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Algebraic QFT & Local Gauge Invariance

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1 What is aimed at here

It has been common to discuss local gauge invariance in close relation with indefinite inner product of a state vector space which violates the basic hypotheses for probabilistic interpretation in QFT. In sharp contrast, such basic structures as the validity of ***Maxwell equation*** can be determined algebraically and/or categorically in algebraic QFT without reference to the concept of state vector spaces (with or without indefinite inner products) as seen below.

1.1 Quadrality scheme related with "5W1H"

To this end, we start our discussion here with our theoretical framework in the context of Micro-Macro duality in quadrality scheme as follows. For the purpose of ensuring

1) ***bi-directionality in inductive & deductive*** arguments,

we need a framework to accommodate induction processes theoretically

$\Longrightarrow$ a possible candidate for such a universal theoretical framework can be found in ***quadrality scheme***:

$\nearrow$	***Spec***	
States	$\leftrightarrows\upharpoonleft$ (***Rep***) $\downharpoonright\leftrightarrows$	***Alg***
	Dyn	$\nearrow$

, consisting of the following 4(+1) basic ingredients adapted to "5W1H":

Spec [When & Where= classifying space to specify events],

States ["Who"= "subject" to specify context],

Alg(ebra of variables) [What= objects to be described],

(***Rep*** (of Alg)) [How= modus of phenomena = "modules"],

Dyn(amics) [Why= causes of process].

1.2 Emergence & quantum fields in quadrality scheme

2) Bi-directional relations become effective between phenomenogical ***visible***

	Spec		***Alg***
Macro=	$\nearrow$	$\rightleftarrows$	$\nearrow$
	States		***Dyn***

= ***invisible Micro*** in theory via ***quantum fields*** as a local net functor ***Spec*** $\to$ ***Alg*** to *extend logically constant* ***Alg*** into ***Alg***-valued *variables* on ***Spec***:

Macro:	***Spec***	
$\nearrow$		$\searrow$: ***quantum fields***
States	$\leftrightarrows$ ***Rep*** $\leftrightarrows$	***Alg***
		$\nearrow$
	Dyn	: Micro

1.3 Quadrality scheme combined with various dualities (1)

3) Combined with *horizontal Fourier-type dualities*, ***States***$\leftrightarrows$ ***Rep***$\leftrightarrows$ ***Alg***, due to operator algebra theory, two non-trivial ingredients in the scheme,

physical emergence of Spec from States

& ***quantum fields on emergent Spec***,

entail the following *network of connections* over this quadrality scheme:

	Spec	
emergence$\nearrow \swarrow$	$V\uparrow \ \downarrow I$	$\nwarrow \searrow$ quantum fields
States $(\sim L^1)$	$\leftrightarrows$ ***Rep*** $(\sim L^2)$ $\leftrightarrows$	***Alg*** $(\sim L^\infty)$
dual fields $\searrow\nwarrow$	$\uparrow \ \downarrow$ Galois	$\swarrow \nearrow$ co-emergence
	Dyn	

1.4 Quadrality scheme combined with various dualities (2)

The actual meaning of ***Spec***= [When & Where] to specify *event localizations* is ensured by its origin of the emergence processes, physically as ***phase separations,*** and mathematically by ***forcing method*** of identifying the *extended semantic space* (of multi-valued logic). Owing to duality $(\boldsymbol{Alg})^*$ =***States***, $(\boldsymbol{States})^*$ =***Alg***, the *emergence* arrow, ***States*** $\to$ ***Spec***, implies the dual arrow of *co-emergence* to create objects in ***Alg*** from the dynamical flow ***Dyn***, together with four other *upward* arrows:

	Spec	
emergence$\nearrow$	$\uparrow$	$\nwarrow$
States	$\leftrightarrows$ ***Rep*** $\leftrightarrows$	***Alg***
$\nwarrow$	$\uparrow$	$\nearrow$ co-emergence
	Dyn	

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1.5 Quadrality scheme combined with various dualities (3)

In the opposite direction,

	Spec	
$\swarrow$	$\downarrow$	$\searrow$ quantum fields
States	$\leftrightarrows$ ***Rep*** $\leftrightarrows$	***Alg***
dual fields $\searrow$	$\downarrow$ Galois	$\swarrow$
	Dyn	

,

local net, ***Spec*** $\rightarrow$ ***Alg***, of *quantum fields* triggers induction processes, inducing five *downward* arrows, among which Galois functor identifies group in ***Dyn*** from the representation contents ***Rep***.

