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Iterative Methods for Split Common Fixed Point Problems in Banach Spaces and Applications

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Abstract. In this article, motivated by split feasibility problems and split common null point problems in Hilbert spaces, we first introduce the concept of nonlinear operators in Banach spaces which covers strict pseud-contractions and generalized hybrid mappings in Hilbert spaces, and the metric projections and the metric resolvents in Banach spaces. Then we consider split common fixed point problems with the operators in Banach spaces. Using hybrid methods, Mann's type iterations and Halpern's type iterations, we prove weak and strong convergence theorems for finding solutions of split common fixed point problems in Banach spaces. Furthermore, using these results, we get well-known and new results which are connected with split feasibility problems and split common null point problems in Banach spaces.

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1 Introduction

Let H_1 and H_2 be two real Hilbert spaces. Let D and Q be nonempty, closed and convex subsets of H_1 and H_2, respectively. Let $A : H_1 \to H_2$ be a bounded linear operator. Then the *split feasibility problem* [7] is to find $z \in H_1$ such that $z \in D \cap A^{-1}Q$. Byrne, Censor, Gibali and Reich [6] also considered the following problem: Given set-valued mappings $A : H_1 \to 2^{H_1}$, and $B : H_2 \to 2^{H_2}$, respectively, and a bounded linear operator $T : H_1 \to H_2$, the *split common null point problem* is to find a point $z \in H_1$ such that

$$z \in A^{-1}0 \cap B^{-1}0,$$

where $A^{-1}0$ and $B^{-1}0$ are null point sets of A and B, respectively. Defining $U = A^*(I - P_Q)A$ in the split feasibility problem, we have that $U : H_1 \to H_1$ is an inverse strongly monotone operator [1], where A^* is the adjoint operator of A and P_Q is the metric projection of H_2 onto

Q. Furthermore, if $D \cap A^{-1}Q$ is nonempty, then $z \in D \cap A^{-1}Q$ is equivalent to

$$z = P_D(I - \lambda A^*(I - P_Q)A)z,$$

where $\lambda > 0$ and P_D is the metric projection of H_1 onto D. By using such results regarding nonlinear operators and fixed points, many authors have studied split feasibility problems and split common null point problems in Hilbert spaces, for instance, [1, 6, 8, 28].

In this article, motivated by split feasibility problems and split common null point problems in Hilbert spaces, we first introduce the concept of nonlinear operators in Banach spaces which covers strict pseud-contractions and generalized hybrid mappings in Hilbert spaces, and the metric projections and the metric resolvents in Banach spaces. Then we consider split common fixed point problems with the operators in Banach spaces. Using hybrid methods, Mann's type iterations and Halpern's type iterations, we prove weak and strong convergence theorems for finding solutions of split common fixed point problems in Banach spaces. Furthermore, using these results, we get well-known and new results which are connected with split feasibility problems and split common null point problems in Banach spaces.

2 Preliminaries

Let E be a real Banach space with norm $\|\cdot\|$ and let E^* be the dual space of E. We denote the value of $y^* \in E^*$ at $x \in E$ by $\langle x, y^* \rangle$. When $\{x_n\}$ is a sequence in E, we denote the strong convergence of $\{x_n\}$ to $x \in E$ by $x_n \to x$ and the weak convergence by $x_n \rightharpoonup x$. The modulus δ of convexity of E is defined by

$$\delta(\epsilon) = \inf \left\{ 1 - \frac{\|x+y\|}{2} : \|x\| \leq 1, \|y\| \leq 1, \|x-y\| \geq \epsilon \right\}$$

for every ϵ with $0 \leq \epsilon \leq 2$. A Banach space E is said to be uniformly convex if $\delta(\epsilon) > 0$ for every $\epsilon > 0$. A uniformly convex Banach space is strictly convex and reflexive. We also know that a uniformly convex Banach space has the Kadec-Klee property, that is, $x_n \rightharpoonup u$ and $\|x_n\| \to \|u\|$ imply $x_n \to u$.

