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The developments of discrete-types of NCP-functions

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Abstract. In this paper, we summarize the developments of some new discrete-types of NCP-functions, which are recently proposed by the author and his team. The behind idea is explained, the related properties are presented, and future directions based on such discrete NCP-functions are discussed as well.

KEYWORDS. NCP-function; Complementarity.

1 Introduction

The nonlinear complementarity problem (NCP) [12, 18] is to find a point $x \in \mathbb{R}^n$ such that

$$x \geq 0,\ \ F(x) \geq 0, \langle x,\ \ F(x)\rangle = 0$$

where $\langle\cdot,\cdot\rangle$ is the Euclidean inner product and $F=(F_1,\cdots,F_n)^T$ maps from $\mathbb{R}^n$ to $\mathbb{R}^n$. The NCP has attracted much attention due to its various applications in operations research, economics, and engineering, see [9, 12, 18] and references therein. There have been many methods proposed for solving the NCP. Among which, one of the most popular and powerful approaches that has been studied intensively recently is to reformulate the NCP as a system of nonlinear equations [17] or as an unconstrained minimization problem [8, 10, 14]. Such a function that can constitute an equivalent unconstrained minimization problem for the NCP is called a merit function. In other words, a merit function is a function whose global minima are coincident with the solutions of the original NCP. For constructing a merit function, the class of functions, so-called NCP-functions plays an important role.

A function $\phi : \mathbb{R}^2 \to \mathbb{R}$ is called an NCP-function if it satisfies

$$\phi(a,b) = 0 \iff a \geq 0,\ \ b \geq 0,\ \ ab = 0. \tag{1}$$

Many NCP-functions and merit functions have been explored and proposed in many literature, see [11] for a survey. Among them, the Fischer-Burmeister (FB) function and the Natural-Residual (NR) function are two effective NCP-functions. The FB function $\phi_{\text{FB}} : \mathbb{R}^2 \to \mathbb{R}$ is defined by

$$\phi_{\text{FB}}(a,b) = \sqrt{a^2+b^2} - (a+b), \tag{2}$$

and the NR function $\phi_{\text{NR}} : \mathbb{R}^2 \to \mathbb{R}$ is defined by

$$\phi_{\text{NR}}(a,b) = a - (a-b)_+ = \min\{a,b\}, \tag{3}$$

where $(t)_+$ means $\max\{0,t\}$ for any $t \in \mathbb{R}$.

Recently, the generalized Fischer-Burmeister function ϕ^p_{FB} which includes the Fischer-Burmeister as a special case was considered in [2, 3, 4, 6, 19]. Indeed, the function ϕ^p_{FB} is a natural extension of the ϕ_{FB} function, in which the 2-norm in ϕ_{FB} is replaced by general p-norm. In other words, $\phi^p_{\text{FB}} : \mathbb{R}^2 \to \mathbb{R}$ is defined as

$$\phi^p_{\text{FB}}(a,b) = \|(a,b)\|_p - (a+b), \tag{4}$$

where $p > 1$ and $\|(a,b)\|_p = \sqrt[p]{|a|^p + |b|^p}$. The detailed geometric view of ϕ^p_{FB} is depicted in [19]. Corresponding to ϕ^p_{FB}, there is a merit function $\psi^p_{\text{FB}} : \mathbb{R}^2 \to \mathbb{R}_+$ given by

$$\psi^p_{\text{FB}}(a,b) = \frac{1}{2}\left|\phi^p_{\text{FB}}(a,b)\right|^2. \tag{5}$$

For any given $p > 1$, the function ψ^p_{FB} is a nonnegative NCP-function and smooth on $\mathbb{R}^2$. Note that ϕ^p_{FB} is a natural "continuous" type of generalization of the FB function ϕ_{FB}. The graphs of ϕ^p_{FB} with different p are depicted in Figure 1.

To the contrast, what does "generalized natural-residual function" look like? This has been a long-standing open question. In other words, we want to know

$$\phi_{\text{FB}}(a,b) = \|(a,b)\|_2 - (a+b) \quad \longrightarrow \quad \phi^p_{\text{FB}}(a,b) = \|(a,b)\|_p - (a+b)$$

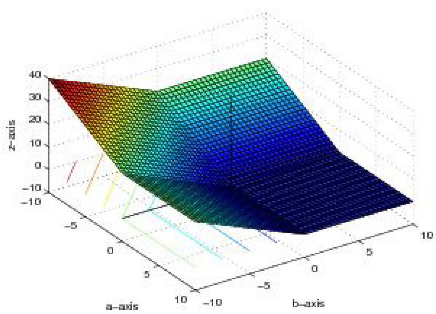
$$\phi_{\text{NR}}(a,b) = \min\{a,b\} \quad \longrightarrow \quad ???$$

In [5], Chen *et al.* give an answer to the long-standing open question. More specifically, the generalized natural-residual function, denoted by ϕ^p_{NR}, is defined by

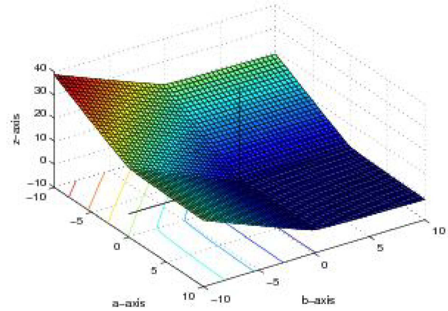
$$\phi^p_{\text{NR}}(a,b) = a^p - (a-b)^p_+ \tag{6}$$

with $p > 1$ being a positive odd integer. As remarked in [5], the main idea to create it relies on "discrete generalization", not the "continuous generalization". Note that when $p = 1$, ϕ^p_{NR} reduces to the natural residual function ϕ_{NR}. The graphs of ϕ^p_{NR} with different p are depicted in Figure 2.