2 Emergence of Spec as sector-classifying space

In this way, we have learned the crucial roles of emergence process in creating ***Spec*** via ***Bottom-Up***. This process can be formulated as follows in terms of the concepts of ***sectors*** in the case of QFT:

1) ***Sectors= pure phases*** parametrized by ***order parameter***[= macroscopic central observables $\mathfrak{Z}_\pi(\mathcal{X}) = \pi(\mathcal{X})'' \cap \pi(\mathcal{X})'$ commuting with all physical variables $\pi(\mathcal{X})''$ in a generic representation π of algebra $\mathcal{X}$ of physical variables]: mathematically, a ***sector***(= ***pure phase***) $\stackrel{\text{def}}{=}$ a ***quasi-equivalence class of factor*** states (& representations π_γ) of (C*-)algebra $\mathcal{X}$ of physical variables, as a *minimal unit* of representations characterized by ***trivial centre*** $\pi_\gamma(\mathcal{X})'' \cap \pi_\gamma(\mathcal{X})' =: \mathfrak{Z}_{\pi_\gamma}(\mathcal{X}) = \mathbb{C}1$.

*) *Important remark*: in the usual quantum mechanics with *finite degrees of freedom*, sectors are replaced by *irreducible representations* & *pure states* with *Spec*={*one point*}*!* They become meaningless, however, in the general contexts involving quantum fields with *infinite degrees of freedom* which play crucial roles in connecting *invisible Micro* and *visible Macro*.

2.1 Micro-Macro Duality of Intra- vs. Inter-sectorial levels

2) The roles of ***sectors as Micro-Macro boundary***: seen in ***Micro-Macro duality*** [1, 2] as a mathematical version of "***Quantum-Classical correpsondence***" between microscopic ***intra-sectorial*** & macroscopic ***inter-sectorial*** levels described by geometry on central spectrum $Sp(\mathfrak{Z}) := Spec(\mathfrak{Z}_\pi(\mathcal{X}))$:

$\longleftarrow$	Visible ***Macro***	of	***Spec*** =	***classifying space***	$\longrightarrow$	***Inter-sectorial***
$\cdots$	γ_N	$\cdots$	***sectors*** γ	γ_2	γ_1	$Sp(3)$
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	***Intra-sectorial***
$\cdots$	π_{γ_N}	$\cdots$	π_γ	π_{γ_2}	π_{γ_1}	$\Vert$
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	invisible ***Micro***

2.2 Inter-sectorial relations & Symmetry Breaking

3) Mutual relations among different sectors:
disjoint w.r.t. ***unbroken*** symmetry

Different sectors are ***connected*** by the actions of ***broken*** symmetries

: as explained later, this contrast is shared even by D(H)R theory of *unbroken* symmetry!

4) ***Emergence process*** [Macro $\Longleftarrow$ Micro] of Spec = sector-classifying space via ***forcing*** along (generic) filters

This is controlled mathematically by ***Tomita theorem*** to decompose a Hilbert bimodule ${}_{\pi(\mathcal{X})''}\widetilde{\mathcal{X}}_{L^\infty(E_\mathcal{X})} := \pi(\mathcal{X})''\otimes L^\infty(E_\mathcal{X})$ with left $\pi(\mathcal{X})''$ & right $L^\infty(E_\mathcal{X},\mu)$ actions, via ***central measure*** μ supported by ***Spec***= supp(μ) = $Sp(3) \subset F_\mathcal{X}$: factor states ($\subset E_\mathcal{X}$: state space of $\mathcal{X}$).

$\Longrightarrow$ Applications to statistical inference based on large deviation principle [3] and to derivation of Born rule [4].

2.3 Simplex vs. complex/ short vs. long exact sequences

5) In homological algebra, distinctions between individual modules and ***complexes*** of modules and between ***short*** and ***long exact sequences*** are well known to be important.