The duality mapping J from E into 2^{E^*} is defined by

$$Jx = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\}$$

for every $x \in E$. Let $U = \{x \in E : \|x\| = 1\}$. The norm of E is said to be Gâteaux differentiable if for each $x, y \in U$, the limit

$$\lim_{t \to 0} \frac{\|x+ty\| - \|x\|}{t}$$

exists. In the case, E is called smooth. We know that E is smooth if and only if J is a single-valued mapping of E into E^*. We also know that E is reflexive if and only if J is surjective, and E is strictly convex if and only if J is one-to-one. Therefore, if E is a smooth, strictly convex and reflexive Banach space, then J is a single-valued bijection and in this case, the inverse mapping J^{-1} coincides with the duality mapping J_* on E^*. For more details, see [18] and [19]. We know the following result:

Lemma 2.1 ([18]). *Let E be a smooth Banach space and let J be the duality mapping on E. Then, $\langle x-y, Jx-Jy\rangle \geq 0$ for all $x, y \in E$. Furthermore, if E is strictly convex and $\langle x-y, Jx-Jy\rangle = 0$, then $x=y$.*

Let C be a nonempty, closed and convex subset of a strictly convex and reflexive Banach space E. Then we know that for any $x \in E$, there exists a unique element $z \in C$ such that $\|x-z\| \leq \|x-y\|$ for all $y \in C$. Putting $z = P_C x$, we call such a mapping P_C the metric projection of E onto C.

Lemma 2.2 ([18]). *Let E be a smooth, strictly convex and reflexive Banach space. Let C be a nonempty, closed and convex subset of E and let $x_1 \in E$ and $z \in C$. Then, the following conditions are equivalent:*

(1) $z = P_C x_1$;
(2) $\langle z-y, J(x_1-z)\rangle \geq 0, \quad \forall y \in C$.

Let E be a Banach space and let A be a mapping of E into 2^{E^*}. The effective domain of A is denoted by $\mathrm{dom}(A)$, that is, $\mathrm{dom}(A) = \{x \in E : Ax \neq \emptyset\}$. A multi-valued mapping A on E is said to be monotone if $\langle x-y, u^*-v^*\rangle \geq 0$ for all $x, y \in \mathrm{dom}(A)$, $u^* \in Ax$, and $v^* \in Ay$. A monotone operator A on E is said to be maximal if its graph is not properly contained in the graph of any other monotone operator on E. The following theorem is due to Browder [4]; see also [19, Theorem 3.5.4].

Theorem 2.3 ([4]). *Let E be a uniformly convex and smooth Banach space and let J be the duality mapping of E into E^*. Let A be a monotone operator of E into 2^{E^*}. Then A is maximal if and only if for any $r>0$, $R(J+rA) = E^*$, where $R(J+rA)$ is the range of $J+rA$.*

Let E be a uniformly convex Banach space with a Gâteaux differentiable norm and let A be a maximal monotone operator of E into 2^{E^*}. For all $x \in E$ and $r > 0$, we consider the following equation $0 \in J(x_r - x) + rAx_r$. This equation has a unique solution x_r. We define J_r by $x_r = J_r x$. Such $J_r, r>0$ are called the metric resolvents of A. The set of null points of A is defined by $A^{-1}0 = \{z \in E : 0 \in Az\}$. We know that $A^{-1}0$ is closed and convex; see [19].

For a sequence $\{C_n\}$ of nonempty, closed and convex subsets of a Banach space E, define $\text{s-Li}_n C_n$ and $\text{w-Ls}_n C_n$ as follows: $x \in \text{s-Li}_n C_n$ if and only if there exists $\{x_n\} \subset E$ such that $\{x_n\}$ converges strongly to x and $x_n \in C_n$ for all $n \in \mathbb{N}$. Similarly, $y \in \text{w-Ls}_n C_n$ if and only if there exist a subsequence $\{C_{n_i}\}$ of $\{C_n\}$ and a sequence $\{y_i\} \subset E$ such that $\{y_i\}$ converges weakly to y and $y_i \in C_{n_i}$ for all $i \in \mathbb{N}$. If C_0 satisfies

$$C_0 = \underset{n}{\text{s-Li}}\, C_n = \underset{n}{\text{w-Ls}}\, C_n,$$

it is said that $\{C_n\}$ converges to C_0 in the sense of Mosco [14] and we write $C_0 = \text{M-lim}_{n\to\infty} C_n$. It is easy to show that if $\{C_n\}$ is nonincreasing with respect to inclusion, then $\{C_n\}$ converges to $\bigcap_{n=1}^{\infty} C_n$ in the sense of Mosco. For more details, see [14]. The following lemma was proved by Tsukada [30].

