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ON THE KKM THEORY OF LOCALLY p-CONVEX SPACES

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ABSTRACT. In a recent paper [5], Gholizadeh et al. investigated the existence of a fixed point of multimaps on almost p-convex or p-convex subsets of topological vector spaces. Most of their results are originated from some previous works of Park on analytical fixed point theory. In this survey article, we recall such works and compare them with the corresponding ones in [5]. Finally, some general comments to [5] are added.

1. Introduction

In our previous talk [23] at the NACA, Chiang Rai, January 2015, we introduced some *recent* results in analytical fixed point theory based on our previous works. After that, we found a paper by Gholizadeh et al. [5], where a number of fixed point theorems due to the present author were claimed to be generalized. Our principal aim in this article is to introduce our previous works related to those in [5].

Let $0 < p \leq 1$. In [5], its authors investigated the existence of a fixed point of multimaps on almost p-convex or p-convex subsets of topological vector spaces. Most of their results are originated from some previous works of Park on the KKM theory and analytical fixed point theory. In fact, in [5] and [3], their authors extended our results in [7], [9], and [10]. Note that these three papers are based on the KKM theory. In this survey article, we recall such works and compare them with the corresponding ones in [3] and [5]. Finally, some general comments on [5] are added.

This paper is organized as follows. Section 2 is a preliminary on basic concepts of our KKM theory of abstract convex spaces. We recall there that ϕ_A-spaces are KKM spaces. Section 3 devotes to definitions related to p-convex spaces, which are shown to be new ϕ_A-spaces. In Section 4, we introduce general forms of the KKM type theorems due to ourselves. One of them is to obtain a KKM theorem for p-convex spaces and a general Alexandroff-Pasynkoff theorem for abstract convex spaces. Section 5 devotes to compare our previous fixed point theorems with the extended p-convex space versions in [5] and [3]. Finally, in Section 6, we give some further comments on the paper [5].

2. Abstract convex spaces

Multimaps are also called simply maps. Let $\langle D \rangle$ denote the set of all nonempty finite subsets of a set D. Recall the following in [16]:

Definition. An *abstract convex space* $(E, D; \Gamma)$ consists of a topological space E, a nonempty set D, and a multimap $\Gamma : \langle D \rangle \multimap E$ with nonempty values $\Gamma_A := \Gamma(A)$ for $A \in \langle D \rangle$, such that the Γ-*convex hull* of any $D' \subset D$ is denoted and defined by

$$\mathrm{co}_\Gamma D' := \bigcup \{\Gamma_A \mid A \in \langle D' \rangle\} \subset E.$$

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A subset X of E is called a Γ-*convex subset* of $(E, D; \Gamma)$ relative to D' if for any $N \in \langle D' \rangle$, we have $\Gamma_N \subset X$, that is, $\text{co}_\Gamma D' \subset X$.

In case $E = D$, let $(E; \Gamma) := (E, E; \Gamma)$.

Recall that some corrections on [16] appeared in [22].

Definition. Let $(E, D; \Gamma)$ be an abstract convex space and Z a topological space. For a multimap $F : E \multimap Z$ with nonempty values, if a multimap $G : D \multimap Z$ satisfies

$$F(\Gamma_A) \subset G(A) := \bigcup_{y \in A} G(y) \qquad \text{for all} \;\; A \in \langle D \rangle,$$

then G is called a *KKM map* with respect to F. A *KKM map* $G : D \multimap E$ is a KKM map with respect to the identity map 1_E.

A multimap $F : E \multimap Z$ is called a $\mathfrak{KC}$-*map* [resp. a $\mathfrak{KO}$-*map*] if, for any closed-valued [resp. open-valued] KKM map $G : D \multimap Z$ with respect to F, the family $\{G(y)\}_{y \in D}$ has the finite intersection property. In this case, we denote $F \in \mathfrak{KC}(E, Z)$ [resp. $F \in \mathfrak{KO}(E, Z)$].