For a (∞-dimensional) von Neumann algebra $\mathcal{X}$, algebra $\mathcal{X}$ and its commutant $\mathcal{X}'$ are symmetric in ***standard form*** via Tomita-Takesaki modular conjugation $J : \mathcal{X} \ni x \longleftrightarrow JxJ \in \mathcal{X}'$

$\Longrightarrow$ via the degrees of freedom of the commutant $\mathcal{X}'$, a ***complex*** $(X_n)_{n\in\mathbb{N}}$ of $\mathcal{X}$-modules can easily be reduced to an $\mathcal{X}$-module $\underset{n}{\oplus} X_n$:

$$X := \underset{n}{\oplus} X_n \Longleftrightarrow \{X_n = Xp_n \text{ with } p_n \in Proj(\mathcal{X}')\}_{n\in\mathbb{N}}.$$

Thus, ***complexes of modules become redundant*** in the category $Mod_\mathcal{X}$ of $\mathcal{X}$-modules with ∞-dim'al v.N. alg. $\mathcal{X}$, where the essence of ***long exact sequences*** $X_1 \to Y_1 \to Z_1 \to X_2 \to Y_2 \to Z_2 \to \ldots \to X_n \to Y_n \to Z_n \to \ldots$

is reduced to the ***triangulated category*** $X \to Y \to Z \to \mathcal{T}(X)$ of $\mathcal{X}$-modules.

2.4 Ward-Takahashi identities & exact sequences in QFT

Thus the distinctions between individual modules and ***complexes*** of modules and between short and long exact sequences are less important for the sake of classifying representations of the algebra $\mathcal{X}$ of physical variables. Such distinctions are, however, *still meaningful in relation with group-theoretical or geometric aspects* arising from the actions of dynamics and/or symmetries on the physical systems in the following sense:

Short exact sequence: corresponds to ***Ward-Takahashi identities*** for correlation functions to describe ***unbroken symmetry***

Long exact sequence: corresponds to Ward-Takahashi identities describing ***spontaneously broken symmetries*** with ***Goldstone bosons*** to function as ***connecting morphisms***

In this context, the contrast between short vs. exact sequences is related with unbroken vs. broken symmetries and also with the absence or presence of ***connecting morphisms***. This last item is directly related with the fate of Goldstone bosons (at least, for spontaneous breakdown of symmetry).

2.5 Relation between emergence & eventualization

Mutual relation between emergence & eventualization (the latter emphasized by Dr. Saigo): while the former refers to universal transitions (real or virtual) from ***States*** to ***Spec*** as the level of ***classifying spaces*** within discussed contexts, the latter concept, eventualization, means the actual physical processes, typically taking place in experimental situations, which verify the relevance and actuality of the points belonging to ***Spec*** as the realized form of ***events***. This context is described by the expression, "events $\in$ ***Spec***", mainly materialized in the quantum-mechanical measurement processes. In contrast to this quantum-mechanical context, "localization of fields" describes transitions from quantum fields to classical fields.

2.6 Symmetry Breaking & Classifying Space

6) ***Symmetry Breaking & Emergence of Classifying Space***

Sector-classifying space emerges typically from spontaneous breakdown of symmetry of a dynamical system $\mathcal{X} \curvearrowleft G$ with action of a group G ("spontaneous" = no changes in dynamics of the system).

Criterion for Symmetry Breaking ([1] SB criterion, for short): judged by non-triviality of ***central*** dynamical system $\mathfrak{Z}_\pi(\mathcal{X}) \curvearrowleft G$ arising from the original one $\mathcal{X} \curvearrowleft G$

I.e., symmetry G is ***broken in sectors*** $\in Sp(3)$ ***with non-trivial responses to central G-action.***

The G-transitivity assumption with ***unbroken*** subgroup H in broken G leads to such a specific form of sector-classifying space as G/H.

$\Longrightarrow$ ***Classical geometric*** structure on G/H arises physically from ***emergence*** process via ***condensation*** of a family of ***degenerate vacua***, each of which is mutually distinguished by condensed values $\in Sp(3) = G/H$.

2.7 Sector Bundle & Logical Extension from const to variable

In this way, ∞-number of low-energy quanta are condensed into geometry of classical Macro objects $\in G/H$.

In combination with sector structure $\hat{H}$ of unbroken symmetry H (à la DHR-DR theory), total sector structure due to this symmetry breaking is described by a ***sector bundle*** $G \underset{H}{\times} \hat{H}$ with fiber $\hat{H}$ over base space G/H consisting of "***degenerate vacua***" [1, 5].