Lemma 2.4 ([30]). *Let E be a uniformly convex Banach space. Let $\{C_n\}$ be a sequence of nonempty, closed and convex subsets of E. If $C_0 = M\text{-lim}_{n\to\infty} C_n$ exists and nonempty, then for each $x \in E$, $\{P_{C_n}x\}$ converges strongly to $P_{C_0}x$, where P_{C_n} and P_{C_0} are the mertic projections of E onto C_n and C_0, respectively.*

3 Iterative Results by Hybrid Methods

Let E be a smooth, strictly convex and reflexive Banach space and let η be a real number with $\eta \in (-\infty, 1)$. Then a mapping $U : E \to E$ with $F(U) \neq \emptyset$ is called η-demimetric [22] if, for any $x \in E$ and $q \in F(U)$,

$$\langle x - q, J(x - Ux)\rangle \geq \frac{1-\eta}{2}\|x - Ux\|^2,$$

where $F(U)$ is the set of fixed points of U.

Examples. We know examples of η-demimetric mappings from [22].

(1) Let H be a Hilbert space and let k be a real number with $0 \leq k < 1$. Let U be a strict pseud-contraction [5] of H into itself such that $F(U) \neq \emptyset$. Then U is k-demimetric.

(2) Let H be a Hilbert space and let C be a nonempty subset of H. A mapping $U : C \to H$ is called generalized hybrid [10] if there exist $\alpha, \beta \in \mathbb{R}$ such that

$$\alpha\|Ux - Uy\|^2 + (1-\alpha)\|x - Uy\|^2 \leq \beta\|Ux - y\|^2 + (1-\beta)\|x - y\|^2, \quad \forall x, y \in H.$$

Such a mapping U is called (α, β)-generalized hybrid. If U is generalized hybrid and $F(U) \neq \emptyset$, then U is 0-demimetric.

(3) Let E be a strictly convex, reflexive and smooth Banach space and let C be a nonempty, closed and convex subset of E. Let P_C be the metric projection of E onto C. Then P_C is (-1)-demimetric.

(4) Let E be a uniformly convex and smooth Banach space and let B be a maximal monotone operator with $B^{-1}0 \neq \emptyset$. Let $\lambda > 0$. Then the metric resolvent J_λ is (-1)-demimetric.

Furthermore, we know an important result for demimetric mappings in a smooth, strictly convex and reflexive Banach space.

Lemma 3.1 ([22]). *Let E be a smooth, strictly convex and reflexive Banach space and let η be a real number with $\eta \in (-\infty, 1)$. Let U be an η-demimetric mapping of E into itself. Then $F(U)$ is closed and convex.*

Using the hybrid methods in mathematical programming, we prove two strong convergence theorems for finding a solution of the split common fixed point problem in Banach spaces. Let E be a Banach space and let D be a nonempty, closed and convex subset of E. A mapping $U : D \to E$ is called demiclosed if for a sequence $\{x_n\}$ in D such that $x_n \rightharpoonup p$ and $x_n - Ux_n \to 0$, $p = Up$ holds. The following theorems are proved by Takahashi [23].