Definition. The *partial KKM principle* for an abstract convex space $(E, D; \Gamma)$ is the statement $1_E \in \mathfrak{KC}(E, E)$; that is, for any closed-valued KKM map $G : D \multimap E$, the family $\{G(y)\}_{y \in D}$ has the finite intersection property. The *KKM principle* is the statement $1_E \in \mathfrak{KC}(E, E) \cap \mathfrak{KO}(E, E)$; that is, the same property also holds for any open-valued KKM map.

An abstract convex space is called a *(partial) KKM space* if it satisfies the (partial) KKM principle, resp.

In our recent works [11-13], we studied elements or foundations of the KKM theory on abstract convex spaces and noticed there that many important results therein are related to the partial KKM principle.

Example. We gave known examples of (partial) KKM spaces in [16] and the references therein. The following is one of them.

Definition. A ϕ_A-*space* $(X, D; \{\phi_A\}_{A \in \langle D \rangle})$ consists of a topological space X, a nonempty set D, and a family of continuous functions $\phi_A : \Delta_n \to X$ (that is, singular n-simplices) for $A \in \langle D \rangle$ with $|A| = n + 1$. By putting $\Gamma_A := \phi_A(\Delta_n)$ for each $A \in \langle D \rangle$, the triple $(X, D; \Gamma)$ becomes an abstract convex space.

Definition. For a ϕ_A-space $(X, D; \{\phi_A\})$, any multimap $G : D \multimap X$ satisfying

$$\phi_A(\Delta_J) \subset G(J) \;\; \text{for each} \;\; A \in \langle D \rangle \text{ and } J \in \langle A \rangle$$

is called a *KKM map*.

We show that every ϕ_A-space is a KKM space:

Lemma 1. *Let $(X, D; \Gamma)$ be a ϕ_A-space and $G : D \multimap X$ a multimap with nonempty closed [resp. open] values. Suppose that G is a KKM map. Then $\{G(a)\}_{a \in D}$ has the finite intersection property.*

Proof. Let $A = \{a_0, a_1, \ldots, a_n\} \in \langle D \rangle$. Then there exists a continuous function $\phi_A : \Delta_n \to \Gamma_A$ such that, for any $0 \le i_0 < i_1 < \cdots < i_k \le n$, we have

$$\phi_A(\text{co}\{e_{i_0}, e_{i_1}, \ldots, e_{i_k}\}) \subset \Gamma(\{a_{i_0}, a_{i_1}, \ldots, a_{i_k}\}) \cap \phi_A(\Delta_n).$$

Since G is a KKM map, it follows that

$$\begin{aligned} \text{co}\{e_{i_0}, e_{i_1}, \ldots, e_{i_k}\} &\subset \phi_A^{-1}(\Gamma(\{a_{i_0}, a_{i_1}, \ldots, a_{i_k}\}) \cap \phi_A(\Delta_n)) \\ &\subset \bigcup_{j=0}^{k} \phi_A^{-1}(G(a_{i_j}) \cap \phi_A(\Delta_n)). \end{aligned}$$

Since $G(a_{i_j}) \cap \phi_A(\Delta_n)$ is closed [resp. open] in the compact subset $\phi_A(\Delta_n)$ of Γ_A, $\phi_A^{-1}(G(a_{i_j}) \cap \phi_A(\Delta_n))$ is closed [resp. open] in Δ_n. Note that $e_i \multimap \phi_A^{-1}(G(a_i) \cap \phi_A(\Delta_n))$ is a KKM map on $\{e_0, e_1, \ldots, e_n\}$. Hence, by the original KKM theorem, we have

$$\bigcap_{i=0}^{n} \phi_A^{-1}(G(a_i) \cap \phi_A(\Delta_n)) \neq \emptyset,$$

which readily implies $\bigcap_{i=0}^{n} G(a_i) \neq \emptyset$. This completes the proof. □

Now we have the following diagram for triples $(E, D; \Gamma)$:

Simplex $\Longrightarrow$ Convex subset of a t.v.s. $\Longrightarrow$ Lassonde type convex space
$\Longrightarrow$ H-space $\Longrightarrow$ G-convex space $\Longrightarrow$ ϕ_A-space $\Longrightarrow$ KKM space
$\Longrightarrow$ Partial KKM space $\Longrightarrow$ Abstract convex space.