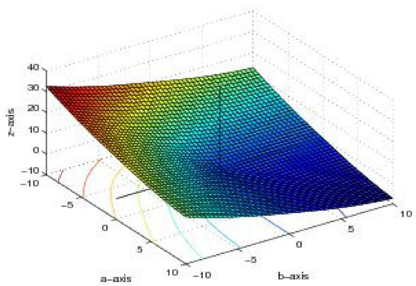
Unlike the surface of ϕ^p_{FB}, the surface of ϕ^p_{NR} is not symmetric which may cause some difficulties in further analysis in designing solution methods. To this end, Chang *et al.*



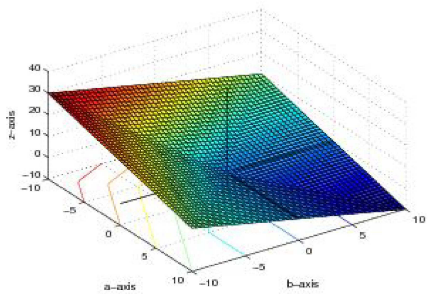
(a) $z = \phi^1_{\text{FB}}(a,b)$



(b) $z = \phi^{1.1}_{\text{FB}}(a,b)$



(c) $z = \phi^3_{\text{FB}}(a,b)$



(d) $z = \phi^{100}_{\text{FB}}(a,b)$

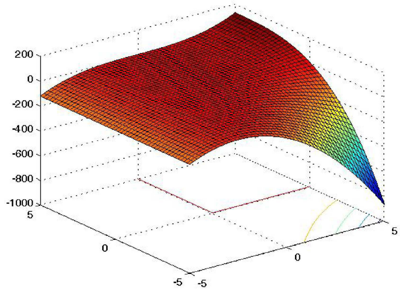
Figure 1: The surface of $z = \phi^p_{\text{FB}}(a,b)$ with different values of p

[1] try to symmetrize the function ϕ^p_{NR}. The first-type symmetrization of ϕ^p_{NR}, denoted by $\phi^p_{\text{S-NR}}$ is proposed as

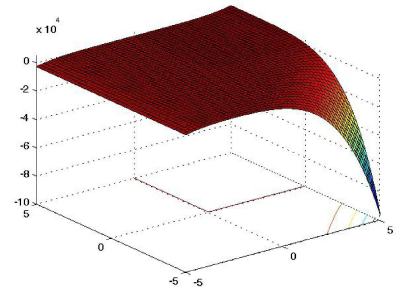
$$\phi^p_{\text{S-NR}}(a,b) = \begin{cases} a^p - (a-b)^p & \text{if} \quad a > b, \\ a^p = b^p & \text{if} \quad a = b, \\ b^p - (b-a)^p & \text{if} \quad a < b, \end{cases} \tag{7}$$

where $p > 1$ being positive odd integer. It is shown in [1] that $\phi^p_{\text{S-NR}}$ is an NCP-function with symmetric surface, but it is not differentiable. The graphs of $\phi^p_{\text{S-NR}}$ with different p are depicted in Figure 3.

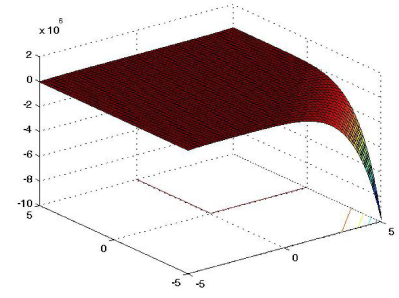
Therefore, it is natural to ask whether there exists another symmetrization function that has not only symmetric surface but also is differentiable. Fortunately, Chang *et al.* [1] also figure out the second symmetrization of ϕ^p_{NR}, denoted by $\psi^p_{\text{S-NR}}$, which is proposed



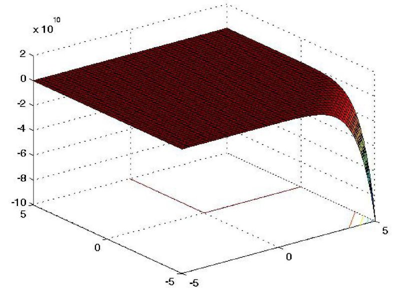
(a) $z = \phi^3_{\mathrm{NR}}(a,b)$



(b) $z = \phi^5_{\mathrm{NR}}(a,b)$



(c) $z = \phi^7_{\mathrm{NR}}(a,b)$



(d) $z = \phi^{11}_{\mathrm{NR}}(a,b)$

Figure 2: The surface of $z = \phi^p_{\mathrm{NR}}(a,b)$ with different values of p

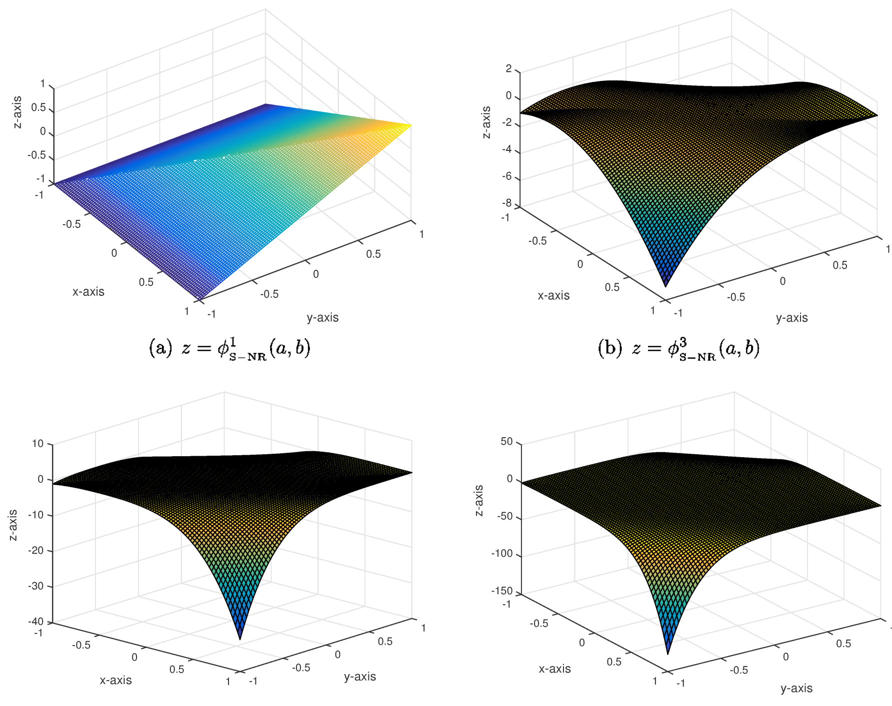
as

$$\psi^p_{\mathrm{S-NR}}(a,b) = \begin{cases} a^p b^p - (a-b)^p b^p & \text{if} \quad a > b, \\ a^p b^p = a^{2p} & \text{if} \quad a = b, \\ a^p b^p - (b-a)^p a^p & \text{if} \quad a < b, \end{cases} \tag{8}$$

where $p > 1$ being positive odd integer. As expected, the function $\psi^p_{\mathrm{S-NR}}$ is not only differentiable but also possesses symmetric surface. The graphs of $\psi^p_{\mathrm{S-NR}}$ with different p are depicted in Figure 4.