Theorem 3.2 ([23]). *Let E and F be uniformly convex and smooth Banach spaces and let J_E and J_F be the duality mappings on E and F, respectively. Let τ and η be real numbers with $\tau, \eta \in (-\infty, 1)$. Let $T : E \to E$ be a τ-demimetric and demiclosed mapping and let $U : F \to F$ be an η-demimetric and demiclosed mapping with $F(U) \neq \emptyset$. Let $A : E \to F$ be a bounded linear operator such that $A \neq 0$ and let A^* be the adjoint operator of A. Suppose* *that $F(T) \cap A^{-1}F(U) \neq \emptyset$. Let $x_1 \in E$ and let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} z_n = x_n - rJ_E^{-1}A^*J_F(Ax_n - UAx_n), \\ y_n = Tz_n, \\ C_n = \{z \in E : \langle z_n - z, J_E(x_n - z_n)\rangle \geq 0\}, \\ D_n = \{z \in E : 2\langle z_n - z, J_E(z_n - y_n)\rangle \geq (1-\tau)\|z_n - y_n\|^2\}, \\ Q_n = \{z \in E : \langle x_n - z, J_E(x_1 - x_n)\rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap D_n \cap Q_n} x_1, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $0 < 2r\|A\|^2 \leq (1-\eta)$. Then $\{x_n\}$ converges strongly to a point $z_1 \in F(T) \cap A^{-1}F(U)$, where $z_1 = P_{F(T)\cap A^{-1}F(U)}x_1$.

Theorem 3.3 ([23]). *Let E and F be uniformly convex and smooth Banach spaces and let J_E and J_F be the duality mappings on E and F, respectively. Let τ and η be real numbers with $\tau, \eta \in (-\infty, 1)$. Let $T : E \to E$ be a τ-demimetric and demiclosed mapping and let $U : F \to F$ be an η-demimetric and demiclosed mapping with $F(U) \neq \emptyset$. Let $A : E \to F$ be a bounded linear operator such that $A \neq 0$ and let A^* be the adjoint operator of A. Suppose that $F(T) \cap A^{-1}F(U) \neq \emptyset$. For $x_1 \in E$ and $C_1 = E$, let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} z_n = x_n - rJ_E^{-1}A^*J_F(Ax_n - UAx_n), \\ y_n = Tz_n, \\ C_{n+1} = \{z \in C_n : \langle z_n - z, J_E(x_n - z_n)\rangle \geq 0 \\ \qquad\qquad\qquad \textit{and} \quad 2\langle z_n - z, J_E(z_n - y_n)\rangle \geq (1-\tau)\|z_n - y_n\|^2\}, \\ x_{n+1} = P_{C_{n+1}} x_1, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $0 < 2r\|A\|^2 \leq (1-\eta)$. Then $\{x_n\}$ converges strongly to a point $z_1 \in F(T) \cap A^{-1}F(U)$, where $z_1 = P_{F(T)\cap A^{-1}F(U)}x_1$.

Using Theorems 3.2 and 3.3, we get strong convergence theorems which are connected with the split common fixed point problems in Banach spaces. We know the following result obtained by Marino and Xu [13]; see also [27].

Lemma 3.4 ([13]). *Let H be a Hilbert space, let C be a nonempty, closed and convex subset of H and k be a real number with $0 \leq k < 1$. Let $U : C \to H$ be a k-strict pseudo-contraction. If $x_n \rightharpoonup z$ and $x_n - Ux_n \to 0$, then $z \in F(U)$.*

Theorem 3.5. *Let H_1 and H_2 be Hilbert spaces. Let k be a real number with $k \in [0, 1)$. Let $T : H_1 \to H_1$ be a nonexpansive mapping and let $U : H_2 \to H_2$ be a k-strict pseud-contraction with $F(U) \neq \emptyset$. Let $A : H_1 \to H_2$ be a bounded linear operator such that $A \neq 0$ and let A^* be the adjoint operator of A. Suppose that $F(T) \cap A^{-1}F(U) \neq \emptyset$. Let $x_1 \in H_1$ and let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} z_n = x_n - rA^*(Ax_n - UAx_n), \\ y_n = Tz_n, \\ C_n = \{z \in H_1 : \langle z_n - z, x_n - z_n\rangle \geq 0\}, \\ D_n = \{z \in H_1 : 2\langle z_n - z, z_n - y_n\rangle \geq \|z_n - y_n\|^2\}, \\ Q_n = \{z \in H_1 : \langle x_n - z, x_1 - x_n\rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap D_n \cap Q_n} x_1, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $0 < 2r\|A\|^2 \leq (1-k)$. *Then* $\{x_n\}$ *converges strongly to a point* $z_1 \in F(T) \cap A^{-1}F(U)$, *where* $z_1 = P_{F(T)\cap A^{-1}F(U)}x_1$.