When this geometric structure is established, all the physical quantities are ***parametrized by condensed values of order parameters*** $\in G/H$

$\Longrightarrow$ "***Logical extension***" of ***constants*** (= global objects) into ***sector-dependent function objects*** (: origin of ***local gauge*** structures)

2.8 Symmetric Space Structure of G/H

This homogeneous space G/H is a ***symmetric space*** with Cartan involution (as shown here) [IO, in preparation].

Lie-bracket relations $[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h}$, $[\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}$ hold for Lie structures $\mathfrak{g}, \mathfrak{h}, \mathfrak{m}$ of $G, H, M := G/H$.

If $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{h}$ is verified, M becomes a symmetric space (at least, locally) equipped with Cartan involution $\mathcal{I}$ with eigenvalues $\mathcal{I} \restriction_{\mathfrak{h}} = +1$ & $\mathcal{I} \restriction_{\mathfrak{m}} = -1$:

Proof of $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{h}$) $[\mathfrak{m}, \mathfrak{m}]=$ ***holonomy*** associated with an infinitesimal loop in ***inter-sectorial space*** $M = Sp(3)$ along ***broken direction***

$\Longrightarrow$ $[\mathfrak{m}, \mathfrak{m}]=$ effect of ***broken*** G transformation along an infinitesimal loop $\overset{\gamma}{\circlearrowleft}$ on M starting from and returning to the same $\gamma \in M$. $\Longrightarrow$ $\mathfrak{m}$-component in $[\mathfrak{m}, \mathfrak{m}]$ is absent by the above SB criterion. Thus, $M = G/H = Sp(3)$ is a symmetric space (at least, locally).

2.9 Example 1: Lorentz boosts

Typical example of this sort can be found for Lorentz group $\mathcal{L}_+^\uparrow =: G$, rotation group $SO(3) =: H$, $G/H = M \cong \mathbb{R}^3$: symmetric space of Lorentz frames connected by Lorentz boosts.

For $\mathfrak{h} := \{M_{ij}; i,j = 1,2,3, i < j\}$, $\mathfrak{m} := \{M_{0i}; i = 1,2,3\}$, the relations $[\mathfrak{h},\mathfrak{h}] = \mathfrak{h}$, $[\mathfrak{h},\mathfrak{m}] = \mathfrak{m}$, $[\mathfrak{m},\mathfrak{m}] \subset \mathfrak{h}$ follow from the basic Lie algebra structure:
$[iM_{\mu\nu}, iM_{\rho_\sigma}] = -(\eta_{\nu\rho} iM_{\mu\sigma} - \eta_{\nu\sigma} iM_{\mu\rho} - \eta_{\mu\rho} iM_{\nu\sigma} + \eta_{\mu\sigma} iM_{\nu\rho})$.

In contrast to the usual interpretation of unbroken $\mathfrak{h}$ & $\mathfrak{m}$, ***unbroken*** Lorentz boosts $\mathfrak{m}$ is ***speciality of the vacuum situation***, which is due to such results as Borchers-Arveson theorem (: Poincaré generators can be physical observables only in vacuum representation) & as the spontaneous breakdown of Lorentz boosts at $T \neq 0K$ [6].

Thus Lorentz frames $M \cong \mathbb{R}^3$ with [boost, boost] = rotation, give a typical example of symmetric space structure emerging from symmetry breaking.

2.10 Example 2: 2nd Law of Thermodynamics

Along this line, ***chiral symmetry*** with current algebra structure $[V,V] = V, [V,A] = A, [A,A] = V$ and ***conformal symmetry*** also provide typical examples.

Physically more interesting example can be found in ***thermodynamics***:
1st law of thermodynamics $\Longrightarrow$ $\Delta' Q \hookrightarrow \Delta E = \Delta' Q + \Delta' W \twoheadrightarrow \Delta' W$: exact sequence corresponding to $\mathfrak{h} \hookrightarrow \mathfrak{g} \twoheadrightarrow \mathfrak{m} = \mathfrak{g}/\mathfrak{h}$.