3. New KKM spaces

Let $0 < p \leq 1$. Recall the definitions given by Bayoumi [4, 5]:

Definition. (*p-convex set*) A set A in a vector space V is said to be *p-convex* if, for any $x, y \in A,\ s, t \geq 0$, we have

$$(1-t)^{1/p}x + t^{1/p}y \in A, \quad \text{whenever } 0 \leq t \leq 1.$$

Definition. (*p-convex hull*) If X is a topological vector space and $A \subset X$, the closed p-convex hull of A denoted by $\overline{C}_p(A)$ is the smallest closed p-convex set containing A.

Definition. (*p-convex combination*) Let A be p-convex and $x_1, \cdots, x_n \in A$, and $t_i \geq 0, \sum_1^n t_i^p = 1$. Then $\sum_1^n t_i x_i$ is called a p-convex combination of $\{x_i\}$. If $\sum_1^n |t_i|^p \leq 1$, then $\sum_1^n t_i x_i$ is called an absolutely p-convex combination. It is easy to see that $\sum_1^n t_i x_i \in A$ for a p-convex set A.

Definition. (*locally p-convex space*) A topological vector space is said to be locally p-convex if the origin has a fundamental set of absolutely p-convex 0-neighborhoods. This topology can be determined by p-seminorms which are defined in the obvious way.

Using these concepts, in [5], definitions of almost p-convex sets and the p-convexly almost fixed point property are introduced as generalizations of almost convex sets (due to Himmelberg) and the almost fixed point property, resp.

Now we have a new KKM space:

Lemma 2. *Suppose that X is a subset of a topological vector space E and D is a nonempty subset of X such that $C_p(D) \subset X$. Let $\Gamma_N := C_p(N)$ for each $N \in \langle D \rangle$. Then $(X, D; \Gamma)$ is a ϕ_A-space.*

Proof. Since $C_p(D) \subset X$, Γ_N is well-defined. For each $N = \{x_0, x_1, \cdots, x_n\} \subset D$, define $\phi_N : \Delta_n \to \Gamma_N$ by

$$\sum_{i=0}^{n} t_i e_i \mapsto \sum_{i=0}^{n} (t_i)^{\frac{1}{p}} x_i.$$

Then clearly $(X, D; \Gamma)$ is a ϕ_A-space. □

4. General KKM theorems

The following whole intersection property for the map-values of a KKM map is a standard form of the KKM type theorems [15,16,18]:

Theorem 1. *Let $(E, D; \Gamma)$ be a partial KKM space [resp. a KKM space] and $G : D \multimap E$ a multimap satisfying*
(1) G has closed [resp. open] values; and
(2) $\Gamma_N \subset G(N)$ for any $N \in \langle D \rangle$ (that is, G is a KKM map).
Then $\{G(z)\}_{z \in D}$ has the finite intersection property.
Further, if
(3) $\bigcap_{z \in M} \overline{G(z)}$ is compact for some $M \in \langle D \rangle$,
then we have

$$\bigcap_{y \in D} \overline{G(y)} \neq \emptyset.$$

Consider the following related four conditions for a map $G : D \multimap E$:

(a) $\bigcap_{z \in D} \overline{G(z)} \neq \emptyset$ implies $\bigcap_{z \in D} G(z) \neq \emptyset$.

(b) $\bigcap_{z \in D} \overline{G(z)} = \overline{\bigcap_{z \in D} G(z)}$ (G is *intersectionally closed-valued* in the sense of Luc et al).

(c) $\bigcap_{z \in D} \overline{G(z)} = \bigcap_{z \in D} G(z)$ (G is *transfer closed-valued*).