Theorem 3.6. *Let* E *and* F *be uniformly convex and smooth Banach spaces and let* J_E *and* J_F *be the duality mappings on* E *and* F*, respectively. Let* C *and* D *be nonempty, closed and convex subsets of* E *and* F*, respectively. Let* P_C *and* P_D *be the metric projections of* E *onto* C *and* F *onto* D*, respectively. Let* $A : E \to F$ *be a bounded linear operator such that* $A \neq 0$ *and let* A^* *be the adjoint operator of* A*. Suppose that* $C \cap A^{-1}D \neq \emptyset$*. For* $x_1 \in E$ *and* $C_1 = E$*, let* $\{x_n\}$ *be a sequence generated by*

$$\begin{cases} z_n = x_n - rJ_E^{-1}A^*J_F(Ax_n - P_DAx_n), \\ y_n = P_Cz_n, \\ C_{n+1} = \{z \in C_n : \langle z_n - z, J_E(x_n - z_n)\rangle \geq 0 \\ \qquad\qquad\qquad \textit{and} \quad \langle z_n - z, J_E(z_n - y_n)\rangle \geq \|z_n - y_n\|^2\}, \\ x_{n+1} = P_{C_{n+1}}x_1, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $0 < r\|A\|^2 \leq 1$*. Then* $\{x_n\}$ *converges strongly to a point* $z_1 \in C \cap A^{-1}D$*, where* $z_1 = P_{C\cap A^{-1}D}x_1$.

Theorem 3.7. *Let* E *and* F *be uniformly convex and smooth Banach spaces and let* J_E *and* J_F *be the duality mappings on* E *and* F*, respectively. Let* G *and* B *be maximal monotone operators of* E *into* E^* *and* F *into* F^**, respectively. Let* J_λ *and* Q_μ *be the metric resolvents of* G *for* $\lambda > 0$ *and* B *for* $\mu > 0$*, respectively. Let* $A : E \to F$ *be a bounded linear operator such that* $A \neq 0$ *and let* A^* *be the adjoint operator of* A*. Suppose that* $G^{-1}0 \cap A^{-1}(B^{-1}0) \neq \emptyset$*. For* $x_1 \in E$ *and* $C_1 = E$*, let* $\{x_n\}$ *be a sequence generated by*

$$\begin{cases} z_n = x_n - rJ_E^{-1}A^*J_F(Ax_n - Q_\mu Ax_n), \\ y_n = J_\lambda z_n, \\ C_{n+1} = \{z \in C_n : \langle z_n - z, J_E(x_n - z_n)\rangle \geq 0 \\ \qquad\qquad\qquad \textit{and} \quad \langle z_n - z, J_E(z_n - y_n)\rangle \geq \|z_n - y_n\|^2\}, \\ x_{n+1} = P_{C_{n+1}}x_1, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $0 < r\|A\|^2 \leq 1$ *and* $\lambda, \mu > 0$*. Then the sequence* $\{x_n\}$ *converges strongly to a point* $z_1 \in G^{-1}0 \cap A^{-1}(B^{-1}0)$*, where* $z_1 = P_{G^{-1}0\cap A^{-1}(B^{-1}0)}x_1$.

4 Iterative Results by Mann and Halpern Iterations

In this section, we first prove a weak convergence theorem [24] of Mann's type iteration for the split common fixed point problem in Banach spaces.