With respect to Cartan involution with $+$ assigned to heat production $\Delta' Q$ and $-$ to macroscopic work $\Delta' W$, the holonomy $[\mathfrak{m},\mathfrak{m}] \subset \mathfrak{h}$ corresponding to a loop in the space M of thermodynamic variables becomes just

Kelvin's version of 2nd law of thermodynamics

namely, holonomy $[\mathfrak{m},\mathfrak{m}]$ in the cyclic process with $\Delta E = \Delta' Q + \Delta' W = 0$, describes heat production $\Delta' Q \geq 0$: $-\Delta' W = -[\mathfrak{m},\mathfrak{m}] = \Delta' Q > 0$ (from system to outside)

2.11 Sector Bundle & Holonomy

In use of sector bundle $\hat{H} \hookrightarrow G \underset{H}{\times} \hat{H} \twoheadrightarrow G/H$, physical origin of space-time concept can be seen in its ***physical emergence process*** [7].

For simplicity, we assume here that a group G of broken internal symmetry be extended by a group $\mathcal{R}$ of space-time symmetry (typically translations) into a larger group $\Gamma = \mathcal{R} \dot{\times} G$ defined by a semi-direct product of $\mathcal{R}$ & G with $\Gamma/G = \mathcal{R}$.

In this case, the sector bundles have a double fibration structure:

$$\begin{array}{ccccc} \hat{H} & \hookrightarrow & G \underset{H}{\times} \hat{H} & \hookrightarrow & \Gamma \underset{G}{\times} (G \underset{H}{\times} \hat{H}) = \Gamma \underset{H}{\times} \hat{H} \\ & & \downarrow & & \downarrow \\ & & G/H & & \Gamma/G = \mathcal{R} \end{array}$$

2.12 Holonomy along Goldstone condensates

$\Longrightarrow$ Three different axes on different levels in Spec= sector-classifying space:
a) sectors $\widehat{H}$ of *unbroken* symmetry H,
b) deg. vacua $G/H = M$ due to *broken internal* symmetry [1, 5],
c) $\Gamma/G = \mathcal{R}$ as emergent *space-time* [7] in broken external symmetry.

These axes arise in a series of structure-group contractions $H \leftarrow G \leftarrow \Gamma$ of principal bdles $P_H \hookrightarrow P_G \hookrightarrow P_\Gamma$ over $\mathcal{R}$, specified by ***solderings*** as bdle sections, $\mathcal{R} \overset{\rho}{\hookrightarrow} P_G/H = P_H \underset{H}{\times} (G/H)$, $\mathcal{R} \overset{\tau}{\hookrightarrow} P_\Gamma/G = P_G \underset{G}{\times} (\Gamma/G)$
$= P_G \underset{G}{\times} \mathcal{R}$, corresponding physically to ***Goldstone modes***:

$$\begin{array}{ccccc}
P_H & \hookrightarrow & P_G & \hookrightarrow & P_\Gamma \\
H\downarrow & \circlearrowleft & \downarrow H & \circlearrowleft & \downarrow H \\
\mathcal{R} & \overset{\rho}{\hookrightarrow} & P_G/H & \overset{\sigma}{\hookrightarrow} & P_\Gamma/H \\
 & \searrow\!\!\searrow \circlearrowleft & \downarrow G/H & \circlearrowleft & \downarrow G/H \\
 & & \mathcal{R} & \overset{\tau}{\hookrightarrow} & P_\Gamma/G \\
 & & & \searrow\!\!\searrow \circlearrowleft & \downarrow \mathcal{R} \\
 & & & & \mathcal{R}
\end{array} .$$

2.13 Helgason duality with Hecke algebra

From algebraic viewpoint (which is dual to the ***Helgason duality*** $K\backslash G \leftrightarrow G/H$:

$$\begin{array}{ccccc}
 & \nearrow & K\backslash G/H & \nwarrow & \\
K\backslash G & & \leftrightarrow & & G/H \\
 & \nwarrow & G & \nearrow &
\end{array}$$

with Radon transforms & ***Hecke algebra*** $K\backslash G/H$), the essence of the relevant structures can be viewed as the "***stereo-graphic***" ***extension*** of such *planar* diagrams as controlling "augmented algebras" [1] of crossed products to describe symmetry breaking:

$$\begin{array}{ccccc}
{}_{G/H}\swarrow & \mathcal{X}^H = \widetilde{\mathcal{X}}^G & \searrow_H \\
\widetilde{\mathcal{X}}^H & \Downarrow & \mathcal{X} \\
\downarrow_H \searrow\!\!\searrow & \widetilde{\mathcal{X}} & \swarrow_{G/H} \downarrow \\
\downarrow\ \swarrow & \Downarrow & \searrow\!\!\searrow\ \downarrow \\
\widehat{H\backslash G} \hookrightarrow & \widehat{G} & \twoheadrightarrow \widehat{H}
\end{array}
\quad \rightleftarrows \quad
\begin{array}{ccc}
{}_{\mathcal{R}}\swarrow & \mathcal{O}_\rho = \mathcal{O}_d^H & \searrow_H \\
\mathcal{A}(\mathcal{R}) & \Downarrow & \mathcal{O}_d \\
\downarrow_H \searrow\!\!\searrow & \mathcal{X}(\mathcal{R}) & \swarrow_{\mathcal{R}} \downarrow \\
\downarrow\ \swarrow & \Downarrow & \searrow\!\!\searrow\ \downarrow \\
\widehat{\mathcal{R}} \hookrightarrow & \widehat{\Gamma} & \twoheadrightarrow \widehat{H}
\end{array}
\quad : \quad \text{[same sort of lines are in the same exact seq]}$$

Note that push-out diagram in DR reconstruction of field algebra $\mathcal{X}(\mathcal{R})$ shows up here (right) in spite of its unbroken symmetry.

3 Symmetric space structure & Maxwell-type equations

Symmetric space structures of $G/H = M$ & $\Gamma/G = \mathcal{R}$ due to symmetry breaking$\Leftrightarrow$ equation of type $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{h}$, which connects holonomy $[\mathfrak{m}, \mathfrak{m}]$ (in terms of curvature) with generators $\mathfrak{h}$ of unbroken subgroup.

Note that this feature is shared in common by Maxwell & Einstein equations of electromagnetism and of gravity, respectively:
LHS: (curvature $F_{\mu\nu}$ or $R_{\mu\nu}$) = (source current J_μ or $T_{\mu\nu}$) : RHS.

According to 2nd Noether theorem (developed in the theory of invariants), Maxwell equation is an identity following from the invariance of action integral under space-time dependent transformations.

In contrast, *no such classical quantities as action integrals nor Lagrangian densities* are available in our algebraic & categorical formulation of quantum fields.

3.1 Galois Functor in Doplicher-Roberts reconstruction of symmetry

The expected roles of action integral: to determine representation contents of a theory $\Longrightarrow$ can be substituted by categorical data concerning Galois group due to Doplicher & Roberts (DR), in terms of DR category $\mathcal{T}$ of modules of local excitations:

Obj($\mathcal{T}$) : local endomorphisms $\rho \in End(\mathcal{A})$ of observable alg. $\mathcal{A}$, selected by DHR localization criterion $\pi_0 \circ \rho \upharpoonright_{\mathcal{A}(\mathcal{O}')} \cong \pi_0 \upharpoonright_{\mathcal{A}(\mathcal{O}')}$,

Mor($\mathcal{T}$): $T \in \mathcal{T}(\rho \leftarrow \sigma) \subset \mathcal{A}$ intertwining $\rho, \sigma \in \mathcal{T}$: $\rho(A)T = T\sigma(A)$.

The group H of unbroken internal symmetry arises as the group $H = End_\otimes(V)$ of unitary tensorial (=monoidal) natural transformations $u : V \dot{\leftarrow} V$ with the representation functor $V : \mathcal{T} \hookrightarrow Hilb$ to embed $\mathcal{T}$ into category $Hilb$ of Hilbert spaces with morphisms as bounded linear maps.

3.2 Galois Functor in Category & its Local gauge invariance

In view of commutativity diagrams: $\begin{array}{ccc} V(\rho) & \overset{v_\rho}{\leftarrow} & W(\rho) \\ V(T)\uparrow & \circlearrowleft & \uparrow W(T) \\ V(\sigma) & \underset{v_\sigma}{\leftarrow} & W(\sigma) \end{array}$, i.e., $v_\rho W(T) = V(T) v_\sigma$ with $T \in \mathcal{T}(\rho \leftarrow \sigma)$, in the definition of natural transformation $v : V \dot{\leftarrow} W$, we try here to reinterpret it as a categorical definition of a *local gauge transformation* $W \overset{\tau_v}{\to} \tau_v(W) = V$ of a functor W into V on the basis of definition:

$$\tau_v(W)(T) := v_\rho W(T) v_\sigma^{-1} \quad \text{for } T \in \mathcal{T}(\rho \leftarrow \sigma).$$

Similar formula can be found for gauge links in lattice gauge theory.