The idea of "discrete generalization" looks simple, but it is novel and important. In fact, the author also apply such idea to construct more NCP-functions. For example, the authors apply it to the Fischer-Burmeister function to obtain $\phi^p_{\mathrm{D-FB}} : \mathbb{R}^2 \to \mathbb{R}$ given by

$$\phi^p_{\mathrm{D-FB}}(a,b) = \left(\sqrt{a^2+b^2}\right)^p - (a+b)^p \tag{9}$$



(a) $z = \phi^1_{\text{S-NR}}(a,b)$ (b) $z = \phi^3_{\text{S-NR}}(a,b)$

(c) $z = \phi^5_{\text{S-NR}}(a,b)$ (d) $z = \phi^7_{\text{S-NR}}(a,b)$

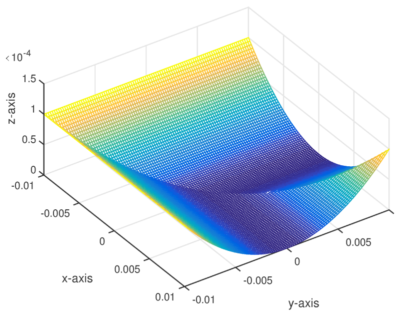
Figure 3: The surface of $z = \phi^p_{\text{S-NR}}(a,b)$ with different values of p

where $p > 1$ is a positive odd integer. This function is proved as an NCP-function in [16]. The graphs of $\phi^p_{\text{D-FB}}$ with different p are depicted in Figure 5.

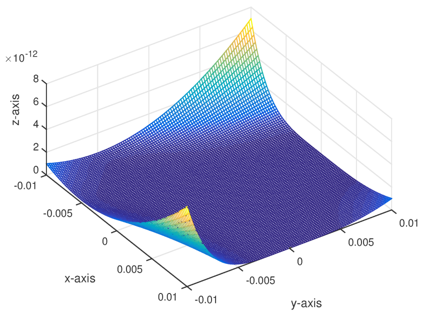
The aforementioned four types of discrete NCP-functions are newly discovered. Unlike the existing NCP-functions, we know that they are discrete-oriented. Even though we have the feature of differentiability, we point out that the Newton method may not applied directly because the Jacobian at a degenerate solution to NCP is singular (see [14, 15]). Nonetheless, this feature may enable that many methods like derivative-free algorithm can be employed directly for solving NCP.

To close this section, we present some well-known properties of ϕ^p_{FB} and ψ^p_{FB}, defined as in (4) and (5), respectively, that are important for designing a descent algorithm that is indeed derivative-free method.

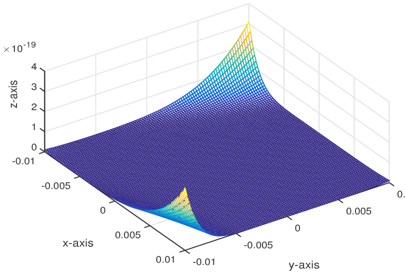
Property 1.1. ([6, Propostion 3.1]) *Let ϕ^p_{FB} be defined as in (4). Then, the following*



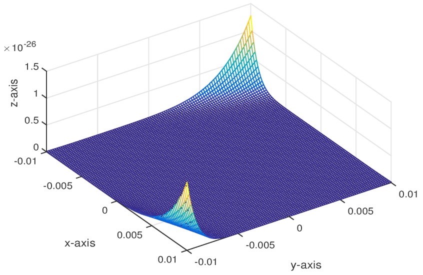
(a) $z = \psi^1_{\text{S-NR}}(a, b)$



(b) $z = \psi^3_{\text{S-NR}}(a, b)$



(c) $z = \psi^5_{\text{S-NR}}(a, b)$



(d) $z = \psi^7_{\text{S-NR}}(a, b)$

Figure 4: The surface of $z = \psi^p_{\text{S-NR}}(a, b)$ with different values of p

hold.

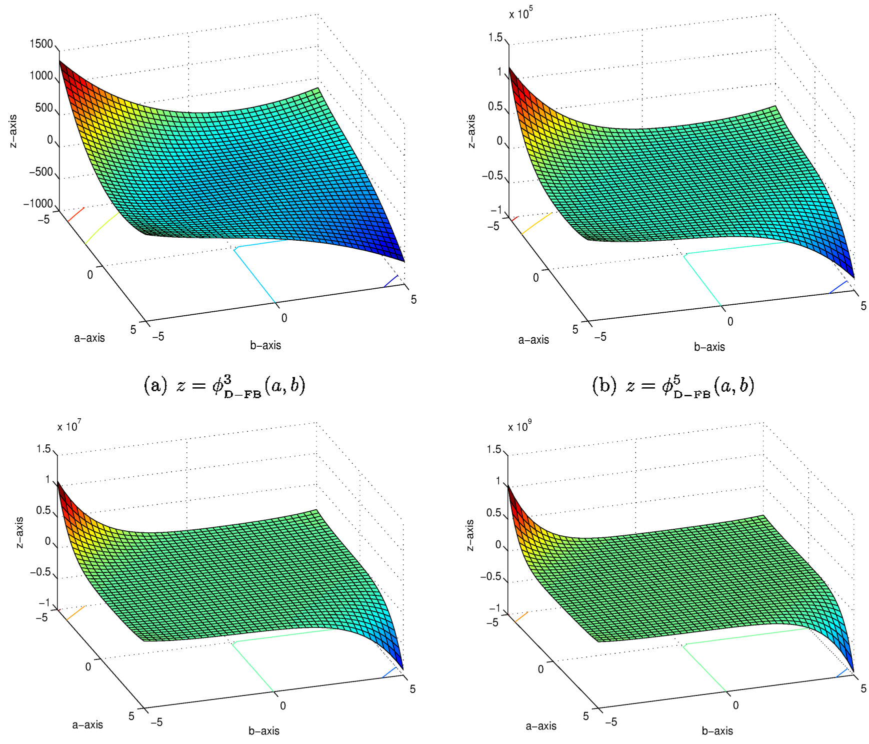
(a) ϕ^p_{FB} *is a NCP-function, i.e., it satisfies (1).*

(b) ϕ^p_{FB} *is sub-additive, i.e.,* $\phi_p(w + w') \leq \phi_p(w) + \phi_p(w') \quad \forall w, w' \in \mathbb{R}^2$.