Theorem 4.1 ([24]). *Let* H *be a Hilbert space and let* F *be a smooth, strictly convex and smooth Banach space. Let* J_F *be the duality mapping on* F *and let* η *be a real number with* $\eta \in (-\infty, 1)$*. Let* $T : H \to H$ *be a nonexpansive mapping and let* $U : F \to F$ *be an* η*-demimetric and demiclosed mapping with* $F(U) \neq \emptyset$*. Let* $A : H \to F$ *be a bounded linear operator such that* $A \neq 0$ *and let* A^* *be the adjoint operator of* A*. Suppose* $F(T) \cap A^{-1}F(U) \neq \emptyset$*. For any* $x_1 = x \in H$*, define*

$$x_{n+1} = \beta_n x_n + (1-\beta_n)T\big(I - rA^*J_F(A - UA)\big)x_n, \quad \forall n \in \mathbb{N},$$

where $\{\beta_n\} \subset [0,1]$ *and* $r \in (0,\infty)$ *satisfy the following:*

$$0 < a \leq \beta_n \leq b < 1 \quad \text{and} \quad 0 < r\|AA^*\| < (1-\eta)$$

for some $a, b \in \mathbb{R}$*. Then* $\{x_n\}$ *converges weakly to a point* $z_0 \in F(T) \cap A^{-1}F(U)$*, where* $z_0 = \lim_{n\to\infty} P_{F(T)\cap A^{-1}F(U)}x_n$*.*

Next, we prove a strong convergence theorem [24] of Halpern's type iteration for the split common fixed point problem in Banach spaces.

Theorem 4.2 ([24]). *Let* H *be a Hilbert space and let* F *be a smooth, strictly convex and smooth Banach space. Let* J_F *be the duality mapping on* F *and let* η *be a real number with* $\eta \in (-\infty, 1)$*. Let* $T : H \to H$ *be a nonexpansive mapping and let* $U : F \to F$ *be an* η*-demimetric and demiclosed mapping with* $F(U) \neq \emptyset$*. Let* $A : H \to F$ *be a bounded linear operator such that* $A \neq 0$ *and let* A^* *be the adjoint operator of* A*. Suppose* $F(T) \cap A^{-1}F(U) \neq \emptyset$*. Let* $\{u_n\}$ *be a sequence in* H *such that* $u_n \to u$*. For* $x_1 = x \in H$*, let* $\{x_n\} \subset H$ *be a sequence generated by*

$$x_{n+1} = \beta_n x_n + (1-\beta_n)\big(\alpha_n u_n + (1-\alpha_n)T(x_n - rA^*J_F(I-U)Ax_n)\big)$$

for all $n \in \mathbb{N}$*, where* $r \in (0,\infty)$*,* $\{\alpha_n\} \subset (0,1)$ *and* $\{\beta_n\} \subset (0,1)$ *satisfy*

$$0 < r\|AA^*\| < (1-\eta), \quad \lim_{n\to\infty} \alpha_n = 0, \quad \sum_{n=1}^{\infty} \alpha_n = \infty \quad \text{and} \quad 0 < a \leq \beta_n \leq b < 1$$

for some $a, b \in \mathbb{R}$*. Then* $\{x_n\}$ *converges strongly to a point* $z_0 \in F(T) \cap A^{-1}F(U)$*, where* $z_0 = P_{F(T)\cap A^{-1}F(U)}u$*.*

Using Theorems 4.1 and 4.2, we get weak and strong convergence theorems which are connected with the split common fixed point problems in Banach spaces. We also know the following result from Takahashi, Yao and Kocourek [29]; see also [10].

Lemma 4.3 ([29]). *Let* H *be a Hilbert space, let* C *be a nonempty, closed and convex subset of* H *and let* $U : C \to H$ *be generalized hybrid. If* $x_n \rightharpoonup z$ *and* $x_n - Ux_n \to 0$*, then* $z \in F(U)$*.*

Theorem 4.4. *Let* H_1 *and* H_2 *be Hilbert spaces. Let* k *be a real number with* $k \in [0,1)$*. Let* $T : H_1 \to H_1$ *be a nonexpansive mapping with* $F(T) \neq \emptyset$ *and let* $U : H_2 \to H_2$ *be a* k*-strict pseud-contraction with* $F(U) \neq \emptyset$*. Let* $A : H_1 \to H_2$ *be a bounded linear operator such that* $A \neq 0$ *and let* A^* *be the adjoint operator of* A*. Suppose* $F(T) \cap A^{-1}F(U) \neq \emptyset$*. For any* $x_1 = x \in H_1$*, define*

