &= \frac{1}{r^2}\Big[\int_{t_0-r^2}^{t_0}\Big(\int_{B(x_0,r)} |p(x,t)-[h]_{x_0,r}(t)|^{1+\alpha} dx\Big)^{\frac{1}{2\alpha}} dt\Big]^{\frac{2\alpha}{1+\alpha}}
\end{aligned}
$$

Here we follow the setting and notation in [4], and we set α as $\frac{1}{27}$ to avoid the unnecessary complexity. Before starting the proof of theorems we prepare some useful lemmata, some of which are from [4] so if you are interested in these lemmmata and how to prove them, please see their paper. More precisely, Lemmma 2.1 is well-known Poincare inequality in a ball, Lemmma 2.2 is an interpolation theorem, Lemma 2.3 to Lemma 2.5 are proved in [4], and Lemma 2.6 and Lemma 2.7 were proved in my recent work [23]. Let me omit the proof of these two lemmata here.

Lemma 2.1 (Poincare inequality in a ball). *Let $f \in W_p^1(\mathbb{R}^n)$, $1 \le p < \infty$ then the following inequality holds*

$$\int_{B(x_0,r)} |f-[f]_{x_0,r}|^p dx \le N r^p \int_{B(x_0,r)} |\nabla f|^p dx$$

where the constant N depends only on n and p.

Lemma 2.2 ([4], Lemma 2.6.). *For any functions $u \in H^1(\mathbb{R}^4)$ and real numbers $q \in [2,4]$ and $r > 0$,*

$$\int_{B(r)} |u|^q dx \le N(q)\Big[\Big(\int_{B(r)} |\nabla u|^2 dx\Big)^{q-2}\Big(\int_{B(r)} |u|^2 dx\Big)^{2-\frac{q}{2}} + \Big(\frac{1}{r}\Big)^{2(q-2)}\Big(\int_{B(r)} |u|^2 dx\Big)^{\frac{q}{2}}\Big]$$

Lemma 2.3 ([4], Lemma 2.8.). *Suppose $\gamma \in (0,1)$, $r > 0$ are constants and $Q(z_0,r) \subset \mathbb{R}^4 \times (0,T)$. Then we have*

$$C(\gamma r) \le N\Big[\Big(\frac{1}{\gamma}\Big)^3 A^{\frac{1}{2}}(r)E(r) + \Big(\frac{1}{\gamma}\Big)^{\frac{9}{2}} A^{\frac{3}{4}}(r)E^{\frac{3}{4}}(r) + \gamma C(r)\Big]$$

where N is a constant independent of γ, r, z_0.

Lemma 2.4 ([4], Lemma 2.9.). *Suppose $\alpha \in (0, \frac{1}{2}]$, $\gamma \in (0, \frac{1}{3}]$, $r > 0$ are constants and $Q(z_0, r) \subset \mathbb{R}^4 \times (0, T)$. Then we have*

$$F(\gamma r) \leq N(\alpha)[(\frac{1}{\gamma})^2 A^{\frac{1-\alpha}{1+\alpha}}(r) E^{\frac{2\alpha}{1+\alpha}}(r) + \gamma^{\frac{3-\alpha}{1+\alpha}} F(r)]$$

where $N(\alpha)$ is a constant independent of γ, r, z_0. In particular, for $\alpha = \frac{1}{2}$ we have,

$$D(\gamma r) \leq N[(\frac{1}{\gamma})^3 A^{\frac{1}{2}}(r) E(r) + \gamma^{\frac{5}{2}} D(r)]$$

Moreover, it holds that

$$D(\gamma r) \leq N(\alpha)[(\frac{1}{\gamma})^3 (A(r) + E(r))^{\frac{3}{2}} + \gamma^{\frac{9-3\alpha}{2+2\alpha}} F^{\frac{3}{2}}(r)]$$

Lemma 2.5 ([4], Lemma 2.10.). *Suppose $\theta \in (0, \frac{1}{2}]$, $r > 0$ are constants and $Q(z_0, r) \subset \mathbb{R}^4 \times (0, T)$. Then we have*

$$A(\theta r) + E(\theta r) \leq N(\frac{1}{\theta})^2 [C^{\frac{2}{3}}(r) + C(r) + C^{\frac{1}{3}}(r) D^{\frac{2}{3}}(r)]$$

In particular, when $\theta = \frac{1}{2}$ we have

$$A(\frac{r}{2}) + E(\frac{r}{2}) \leq N[C^{\frac{2}{3}}(r) + C(r) + C^{\frac{1}{3}}(r) D^{\frac{2}{3}}(r)]$$

Lemma 2.6 ([23]). *For any $z_0 \in \mathbb{R}^4 \times (0, T]$, $r > 0$ which satisfy $Q(z_0, r) \in \mathbb{R}^4 \times (0, T)$, the following inequality holds.*

$$C(r) + D(r) + F(r) < N_0(\frac{1}{r^2} + \frac{1}{r^3})$$

where N_0 is a positive constant which is independent of r, x_0.