(c) ϕ^p_{FB} *is positive homogeneous, i.e.,* $\phi_p(\alpha w) = \alpha\phi_p(w)$ *for all* $w \in \mathbb{R}^2$ *and* $\alpha \geq 0$.

(d) ϕ^p_{FB} *is convex, i.e.,* $\phi_p(\alpha w + (1-\alpha)w') \leq \alpha\phi_p(w) + (1-\alpha)\phi_p(w')$ *for all* $w, w' \in \mathbb{R}^2$ *and* $\alpha \geq 0$.

(e) ϕ^p_{FB} *is Lipschitz continuous with* $\kappa_1 = \sqrt{2} + 2^{(1/p-1/2)}$ *when* $1 < p < 2$, *and with* $\kappa_2 = 1 + \sqrt{2}$ *when* $p \geq 2$. *In other words,* $|\phi^p_{\text{FB}}(w) - \phi^p_{\text{FB}}(w')| \leq \kappa_1\|w - w'\|$ *when* $1 < p < 2$ *and* $|\phi^p_{\text{FB}}(w) - \phi^p_{\text{FB}}(w')| \leq \kappa_2\|w - w'\|$ *when* $p \geq 2$ *for all* $w, w' \in \mathbb{R}^2$.



(a) $z = \phi^{3}_{\mathrm{D-FB}}(a,b)$ (b) $z = \phi^{5}_{\mathrm{D-FB}}(a,b)$

(c) $z = \phi^{7}_{\mathrm{D-FB}}(a,b)$ (d) $z = \phi^{9}_{\mathrm{D-FB}}(a,b)$

Figure 5: The surface of $z = \phi^{p}_{\mathrm{D-FB}}(a,b)$ with different values of p

Property 1.2. ([6]) *Let ϕ^p_{FB} be defined as in (4). Then, the following variants of ϕ^p_{FB} are also NCP-functions.*

$$
\begin{aligned}
\widetilde{\phi^p_{\mathrm{FB}\,1}}(a,b) &= \phi^p_{\mathrm{FB}}(a,b) - \alpha(a)_+(b)_+, \ \ \alpha > 0. \\
\widetilde{\phi^p_{\mathrm{FB}\,2}}(a,b) &= \phi^p_{\mathrm{FB}}(a,b) - \alpha(ab)_+, \ \ \alpha > 0. \\
\widetilde{\phi^p_{\mathrm{FB}\,3}}(a,b) &= \sqrt{[\phi^p_{\mathrm{FB}}(a,b)]^2 + \alpha\left((a)_+(b)_+\right)^2}, \ \ \alpha > 0. \\
\widetilde{\phi^p_{\mathrm{FB}\,4}}(a,b) &= \sqrt{[\phi^p_{\mathrm{FB}}(a,b)]^2 + \alpha\left[(ab)_+\right]^2}, \ \ \alpha > 0.
\end{aligned}
$$

Property 1.3. ([7, Lemma 2.2]) *Let ϕ^p_{FB} be defined as in (4). Then, the generalized* *gradient* $\partial\phi^p_{\mathrm{FB}}(a,b)$ *of* ϕ^p_{FB} *at a point* (a,b) *is equal to the set of all* (v_a,v_b) *such that*

$$(v_a,v_b)=\begin{cases}\left(\dfrac{\operatorname{sgn}(a)\cdot|a|^{p-1}}{\|(a,b)\|_p^{p-1}}-1,\dfrac{\operatorname{sgn}(b)\cdot|b|^{p-1}}{\|(a,b)\|_p^{p-1}}-1\right) & \text{if} \quad (a,b)\neq(0,0),\\ (\xi-1,\zeta-1) & \text{if} \quad (a,b)=(0,0),\end{cases}$$

where (ξ,ζ) *is any vector satisfying* $|\xi|^{\frac{p}{p-1}}+|\zeta|^{\frac{p}{p-1}}\leq 1$.

Property 1.4. ([6, Propostion 3.2]) *Let* ϕ^p_{FB}, ψ^p_{FB} *be defined as in (4) and (5), respectively. Then, the following hold.*

(a) ψ^p_{FB} *is a NCP-function, i.e., it satisfies (1).*

(b) $\psi^p_{\mathrm{FB}}(a,b)\geq 0$ *for all* $(a,b)\in\mathbb{R}^2$.

(c) ψ^p_{FB} *is continuously differentiable everywhere.*

(d) $\nabla_a\psi^p_{\mathrm{FB}}(a,b)\cdot\nabla_b\psi^p_{\mathrm{FB}}(a,b)\geq 0$ *for all* $(a,b)\in\mathbb{R}^2$. *The equality holds if and only if* $\phi^p_{\mathrm{FB}}(a,b)=0$.

(e) $\nabla_a\psi^p_{\mathrm{FB}}(a,b)=0\Longleftrightarrow\nabla_b\psi^p_{\mathrm{FB}}(a,b)=0\Longleftrightarrow\phi^p_{\mathrm{FB}}(a,b)=0$.

2 Properties of the function ϕ^p_{NR}

In this section, we present all the properties of ϕ^p_{NR} defined as in (6).