(d) G is closed-valued.

From the partial KKM principle we have a whole intersection property of the Fan type. The following is given in [18,19]:

Theorem 2. *Let $(E, D; \Gamma)$ be a partial KKM space and $G : D \multimap E$ a map such that*
(1) $\overline{G}$ is a KKM map [that is, $\Gamma_A \subset \overline{G}(A)$ for all $A \in \langle D \rangle$]; and
(2) there exists a nonempty compact subset K of E such that either
(i) $\bigcap_{z \in M} \overline{G(z)} \subset K$ for some $M \in \langle D \rangle$; or
(ii) for each $N \in \langle D \rangle$, there exists a compact Γ-convex subset L_N of E relative to some $D' \subset D$ such that $N \subset D'$ and

$$\overline{L_N} \cap \bigcap_{z \in D'} \overline{G(z)} \subset K.$$

Then we have $K \cap \bigcap_{z \in D} \overline{G(z)} \neq \emptyset$.
Furthermore,
(α) if G is transfer closed-valued, then $K \cap \bigcap \{G(z) \mid z \in D\} \neq \emptyset$;
(β) if G is intersectionally closed-valued, then $\bigcap \{G(z) \mid z \in D\} \neq \emptyset$.

We give some consequences of Theorem 1:

Theorem 3. [5] *Suppose that X is a subset of a topological vector space E and D is a nonempty subset of X such that $C_p(D) \subset X$. Also suppose that $G : D \multimap X$ is a multimap satisfying*
(a) $G(x)$ is closed [resp. open] in X for all $x \in D$.
(b) $C_p(N) \subset G(N)$ for each $N \in \langle D \rangle$.
Then $\{G(x) \mid x \in D\}$ has the finite intersection property.

Proof. By putting $\Gamma_N := C_p(N)$, $(X, D; \Gamma)$ is a KKM space by Lemma 2. Now the conclusion follows from Theorem 1. □

From Theorem 1, we have the following generalization of the Alexandroff-Pasynkoff theorem [1]:

Theorem 4. *Let $(E, D; \Gamma)$ be a partial KKM space [resp. a KKM space], $A \subset E$, $\{A_0, A_1, \ldots, A_N\}$ be a family of closed [resp. open] subsets of E such that $A \subset \bigcup_{i=0}^n A_i$, and $N = \{z_0, z_1, \ldots, z_n\}$ be a family of points in D such that $\Gamma(N) \subset A$. If $\Gamma(N \setminus \{z_i\}) \subset A_i$ for each $i = 0, 1, \ldots, n$, then $\bigcap_{i=0}^n A_i \neq \emptyset$.*

Proof. Let $C_0 = \Gamma(N \setminus \{z_n\})$ and for $i = 1, 2, \ldots, n$, let $C_i = \Gamma(N \setminus \{z_{i-1}\})$. Define a multimap $F : D \multimap X$ by $F(z_0) = A_n$, $F(z_i) = A_{i-1}$ for $i = 1, 2, \ldots, n$, and $F(z) = X$ for all $z \in D \setminus N$. We claim that F is a KKM map. To see this, we note that $\Gamma(N) \subset A \subset \bigcup_{i=0}^n A_i = F(N)$ and for any proper subset $z_{i_0}, z_{i_1}, \ldots, z_{i_k}$ of N with $0 \leq k < n$ and $0 \leq i_0 < i_1 < \cdots < i_k \leq n$, we have

$$\Gamma(\{z_{i_0}, z_{i_1}, \ldots, z_{i_k}\}) \subset C_{i_j} \subset A_{i_j - 1} = F(z_{i_j})$$

for some $j \in \{0, 1, \ldots, k\}$. Note that $i_j = 0$ if and only if $i_j - 1 = n$, and so $\Gamma(\{z_{i_0}, z_{i_1}, \ldots, z_{i_k}\}) \subset \bigcup_{j=0}^k F(z_{i_j})$. Now by Theorem 1 we have $\bigcap_{i=0}^n A_i \neq \emptyset$. □

Remarks. 1. If we adopt Theorem 2 instead of Theorem 1, we may have another version of Theorem 4.