Lemma 2.7 ([23]). *We define $X(r)$ as follows*

$$X(r) = C(r) + D(r) + F(r)$$

Then, for any $z_0 \in \mathbb{R}^4 \times (0, T]$, $\gamma \in (0, \frac{1}{9})$, and $r > 0$ which satisfies $Q(z_0, r) \in Q_T$, the following inequality holds.

$$\begin{aligned} X(\gamma r) &\leq N_1[\gamma X(\frac{r}{3}) + (\frac{1}{\gamma})^3 A^{\frac{1}{2}}(r) X(r) + \frac{2}{3}\gamma X(r) + \frac{1}{3}(\frac{1}{\gamma})^{11} A^{\frac{3}{2}}(r) + \frac{3}{4}(\frac{1}{\gamma})^6 A^{\frac{2}{3}}(r) X(r) \\ &+ \frac{1}{4}A(r) + \frac{1}{2}\gamma X(r) + \frac{1}{4}(\frac{1}{\gamma})^{20} A^2(r) + \frac{13}{14} A^{\frac{12}{13}}(r) + \frac{1}{21}\gamma X(r) + \frac{1}{14}(\frac{1}{\gamma})^{28} A(r) X(r) \\ &+ \frac{1}{42}(\frac{1}{\gamma})^{86} A^3(r)] \end{aligned}$$

where N_1 is a positive constant which is independent of r, x_0, γ.

3 Proof of the Theorems

3.1 Proof of Theorem1.1.

Let me omit the proof for 3D case due to limitations of space.

proof of Theorem1.1. Let $z_0 = (x_0, T) \in \mathbb{R}^4 \times T$ be given. We take $\gamma \in (0, \frac{1}{9})$ so small and then $0 < \epsilon < 1$ so small that the following inequalities hold.

$$(1+\frac{2}{3}+\frac{1}{2}+\frac{1}{21})N_1\gamma \leq \frac{1}{108}$$

$$(\frac{1}{\gamma})^3\epsilon^{\frac{1}{2}}+\frac{3}{4}(\frac{1}{\gamma})^6\epsilon^{\frac{2}{3}}+\frac{1}{14}(\frac{1}{\gamma})^{28}\epsilon < \gamma$$

$$N_1[\frac{1}{3}(\frac{1}{\gamma})^{11}\epsilon^{\frac{3}{2}}+\frac{1}{4}\epsilon+\frac{1}{4}(\frac{1}{\gamma})^{20}\epsilon^2+\frac{13}{14}\epsilon^{\frac{12}{13}}+\frac{1}{42}(\frac{1}{\gamma})^{86}\epsilon^3] < \epsilon^{\frac{1}{2}}$$

When $0 < r < R$, we get

$$\begin{aligned}
X(\gamma r) &\leq N_1[\gamma X(\frac{r}{3})+(\frac{1}{\gamma})^3A^{\frac{1}{2}}(r)X(r)+\frac{2}{3}\gamma X(r)+\frac{1}{3}(\frac{1}{\gamma})^{11}A^{\frac{3}{2}}(r)+\frac{3}{4}(\frac{1}{\gamma})^6A^{\frac{2}{3}}(r)X(r)\\
&+\frac{1}{4}A(r)+\frac{1}{2}\gamma X(r)+\frac{1}{4}(\frac{1}{\gamma})^{20}A^2(r)+\frac{13}{14}A^{\frac{12}{13}}(r)+\frac{1}{21}\gamma X(r)+\frac{1}{14}(\frac{1}{\gamma})^{28}A(r)X(r)\\
&+\frac{1}{42}(\frac{1}{\gamma})^{86}A^3(r)]\\
&\leq N_1[\gamma X(\frac{r}{3})+\left\{(\frac{1}{\gamma})^3A^{\frac{1}{2}}(r)+\frac{2}{3}\gamma+\frac{3}{4}(\frac{1}{\gamma})^6A^{\frac{2}{3}}(r)+\frac{1}{2}\gamma+\frac{1}{21}\gamma+\frac{1}{14}(\frac{1}{\gamma})^{28}A(r)\right\}X(r)\\
&+\frac{1}{3}(\frac{1}{\gamma})^{11}A^{\frac{3}{2}}(r)+\frac{1}{4}A(r)+\frac{1}{4}(\frac{1}{\gamma})^{20}A^2(r)+\frac{13}{14}A^{\frac{12}{13}}(r)+\frac{1}{42}(\frac{1}{\gamma})^{86}A^3(r)]
\end{aligned}$$