Proposition 2.1. ([5, Proposition 2.1]) *Let* ϕ^p_{NR} *be defined as in (6) with* $p>1$ *being a positive odd integer. Then,* ϕ^p_{NR} *is an NCP-function.*

Proposition 2.2. ([5, Proposition 2.2]) *Let* ϕ^p_{NR} *be defined as in (6) with* $p>1$ *being a positive odd integer, and let* $p=2k+1$ *where* $k=1,2,3,\cdots$. *Then, the following hold.*

(a) *An alternative expression of* ϕ^p_{NR} *is*

$$\phi^p_{\mathrm{NR}}(a,b)=a^{2k+1}-\frac{1}{2}\left((a-b)^{2k+1}+(a-b)^{2k}|a-b|\right).$$

(b) *The function* ϕ^p_{NR} *is continuously differentiable with*

$$\nabla\phi^p_{\mathrm{NR}}(a,b)=p\begin{bmatrix}a^{p-1}-(a-b)^{p-2}(a-b)_+\\(a-b)^{p-2}(a-b)_+\end{bmatrix}.$$

(c) *The function* $\phi^p_{\rm NR}$ *is twice continuously differentiable with*

$$\nabla^2 \phi^p_{\rm NR}(a,b) = p(p-1)\begin{bmatrix} a^{p-2} - (a-b)^{p-3}(a-b)_+ & (a-b)^{p-3}(a-b)_+ \\ (a-b)^{p-3}(a-b)_+ & -(a-b)^{p-3}(a-b)_+ \end{bmatrix}.$$

Proposition 2.3. ([5, Proposition 2.4]) *Let* $\phi^p_{\rm NR}$ *be defined as in (6) with* $p > 1$ *being a positive odd integer. Then, the following variants of* $\phi^p_{\rm NR}$ *are also NCP-functions.*

$$\begin{aligned}
\widetilde{\phi^p_{\rm NR\,1}}(a,b) &= \phi^p_{\rm NR}(a,b) + \alpha(a)_+(b)_+, \ \alpha > 0. \\
\widetilde{\phi^p_{\rm NR\,2}}(a,b) &= \phi^p_{\rm NR}(a,b) + \alpha\left((a)_+(b)_+\right)^2, \ \alpha > 0. \\
\widetilde{\phi^p_{\rm NR\,3}}(a,b) &= [\phi^p_{\rm NR}(a,b)]^2 + \alpha\left((ab)_+\right)^4, \ \alpha > 0. \\
\widetilde{\phi^p_{\rm NR\,4}}(a,b) &= [\phi^p_{\rm NR}(a,b)]^2 + \alpha\left((ab)_+\right)^2, \ \alpha > 0.
\end{aligned}$$

Proposition 2.4. ([13, Proposition 3.4]) *Let* $\phi^p_{\rm NR}$ *be defined as in (6) with* $p > 1$ *being a positive odd integer. Then, the following hold.*

(a) $\phi^p_{\rm NR}(a,b) > 0 \iff a > 0, \ b > 0.$

(b) $\phi^p_{\rm NR}$ *is positive homogeneous of degree* p, *i.e.,* $\phi^p_{\rm NR}(\alpha w) = \alpha^p \phi^p_{\rm NR}(w)$ *for all* $w \in \mathbb{R}^2$ *and* $\alpha \geq 0$.

(c) $\phi^p_{\rm NR}$ *is not Lipschitz continuous.*

(d) $\phi^p_{\rm NR}$ *is not* α*-Hölder continuous for any* $\alpha \in (0,1]$, *that is, the Hölder coefficient*

$$[\phi^p_{\rm NR}]_{\alpha,\mathbb{R}^2} := \sup_{w \neq w'} \frac{|\phi^p_{\rm NR}(w) - \phi^p_{\rm NR}(w')|}{\|w - w'\|^\alpha}$$

is infinite.

(e) $$\nabla_a \phi^p_{\rm NR}(a,b) \cdot \nabla_b \phi^p_{\rm NR}(a,b) \begin{cases} > 0 & \text{on } \{(a,b) \mid a > b > 0 \text{ or } a > b > 2a\}, \\ = 0 & \text{on } \{(a,b) \mid a \leq b \text{ or } a > b = 2a \text{ or } a > b = 0\}, \\ < 0 & \text{otherwise.} \end{cases}$$

(f) $\nabla_a \phi^p_{\rm NR}(a,b) \cdot \nabla_b \phi^p_{\rm NR}(a,b) = 0$ *provided that* $\phi^p_{\rm NR}(a,b) = 0$.

3 Properties of the function $\phi^p_{\mathrm{S-NR}}$

In this section, we present all the properties of $\phi^p_{\mathrm{S-NR}}$ defined as in (7).

Proposition 3.1. ([1, Proposition 2.1]) *Let $\phi^p_{\mathrm{S-NR}}$ be defined as in (7) with $p > 1$ being a positive odd integer. Then, $\phi^p_{\mathrm{S-NR}}$ is an NCP-function and is positive only on the first quadrant $\mathbb{R}^n_{++} := \{(a,b)\,|\,a > 0,\ b > 0\}$.*

Proposition 3.2. ([1, Proposition 2.2]) *Let $\phi^p_{\mathrm{S-NR}}$ be defined as in (7) with $p > 1$ being a positive odd integer. Then, the following hold.*

(a) *An alternative expression of $\phi^p_{\mathrm{S-NR}}$ is*

$$\phi^p_{\mathrm{S-NR}}(a,b) = \begin{cases} \phi^p_{\mathrm{NR}}(a,b) & \text{if } a > b, \\ a^p = b^p & \text{if } a = b, \\ \phi^p_{\mathrm{NR}}(b,a) & \text{if } a < b. \end{cases}$$

(b) *The function $\phi^p_{\mathrm{S-NR}}$ is not differentiable. However, $\phi^p_{\mathrm{S-NR}}$ is continuously differentiable on the set $\Omega := \{(a,b)\,|\,a \neq b\}$ with*

$$\nabla\phi^p_{\mathrm{S-NR}}(a,b) = \begin{cases} p\,[a^{p-1} - (a-b)^{p-1},\ (a-b)^{p-1}]^T & \text{if } a > b, \\ p\,[(b-a)^{p-1},\ b^{p-1} - (b-a)^{p-1}]^T & \text{if } a < b. \end{cases}$$

In a more compact form,

$$\nabla\phi^p_{\mathrm{S-NR}}(a,b) = \begin{cases} p\,[\phi^{p-1}_{\mathrm{NR}}(a,b),\ (a-b)^{p-1}]^T & \text{if } a > b, \\ p\,[(b-a)^{p-1},\ \phi^{p-1}_{\mathrm{NR}}(b,a)]^T & \text{if } a < b. \end{cases}$$