Then, the commutativity, $u_\rho V(T) = V(T)u_\sigma$ for $u \in End_\otimes(V)$, can be interpreted as *local gauge invariance* $\tau_u(V) = V$ of the functor V under *local gauge transformation* $V \to \tau_u(V)$ induced by a natural transformation $u \in H = End_\otimes(V)$.

3.3 Local gauge invariance & Maxwell equation

In the original Doplicher-Roberts theory, local endomorphisms $\rho \in \mathcal{T} \subset End(\mathcal{A})$ have, unfortunately, been regarded *global* constant objects, owing to the emphasis on space-time transportability[1], and hence, the left-right difference of u_ρ and u_σ in $\tau_u(V)(T) := u_\rho V(T) u_\sigma^{-1}$ has not been properly recognized as important signal of local gauge structures.

From the general viewpoint of forcing method, however, the essential features of logical extension ***from constants to variables*** naturally lead to the interpretation of $\tau_u(V)(T) = u_\rho V(T) u_\sigma^{-1} = V(T)$ as the characterization of local gauge invariance of V under local gauge transform $u : \mathcal{T} \ni \rho \longmapsto u_\rho$.

This is in harmony also with the alternative formulation of principal bundles in terms of group-valued Čech cohomologies.

3.4 Symmetry breaking & Maxwell equation

In the above preliminary discussion, the recovered group H of unbroken symmetry is compact in DR theory. So, the space $\hat{H}$ of sector parameters is discrete, which makes it difficult to incorporate differential equations.

To adapt the roles of DR category $\mathcal{T} \subset End(\mathcal{A}) = End(\mathcal{X}^H)$ in determining the factor spectrum $Sp(\mathfrak{Z}(\mathcal{X}^H)) = \hat{H}$ to our present purpose, we need to replace $\mathcal{T}$ by $\tilde{\tilde{\mathcal{T}}} = End(\tilde{\tilde{\mathcal{X}}}^H)$ with $\tilde{\tilde{\mathcal{X}}} = \mathcal{X}^H \rtimes \widehat{\mathcal{R}}$ and with $\Gamma/G = \mathcal{R}$(: space-time) in the two-step construction of augmented algebras associated with the series of group extensions: unbroken $H \hookrightarrow$ broken internal $G \hookrightarrow$broken external Γ.

By repeating the categorical formulation of $End_\otimes(V : \mathcal{T} \hookrightarrow Hilb)$ with $\mathcal{T}$ and V replaced by $\tilde{\tilde{\mathcal{T}}}$ and $\tilde{\tilde{V}}$, we can reproduce the essence of 2nd Noether theorem to connect the local gauge invariance and Maxwell equation.

3.5 Second Noether theorem

In this context, 2nd Noether theorem can be generalized into a form with three type arguments, $x \in \mathcal{R}, \xi \in G/H, a \in \hat{H}$, so as to incorporate low-energy theorem (with "soft pions") due to symmetry breaking.

For simplicity, we repeat its standard form with infinitesimal local gauge transformation $\delta_\Lambda \varphi^a(x) = G^a(x) \cdot \Lambda(x) + T^{a\mu}(x) \cdot \partial_\mu \Lambda(x)$ of fields $\varphi^a(x)$ specified by an "inifinitesimal parameter" $\Lambda = \Lambda(x)$ of a natural transformation depending on sector parameter $x \in \mathcal{R}$.

Then Maxwell-type equation holds identically,

$$\partial_\nu K^{\nu\mu} + J^\mu = 0,$$

[1]This has led to the mathematical definition of "sectors" of $\mathcal{A}$ by $End(\mathcal{A})/Inn(\mathcal{A})$.

when $K^{\nu\mu}$ and J^{μ} are "defined" in relation with the "infinitesimal transforms" of Galois functor V:

$$K^{\nu\mu} \;:\; = T^{a\mu}\frac{\partial}{\partial(\partial_\nu\varphi^a)}V,$$
$$J^{\mu} \;:\; = T^{a\mu}(\frac{\partial}{\partial\varphi^a} - \frac{\partial}{\partial(\partial_\nu\varphi^a)})V + G^a\frac{\partial}{\partial(\partial_\mu\varphi^a)}V.$$

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