By estimating the right-hand side, we obtain

$$X(\gamma r) \leq \frac{1}{108}[X(r)+X(\frac{r}{3})]+\epsilon^{\frac{1}{2}}$$

By induction we eventually have the following inequality for any $k \in \mathbb{N}$.

$$X(\gamma^k r) \leq \frac{1}{108^k}\sum_{j=0}^{k}\binom{k}{j}X(\frac{r}{3^{k-j}})+\sum_{j=0}^{k-1}\frac{2^j}{108^j}\epsilon^{\frac{1}{2}}$$

By using Lemma 2.6 we can estimate the first term as follows.

$$\begin{aligned}
\frac{1}{108^k}\sum_{j=0}^{k}\binom{k}{j}X(\frac{r}{3^{k-j}}) &\leq \frac{1}{108^k}2^kN\sum_{j=0}^{k}(\frac{1}{\frac{r}{3^j}^2}+\frac{1}{\frac{r}{3^j}^3})\\
&\leq \frac{1}{2^k}N(\frac{1}{r^2}+\frac{1}{r^3})
\end{aligned}$$

Then, we can obtain

$$X(\gamma^k r) \leq \frac{1}{2^k}N(\frac{1}{r^2}+\frac{1}{r^3})+2\epsilon^{\frac{1}{2}}$$

For any ϵ_0, we can take the right-hand side smaller than ϵ_0 by taking ϵ sufficiently small and k sufficienty large. Then we have

$$C(r_0) + D(r_0) + F(r_0) < \epsilon_0$$

for some $r_0 > 0$ which is independent of x_0. According to Proposition 2.14 and Lemma 5.1 in [4] and utilizing scaling method, we eventually obtain

$$\max_{\overline{Q}(z_0,\frac{r_0}{4})\cap\overline{Q}_{T-\delta}} (t + \frac{r_0^2}{16} - T)^{\frac{1}{2}}|u(z)| \leq 2$$

for all $\delta \in (0, \frac{r_0^2}{16})$. Now it is easy to see

$$\sup_{z\in Q(z_0,\frac{r_0}{4\sqrt{2}})} |u(z)| \leq \frac{8\sqrt{2}}{r_0} \tag{3.1}$$

□

3.2 Proof of Theorem1.2.

We immediately get to know that the solution u is bounded near the final time T by the Theorem1.1. Because the solution satisfies Leray-Hopf's condition, an easy interpolation argument will yield the following property for any $2 \leq q \leq \infty$.

$$u \in L^\infty(T_0, T; L^q(\mathbb{R}^n)) \tag{3.2}$$

where T_0 depends on R. Now this property enables us to construct another mild solution $\tilde{u}$ from arbitrary time t_0 which satisfies $T_0 < t_0 < T$. More specifically, we consider mild solution $\tilde{u}$ which satisfies following properties with time span T^* for $q > n$ (see Giga[8]).

$$\tilde{u} \in BC([t_0, t_0 + T^*); L^q_\sigma) \cap L^s(t_0, t_0 + T^*; L^p_\sigma), \quad \tilde{u}(t_0) = u(t_0)$$

$$T^* \geq \frac{N}{\|u(t_0)\|_{L_q}^{\frac{2q}{q-n}}} \tag{3.3}$$

with $\frac{2}{s} + \frac{n}{p} = \frac{n}{q}$, $s, p > q$. Actually these two solutions u and $\tilde{u}$ coincide on $[t_0, \min\{T, t_0+T^*\})$ due to the uniqueness theorem of the weak solutions in Kozono-Sohr[11]. By estimating the right-hand term of (3.3) in terms of the initial data, we obtain

$$T^* \geq \frac{N}{\|u(t_0)\|_{L_2}^{\frac{4}{q-n}} \|u(t_0)\|_{L_\infty}^{\frac{2(q-2)}{q-n}}} \geq \frac{N}{\|u_0\|_{L_2}^{\frac{4}{q-n}}} r_0^{\frac{q-n}{2(q-2)}}$$

Since the most right-hand side is independent of t_0 from where the mild solution $\tilde{u}$ starts, we can take t_0 close to the final time T enough so that $t_0 + T^* > T$. This means the classical solution u can be continued beyond T and untill $t_0 + T^*$.

Remark 3.1. *As I wrote in the Remark 1.3, we can prove the same theorem for bounded domains. The difficulty here appears near the boundary and we have to take care of that. Actually the upper bound of the estimate of Theorem1.1 soars when x_0 gets close to the boundary, so obviously we cannot get the property of $u \in L^\infty(\mathbb{R}^n \times [T_0, T))$. However, we can modify that estimate and eventually get $u \in L^\infty(T_0, T; L^q(\mathbb{R}^n))$ for any $2 \leq q < \infty$, and this property is actually enough to continue the time.*

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