(c) *The function $\phi^p_{\mathrm{S-NR}}$ is twice continuously differentiable on the set $\Omega = \{(a,b)\,|\,a \neq b\}$ with*

$$\nabla^2\phi^p_{\mathrm{S-NR}}(a,b) = \begin{cases} p(p-1)\begin{bmatrix} a^{p-2} - (a-b)^{p-2} & (a-b)^{p-2} \\ (a-b)^{p-2} & -(a-b)^{p-2} \end{bmatrix} & \text{if } a > b, \\ p(p-1)\begin{bmatrix} -(b-a)^{p-2} & (b-a)^{p-2} \\ (b-a)^{p-2} & b^{p-2} - (b-a)^{p-2} \end{bmatrix} & \text{if } a < b. \end{cases}$$

In a more compact form,

$$\nabla^2\phi^p_{\mathrm{S-NR}}(a,b) = \begin{cases} p(p-1)\begin{bmatrix} \phi^{p-2}_{\mathrm{NR}}(a,b) & (a-b)^{p-2} \\ (a-b)^{p-2} & -(a-b)^{p-2} \end{bmatrix} & \text{if } a > b, \\ p(p-1)\begin{bmatrix} -(b-a)^{p-2} & (b-a)^{p-2} \\ (b-a)^{p-2} & \phi^{p-2}_{\mathrm{NR}}(b,a) \end{bmatrix} & \text{if } a < b. \end{cases}$$

Proposition 3.3. ([1, Proposition 2.3]) *Let $\phi^p_{\text{S-NR}}$ be defined as in (7) with $p > 1$ being a positive odd integer. Then, the following variants of $\phi^p_{\text{S-NR}}$ are also NCP-functions.*

$$\begin{aligned}
\widetilde{\phi}_1(a,b) &= \phi^p_{\text{S-NR}}(a,b) + \alpha(a)_+(b)_+, \ \alpha > 0. \\
\widetilde{\phi}_2(a,b) &= \phi^p_{\text{S-NR}}(a,b) + \alpha\left((a)_+(b)_+\right)^2, \ \alpha > 0. \\
\widetilde{\phi}_3(a,b) &= [\phi^p_{\text{S-NR}}(a,b)]^2 + \alpha\left((ab)_+\right)^4, \ \alpha > 0. \\
\widetilde{\phi}_4(a,b) &= [\phi^p_{\text{S-NR}}(a,b)]^2 + \alpha\left((ab)_+\right)^2, \ \alpha > 0.
\end{aligned}$$

Proposition 3.4. ([13, Proposition 4.4]) *Let $\phi^p_{\text{S-NR}}$ be defined as in (7) with $p > 1$ being a positive odd integer. Then, the following hold.*

(a) $\phi^p_{\text{S-NR}}(a,b) > 0 \iff a > 0, \ b > 0.$

(b) $\phi^p_{\text{S-NR}}$ *is positive homogeneous of degree p.*

(c) $\phi^p_{\text{S-NR}}$ *is not Lipschitz continnuous.*

(d) $\phi^p_{\text{S-NR}}$ *is not α-Hölder continuous for any $\alpha \in (0,1]$.*

(e) $\nabla_a\phi^p_{\text{S-NR}}(a,b)\cdot\nabla_b\phi^p_{\text{S-NR}}(a,b) > 0$ *on* $\{(a,b)\,|\,a > b > 0\}\bigcup\{(a,b)\,|\,b > a > 0\}$.

(f) $\nabla_a\phi^p_{\text{S-NR}}(a,b)\cdot\nabla_b\phi^p_{\text{S-NR}}(a,b) = 0$ *provided that* $\phi^p_{\text{S-NR}}(a,b) = 0$ *and* $(a,b) \neq (0,0)$.

Lemma 3.1. ([13, Lemma 4.1]) *Let $\phi^p_{\text{S-NR}}$ be defined as in (7) with $p > 1$ being a positive odd integer. Then, $\phi^p_{\text{S-NR}}$ is strictly continuous (locally Lipschitz continuous).*

Proposition 3.5. ([13, Proposition 4.5]) *Let $\phi^p_{\text{S-NR}}$ be defined as in (7) with $p > 1$ being a positive odd integer. Then, the generalized gradient of $\phi^p_{\text{S-NR}}$ is given by*

$$\partial\phi^p_{\text{S-NR}}(a,b) = \begin{cases} p\,[a^{p-1} - (a-b)^{p-1}, \ (a-b)^{p-1}]^T & \text{if } \ a > b, \\ \{p\,[\alpha a^{p-1}, (1-\alpha)a^{p-1}]^T \,|\, \alpha \in [0,1]\} & \text{if } \ a = b, \\ p\,[(b-a)^{p-1}, \ b^{p-1} - (b-a)^{p-1}]^T & \text{if } \ a < b. \end{cases}$$

Lemma 3.2. ([13, Lemma 4.2]) *Let $\phi^p_{\text{S-NR}}$ be defined as in (7) with $p > 1$ being a positive odd integer. Then, $\phi^p_{\text{S-NR}}$ is a directional differentiable function.*

Proposition 3.6. ([13, Proposition 4.6]) *Let $\phi^p_{\text{S-NR}}$ be defined as in (7) with $p > 1$ being a positive odd integer. Then, $\phi^p_{\text{S-NR}}$ is a semismooth function. Moreover, $\phi^p_{\text{S-NR}}$ is strongly semismooth.*

4 Properties of the function $\psi^p_{\text{S-NR}}$

In this section, we present all the properties of $\psi^p_{\text{S-NR}}$ defined as in (8).