2. Note that [5, Theorem 2.2] is a generalized minimal space version of Theorem 4 motivated from the previous work of Park [7].

3. It is well-known that the Alexandroff-Pasynkoff theorem implies the Brouwer fixed point theorem (e.g., see [24]). Therefore, Theorem 4 is also equivalent to the KKM theorem.

5. Original results extended to p-convex spaces

Recall that, in [5] and [3], their authors extended our results in [7], [9], and [10] to p-convex spaces, and these three papers of ours are based on the KKM theory. Now, we give our original results in there, and indicate the corresponding results extended by [5] and [3].

Theorem 5. [7] *Let X be a subset of a Hausdorff topological vector space E and Y an almost convex dense subset of X. Let $T : X \multimap E$ be a lower [resp. upper] semicontinuous multimap such that $T(y)$ is convex for all $y \in Y$. If there is a precompact subset K of X such that $T(y) \cap K \neq \emptyset$ for each $y \in Y$, then for any convex neighborhood U of the origin 0 of E, there exists a point $x_U \in Y$ such that $T(x_U) \cap (x_U + U) \neq \emptyset$.*

Note that Hausdorffness of E is redundant. In [5, Theorem 2.7], all 'convex' is replaced in Theorem 5 by p-convex.

Corollary 6. [7] *Let X be a convex subset of a Hausdorff topological vector space E. Let $T : X \multimap E$ be a lower [resp. upper] semicontinuous multimap such that $T(x)$ is convex for each $x \in X$. If there is a precompact subset K of X such that $T(x) \cap K \neq \emptyset$ for each $x \in X$, then for every convex neighborhood U of the origin 0 of E, there exists a point $x_U \in X$ such that $T(x_U) \cap (x_U + U) \neq \emptyset$.*

Note that Hausdorffness of E is redundant. In [5, Corollary 2.8], all 'convex' in Corollary 6 is replaced by p-convex.

Corollary 7. [7] *Let X be a subset of a locally convex Hausdorff topological vector space E and Y an almost convex dense subset of X. Let $T : X \multimap X$ be a compact upper semicontinuous multimap with closed values such that $T(y)$ is nonempty convex for all $y \in Y$. Then T has a fixed point $x_0 \in X$; that is, $x_0 \in T(x_0)$.*

In [5, Theorem 2.12], all 'convex' in Corollary 7 is replaced by p-convex and Hausdorffness is not assumed, but used in its proof. This means that, in [5], all topological spaces seem to be assumed Hausdorff.

Corollary 8. [7] *Let X be a subset of a Hausdorff topological vector space E and Y an almost convex dense subset of X. Let $T : X \multimap E$ be a multimap such that*

(1) $T^-(z)$ is open for each $z \in E$; and

(2) $T(y)$ is convex for for each $y \in Y$.

If there is a precompact subset K of X such that $T(y) \cap K \neq \emptyset$ for each $y \in Y$, then for any convex neighborhood U of the origin 0 of E, there exists a point $x_U \in Y$ such that $T(x_U) \cap (x_U + U) \neq \emptyset$.

In [5, Corollary 2.9], all 'convex' in Corollary 8 is replaced by p-convex, and Hausdorffness is not assumed.

Corollary 9. [7] *Let X be a convex subset of a Hausdorff topological vector space E, and $T : X \multimap X$ be a compact multimap such that*

(1) $T(x)$ is nonempty and convex for each $x \in X$;

(2) $T^-(y)$ is open for each $y \in X$; and

Then for any convex neighborhood U of the origin 0 of E, there exists a point $x_U \in X$ such that $T(x_U) \cap (x_U + U) \neq \emptyset$.