$$x_{n+1} = \beta_n x_n + (1-\beta_n)T\big(I - rA^*J_F(A-UA)\big)x_n, \quad \forall n \in \mathbb{N},$$

where $\{\beta_n\} \subset [0,1]$ *and* $r \in (0,\infty)$ *satisfy the following:*

$$0 < a \leq \beta_n \leq b < 1 \quad \text{and} \quad 0 < r\|AA^*\| < (1-k)$$

for some $a, b \in \mathbb{R}$*. Then* $\{x_n\}$ *converges weakly to a point* $z_0 \in F(T) \cap A^{-1}F(U)$*, where* $z_0 = \lim_{n\to\infty} P_{F(T)\cap A^{-1}F(U)}x_n$*.*

Theorem 4.5. *Let* H *be a Hilbert space and let* F *be a smooth, strictly convex and reflexive Banach space. Let* J_F *be the duality mapping on* F*. Let* C *and* D *be nonempty, closed and convex subsets of* H *and* F*, respectively. Let* P_C *and* P_D *be the metric projections of* H *onto* C *and* F *onto* D*, respectively. Let* $A : H \to F$ *be a bounded linear operator such that* $A \neq 0$

and let A^* be the adjoint operator of A. Suppose $C \cap A^{-1}D \neq \emptyset$. Let $\{u_n\}$ be a sequence in H such that $u_n \to u$. For $x_1 = x \in H$, let $\{x_n\} \subset H$ be a sequence generated by

$$x_{n+1} = \beta_n x_n + (1-\beta_n)\big(\alpha_n u_n + (1-\alpha_n) P_C(x_n - rA^* J_F(I - P_D)Ax_n)\big)$$

for all $n \in \mathbb{N}$, where $r \in (0,\infty)$, $\{\alpha_n\} \subset (0,1)$ and $\{\beta_n\} \subset (0,1)$ satisfy

$$0 < r\|AA^*\| < 2, \quad \lim_{n\to\infty} \alpha_n = 0, \quad \sum_{n=1}^{\infty} \alpha_n = \infty \quad \text{and} \quad 0 < a \leq \beta_n \leq b < 1$$

for $a, b \in \mathbb{R}$. Then $\{x_n\}$ converges strongly to a point $z_0 \in C \cap A^{-1}D$, where $z_0 = P_{C\cap A^{-1}D}u$.

Theorem 4.6. *Let H be a Hilbert space and let F be a uniformly convex and smooth Banach space. Let J_F be the duality mapping on F. Let T and B be maximal monotone operators of H into H and F into F^*, respectively. Let Q_μ be the resolvent of T for $\mu > 0$ and let J_λ be the metric resolvent of B for $\lambda > 0$, respectively. Let $A : H \to F$ be a bounded linear operator such that $A \neq 0$ and let A^* be the adjoint operator of A. Suppose $T^{-1}0 \cap A^{-1}(B^{-1}0) \neq \emptyset$. Let $\{u_n\}$ be a sequence in H such that $u_n \to u$. For $x_1 = x \in H$, let $\{x_n\} \subset H$ be a sequence generated by*

$$x_{n+1} = \beta_n x_n + (1-\beta_n)\big(\alpha_n u_n + (1-\alpha_n) Q_\mu(x_n - rA^* J_F(I - J_\lambda)Ax_n)\big)$$

for all $n \in \mathbb{N}$, where $r \in (0,\infty)$, $\{\alpha_n\} \subset (0,1)$ and $\{\beta_n\} \subset (0,1)$ satisfy

$$0 < r\|AA^*\| < 2, \quad \lim_{n\to\infty} \alpha_n = 0, \quad \sum_{n=1}^{\infty} \alpha_n = \infty \quad \text{and} \quad 0 < a \leq \beta_n \leq b < 1$$

for some $a, b \in \mathbb{R}$. Then $\{x_n\}$ converges strongly to a point $z_0 \in T^{-1}0 \cap A^{-1}(B^{-1}0)$, where $z_0 = P_{T^{-1}0\cap A^{-1}(B^{-1}0)}u$.

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