Proposition 4.1. ([1, Proposition 3.1]) *Let $\psi^p_{\text{S-NR}}$ be defined as in (8) with $p > 1$ being a positive odd integer. Then, $\psi^p_{\text{S-NR}}$ is an NCP-function and is positive on the set*

$$\Omega' = \{(a,b)\,|\,ab \neq 0\} \cup \{(a,b)\,|\,a < b = 0\} \cup \{(a,b)\,|\,0 = a > b\}.$$

Proposition 4.2. ([1, Proposition 3.2]) *Let $\psi^p_{\text{S-NR}}$ be defined as in (8) with $p > 1$ being a positive odd integer. Then, the following hold.*

(a) *An alternative expression of $\phi^p_{\text{S-NR}}$ is*

$$\psi^p_{\text{S-NR}}(a,b) = \begin{cases} \phi^p_{\text{NR}}(a,b)b^p & \text{if} \quad a > b, \\ a^p b^p = a^{2p} & \text{if} \quad a = b, \\ \phi^p_{\text{NR}}(b,a)a^p & \text{if} \quad a < b. \end{cases}$$

(b) *The function $\psi^p_{\text{S-NR}}$ is continuously differentiable with*

$$\nabla\psi^p_{\text{S-NR}}(a,b) = \begin{cases} p\,[\,a^{p-1}b^p - (a-b)^{p-1}b^p,\ a^p b^{p-1} - (a-b)^p b^{p-1} + (a-b)^{p-1}b^p\,]^T & \text{if} \quad a > b, \\ p\,[\,a^{p-1}b^p, a^p b^{p-1}\,]^T = pa^{2p-1}[1\ ,1\,]^T & \text{if} \quad a = b, \\ p\,[\,a^{p-1}b^p - (b-a)^p a^{p-1} + (b-a)^{p-1}a^p,\ a^p b^{p-1} - (b-a)^{p-1}a^p\,]^T & \text{if} \quad a < b. \end{cases}$$

In a more compact form,

$$\nabla\psi^p_{\text{S-NR}}(a,b) = \begin{cases} p\,[\,\phi^{p-1}_{\text{NR}}(a,b)b^p,\ \phi^p_{\text{NR}}(a,b)b^{p-1} + (a-b)^{p-1}b^p\,]^T & \text{if} \quad a > b, \\ p\,[\,a^{2p-1}, a^{2p-1}\,]^T & \text{if} \quad a = b, \\ p\,[\,\phi^p_{\text{NR}}(b,a)a^{p-1} + (b-a)^{p-1}a^p,\ \phi^{p-1}_{\text{NR}}(b,a)a^p\,]^T & \text{if} \quad a < b. \end{cases}$$

(c) *The function $\psi^p_{\text{S-NR}}$ is twice continuously differentiable with*

$$\nabla^2\psi^p_{\text{S-NR}}(a,b) = \begin{cases} p\begin{bmatrix} (p-1)[a^{p-2}-(a-b)^{p-2}]b^p & \begin{array}{c}(p-1)(a-b)^{p-2}b^p \\ +p[a^{p-1}-(a-b)^{p-1}]b^{p-1}\end{array} \\ \begin{array}{c}(p-1)(a-b)^{p-2}b^p \\ +p[a^{p-1}-(a-b)^{p-1}]b^{p-1}\end{array} & \begin{array}{c}(p-1)[a^p-(a-b)^p]b^{p-2} \\ +2p(a-b)^{p-1}b^{p-1} \\ -(p-1)(a-b)^{p-2}b^p\end{array} \end{bmatrix} & \text{if } a>b, \\ p\begin{bmatrix} (p-1)a^{p-2}b^p & pa^{p-1}b^{p-1} \\ pa^{p-1}b^{p-1} & (p-1)a^pb^{p-2} \end{bmatrix} & \text{if } a=b, \\ p\begin{bmatrix} \begin{array}{c}(p-1)[b^p-(b-a)^p]a^{p-2} \\ +2p(b-a)^{p-1}a^{p-1} \\ -(p-1)(b-a)^{p-2}a^p\end{array} & \begin{array}{c}(p-1)(b-a)^{p-2}a^p \\ +p[b^{p-1}-(b-a)^{p-1}]a^{p-1}\end{array} \\ \begin{array}{c}(p-1)(b-a)^{p-2}a^p \\ +p[b^{p-1}-(b-a)^{p-1}]a^{p-1}\end{array} & (p-1)[b^{p-2}-(b-a)^{p-2}]a^p \end{bmatrix} & \text{if } a<b. \end{cases}$$

Proposition 4.3. [1, Proposition 3.3] *Let $\psi^p_{\text{S-NR}}$ be defined as in (8) with $p>1$ being a positive odd integer. Then the following variants of $\psi^p_{\text{S-NR}}$ are also NCP-functions.*

$$\begin{aligned} \widetilde{\psi}_1(a,b) &= \psi^p_{\text{S-NR}}(a,b) + \alpha(a)_+(b)_+, \ \alpha>0. \\ \widetilde{\psi}_2(a,b) &= \psi^p_{\text{S-NR}}(a,b) + \alpha\left((a)_+(b)_+\right)^2, \ \alpha>0. \\ \widetilde{\psi}_3(a,b) &= \psi^p_{\text{S-NR}}(a,b) + \alpha\left((ab)_+\right)^4, \ \alpha>0. \\ \widetilde{\psi}_4(a,b) &= \psi^p_{\text{S-NR}}(a,b) + \alpha\left((ab)_+\right)^2, \ \alpha>0. \end{aligned}$$

Proposition 4.4. ([13, Proposition 5.4]) *Let $\psi^p_{\text{S-NR}}$ be defined as in (8) with $p>1$ being a positive odd integer. Then, the following hold.*

(a) $\psi^p_{\text{S-NR}}(a,b)\geq 0$ *for all* $(a,b)\in\mathbb{R}^2$.

(b) $\psi^p_{\text{S-NR}}$ *is positive homogeneous of degree* $2p$.

(c) $\psi^p_{\text{S-NR}}$ *is not Lipschitz continuous.*

(d) $\psi^p_{\text{S-NR}}$ *is not α-Hölder continuous for any $\alpha\in(0,1]$.*

(e) $\nabla_a\psi^p_{\text{S-NR}}(a,b)\cdot\nabla_b\psi^p_{\text{S-NR}}(a,b)>0$ *on the first quadrant* $\mathbb{R}^2_{++}$.

(f) $\psi^p_{\text{S-NR}}(a,b)=0 \iff \nabla\psi^p_{\text{S-NR}}(a,b)=0$. *In particular, we have* $\nabla_a\psi^p_{\text{S-NR}}(a,b)\cdot\nabla_b\psi^p_{\text{S-NR}}(a,b)=0$ *provided that* $\psi^p_{\text{S-NR}}(a,b)=0$.

5 Properties of the function $\phi^p_{\text{D-FB}}$

In this section, we present all the properties of $\phi^p_{\text{D-FB}}$ defined as in (9).