Here Hausdorffness is redundant. In [5, Corollary 2.10], all 'convex' in Corollary 9 is replaced by p-convex, and added that T can be assumed u.s.c. instead of (2).

Theorem 10. [9] *Let X be a star-shaped subset of a Hausdorff topological vector space E with the origin O of E as the center. Let $f : X \to X$ be a compact continuous map. Then one of the following holds:*

(i) f has a fixed point $x_0 = f(x_0) \in X$;

(ii) there exist a point $y_0 \in X$ and a $t_0 \in (0, 1)$ such that $O \neq y_0 = t_0 f(y_0)$; or

(iii) $f(O) \neq O$.

In [3], this is extended to a p-star shaped subsets of a topological vector space via Fan-KKM principle in a generalized convex space.

Theorem 11. [10] *Let X be a convex subset of a locally convex Hausdorff t.v.s. E. Then any closed compact multimap $T : X \multimap X$ having the almost fixed property has a fixed point.*

In [5, Theorem 2.14], all 'convex' in Theorem 11 is replaced by p-convex.

Theorem 12. [10] *Let X be a compact convex subset of a t.v.s. E and $T : X \multimap X$ a multimap such that*

(i) T has the almost fixed point property;

(ii) T has closed values; and

(iii) T satisfies condition

$$\bigcap_{U \in \mathcal{V}} \{x \in X \mid x \in T(x) + U\} = \bigcap_{U \in \mathcal{V}} \mathrm{cl}\{x \in X \mid x \in T(x) + \mathrm{coU}\},$$

where $\mathcal{V}$ is a local base of open neighborhoods of 0 in E.

Then T has a fixed point.

Note that [5, Theorem 2.19 and Corollaries 2.20-2.22] are all motivated from Theorem 12 above by replacing all 'convex' by p-convex.

Corollary 13. [10] *Let X be a compact convex subset of a locally convex Hausdorff t.v.s. Then any closed multimap $T : X \multimap X$ having the almost fixed point property has a fixed point.*

Moreover, in [5, Theorem 2.24], all 'convex' in Corollary 13 is replaced by p-convex and the almost fixed point property by the p-convexly almost fixed point property.

6. Further comments on [5]

1. In [5] the authors are based on the KKM type theorems (Theorems 1.3 and 1.4 there) on generalized minimal spaces in [2], and noted that they are generalizations of Theorem 1 in Park [8,6]. However the concept of G-convex spaces are obsolete and we established already much more general theory on abstract convex spaces. Moreover, since any minimal space can be made into a topological space, results on abstract convex minimal spaces can be deduced from the theory of abstract convex spaces; see [14, 31]. Note also that some authors are still publishing papers on minimal spaces.

2. Notice that no consideration on the Hausdorffness of topological vector spaces are given in [5]. Many results there can hold without assuming the Hausdorffness. This can be also stated the original works of Park on which [5] has based. In the present paper we clearly distinguish original results where Hausdorffness is redundant.

3. The following is given in [17]:

Definition. A *γ-convex space* $(E, D; \gamma)$ consists of a topological space E, a nonempty set D, and a multimap $\gamma : D \times D \multimap E$ with nonempty values $\gamma(a, b)$ for any $a, b \in D$.

For any $D' \subset D$, the *γ-convex hull* of D' is denoted and defined by

$$\mathrm{co}_\gamma D' := \bigcup\{\gamma(a, b) \mid a, b \in D'\} \subset E.$$

A subset X of E is called a *γ-convex subset* of $(E, D; \gamma)$ relative to D' if for any $a, b \in D'$, we have $\gamma(a, b) \subset X$, that is, $\mathrm{co}_\gamma D' \subset X$.

In case $E \supset D$, let $(E \supset D; \gamma) := (E, D; \gamma)$ and let $(E; \gamma) := (E, E; \gamma)$.

Note that a p-convex subset X (in the sense of Bayoumi) of a topological vector space E is a γ-convex subset of $(E, X; \gamma)$ relative to X itself.

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