Proposition 5.1. ([16, Proposition 3.1]) *Let $\phi^p_{\text{D-FB}}$ be defined as in (9) where $p > 1$ is a positive odd integer. Then, we have*

(a) *$\phi^p_{\text{D-FB}}$ is a NCP-function;*

(b) *$\phi^p_{\text{D-FB}}$ is positive homogeneous of degree p.*

Proposition 5.2. ([16, Proposition 3.2]) *Let $\phi^p_{\text{D-FB}}$ be defined as in (9). Then, the following hold.*

(a) *For $p > 1$, $\phi^p_{\text{D-FB}}$ is continuously differentiable with*

$$\nabla \phi^p_{\text{D-FB}}(a,b) = p \begin{bmatrix} a(\sqrt{a^2+b^2})^{p-2} - (a+b)^{p-1} \\ b(\sqrt{a^2+b^2})^{p-2} - (a+b)^{p-1} \end{bmatrix}.$$

(b) *For $p > 3$, $\phi^p_{\text{D-FB}}$ is twice continuously differentiable with*

$$\nabla^2 \phi^p_{\text{D-FB}}(a,b) = \begin{bmatrix} \frac{\partial^2 \phi^p_{\text{D-FB}}}{\partial a^2} & \frac{\partial^2 \phi^p_{\text{D-FB}}}{\partial a \partial b} \\ \frac{\partial^2 \phi^p_{\text{D-FB}}}{\partial b \partial a} & \frac{\partial^2 \phi^p_{\text{D-FB}}}{\partial b^2} \end{bmatrix},$$

where

$$\begin{aligned}
\frac{\partial^2 \phi^p_{\text{D-FB}}}{\partial a^2} &= p\left\{[(p-1)a^2+b^2](\sqrt{a^2+b^2})^{p-4} - (p-1)(a+b)^{p-2}\right\}, \\
\frac{\partial^2 \phi^p_{\text{D-FB}}}{\partial a \partial b} &= p[(p-2)ab(\sqrt{a^2+b^2})^{p-4} - (p-1)(a+b)^{p-2}] = \frac{\partial^2 \phi^p_{\text{D-FB}}}{\partial b \partial a}, \\
\frac{\partial^2 \phi^p_{\text{D-FB}}}{\partial b^2} &= p\left\{[a^2+(p-1)b^2](\sqrt{a^2+b^2})^{p-4} - (p-1)(a+b)^{p-2}\right\}.
\end{aligned}$$

Proposition 5.3. ([16, Proposition 3.3]) *Let $\phi^p_{\text{D-FB}}$ be defined as in (9). Then, the following variants of $\phi^p_{\text{D-FB}}$ are also NCP-functions.*

$$\begin{aligned}
\varphi_1(a,b) &= \phi^p_{\text{D-FB}}(a,b) - \alpha(a)_+(b)_+, \ \alpha > 0. \\
\varphi_2(a,b) &= \phi^p_{\text{D-FB}}(a,b) - \alpha\left((a)_+(b)_+\right)^2, \ \alpha > 0. \\
\varphi_3(a,b) &= [\phi^p_{\text{D-FB}}(a,b)]^2 + \alpha\left((ab)_+\right)^4, \ \alpha > 0. \\
\varphi_4(a,b) &= [\phi^p_{\text{D-FB}}(a,b)]^2 + \alpha\left((ab)_+\right)^2, \ \alpha > 0.
\end{aligned}$$

Proposition 5.4. ([13, Proposition 6.4]) *Let $\phi^p_{\text{D-FB}}$ be defined as in (9) with $p > 1$ being a positive odd integer. Then, the following hold.*

(a) $\phi^p_{\text{D-FB}}(a, b) < 0 \iff a > 0,\ b > 0.$

(b) $\phi^p_{\text{D-FB}}$ *is not Lipschitz continuous.*

(c) $\phi^p_{\text{D-FB}}$ *is not α-Hölder continuous for any $\alpha \in (0, 1]$.*

(d) $\nabla_a \phi^p_{\text{D-FB}}(a, b) \cdot \nabla_b \phi^p_{\text{D-FB}}(a, b) > 0$ *on the first quadrant* $\mathbb{R}^2_{++}$.

(e) $\nabla_a \phi^p_{\text{D-FB}}(a, b) \cdot \nabla_b \phi^p_{\text{D-FB}}(a, b) = 0$ *provided that* $\phi^p_{\text{D-FB}}(a, b) = 0$.

6 Final Remarks

In summary, we have obtained 4 new discrete-type of generalizations so far. Besides the continuous generalization ϕ^p_{FB}, we list all of them as below.

$$
\begin{array}{rcl}
\phi^p_{\text{FB}} & : & \text{strong semismooth with symmetric surface} \\
\phi^p_{\text{D-FB}} & : & \text{twice differentiable with symmetric surface} \\
\phi^p_{\text{NR}} & : & \text{twice differentiable} \\
\phi^p_{\text{S-NR}} & : & \text{strong semismooth with symmetric surface} \\
\psi^p_{\text{S-NR}} & : & \text{twice differentiable with symmetric surface}
\end{array}
$$

In addition, we show a diagram which describes the relation between smooth functions and nonsmooth functions. This will help clarify the aforementioned functions.

$$
\begin{array}{ccccccccccc}
C^2 \Rightarrow SC^1 & \Rightarrow & LC^1 & \Rightarrow & C^1 & \Rightarrow & \text{semismooth} & \Rightarrow & \text{locally Lipschitz} \\
 & & \Downarrow & & & & \Uparrow & & \\
 & & \multicolumn{5}{c}{\text{strongly semismooth}} & &
\end{array}
$$

We also list some future research directions as below:

1. What are the extra benefits for NCP-functions with symmetric surfaces?
2. Doing numerical comparisons among ϕ^p_{FB}, ϕ^p_{NR}, $\phi^p_{\text{S-NR}}$, $\psi^p_{\text{S-NR}}$, and $\phi^p_{\text{D-FB}}$ involved in various algorithms.
3. The Newton method may not be applicable even though we have the differentiability for some new NCP-functions because the Jacobian at a degenerate solution is singular (Kanzow and Kleinmichel, Optimization Methods and Software, 1995).
4. Extending these functions as complementarity functions associated with second-order cone and symmetric cone.

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