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The Split Common Fixed Point Problem and the Hybrid Method in Banach Spaces

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Abstract. In this article, we consider the split common fixed point problem in Banach spaces. Using the hybrid method in mathematical programming, we prove a strong convergence theorem for finding a solution of the split common fixed point problem in Banach spaces. Using this result, we get well-known and new results which are connected with the split feasibility problem and the split common null point problem in Banach spaces.

1 Introduction

Let H_1 and H_2 be two real Hilbert spaces. Let D and Q be nonempty, closed and convex subsets of H_1 and H_2, respectively. Let $A: H_1 \to H_2$ be a bounded linear operator. Then the *split feasibility problem* [7] is to find $z \in H_1$ such that $z \in D \cap A^{-1}Q$. Byrne, Censor, Gibali and Reich [6] also considered the following problem: Given set-valued mappings $A_i : H_1 \to 2^{H_1}$, $1 \leq i \leq m$, and $B_j : H_2 \to 2^{H_2}$, $1 \leq j \leq n$, respectively, and bounded linear operators $T_j : H_1 \to H_2$, $1 \leq j \leq n$, the *split common null point problem* [6] is to find a point $z \in H_1$ such that

$$z \in \left(\bigcap_{i=1}^{m} A_i^{-1}0\right) \cap \left(\bigcap_{j=1}^{n} T_j^{-1}(B_j^{-1}0)\right),$$

where $A_i^{-1}0$ and $B_j^{-1}0$ are null point sets of A_i and B_j, respectively. Defining $U = A^*(I-P_Q)A$ in the split feasibility problem, we have that $U : H_1 \to H_1$ is an inverse strongly monotone operator [1], where A^* is the adjoint operator of A and P_Q is the metric projection of H_2 onto Q. Furthermore, if $D \cap A^{-1}Q$ is nonempty, then $z \in D \cap A^{-1}Q$ is equivalent to

$$z = P_D(I - \lambda A^*(I - P_Q)A)z, \tag{1.1}$$

where $\lambda > 0$ and P_D is the metric projection of H_1 onto D. Using such results regarding nonlinear operators and fixed points, many authors have studied the split feasibility problem and the split common null point problem in Hilbert spaces; see, for instance, [1, 6, 8, 16, 17, 31]. However, we have not known such results outside Hilbert spaces. Recently, Takahashi [25] extended the result of (1.1) to Banach spaces. Furthermore, by using the ideas of [18, 19, 21], Takahashi [25, 26] obtained two results for the split feasibility problem and the split common null point problem in Banach spaces.

In this article, we consider the split common fixed point problem in Banach spaces. Using the hybrid method in mathematical programming, we prove a strong convergence theorem for finding a solution of the split common fixed point problem in Banach spaces. Using this result, we get well-known and new results which are connected with the split feasibility problem and the split common null point problem in Banach spaces.

2 Preliminaries

Throughout this paper, we denote by $\mathbb{N}$ the set of positive integers and by $\mathbb{R}$ the set of real numbers. Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$, respectively. For $x, y \in H$ and $\lambda \in \mathbb{R}$, we have from [24] that

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, x+y\rangle;$$
$$\|\lambda x + (1-\lambda)y\|^2 = \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x-y\|^2.$$

Furthermore we have that for $x, y, u, v \in H$,

$$2\langle x-y, u-v\rangle = \|x-v\|^2 + \|y-u\|^2 - \|x-u\|^2 - \|y-v\|^2.$$

Let C be a nonempty, closed and convex subset of a Hilbert space H. The nearest point projection of H onto C is denoted by P_C, that is, $\|x - P_C x\| \leq \|x-y\|$ for all $x \in H$ and $y \in C$. Such P_C is called the metric projection of H onto C. We know that the metric projection P_C is firmly nonexpansive, i.e.,

$$\|P_C x - P_C y\|^2 \leq \langle P_C x - P_C y, x-y\rangle$$

for all $x, y \in H$. Furthermore $\langle x - P_C x, y - P_C x\rangle \leq 0$ holds for all $x \in H$ and $y \in C$; see [22].

Let E be a real Banach space with norm $\|\cdot\|$ and let E^* be the dual space of E. We denote the value of $y^* \in E^*$ at $x \in E$ by $\langle x, y^*\rangle$. When $\{x_n\}$ is a sequence in E, we denote the strong convergence of $\{x_n\}$ to $x \in E$ by $x_n \to x$ and the weak convergence by $x_n \rightharpoonup x$. The modulus δ of convexity of E is defined by

$$\delta(\epsilon) = \inf\left\{1 - \frac{\|x+y\|}{2} : \|x\| \leq 1, \|y\| \leq 1, \|x-y\| \geq \epsilon\right\}$$

for every ϵ with $0 \leq \epsilon \leq 2$. A Banach space E is said to be uniformly convex if $\delta(\epsilon) > 0$ for every $\epsilon > 0$. It is known that a Banach space E is uniformly convex if and only if for any two sequences $\{x_n\}$ and $\{y_n\}$ in E such that

$$\lim_{n\to\infty} \|x_n\| = \lim_{n\to\infty} \|y_n\| = 1 \text{ and } \lim_{n\to\infty} \|x_n + y_n\| = 2,$$

$\lim_{n\to\infty} \|x_n - y_n\| = 0$ holds. A uniformly convex Banach space is strictly convex and reflexive. We also know that a uniformly convex Banach space has the Kadec-Klee property, that is, $x_n \rightharpoonup u$ and $\|x_n\| \to \|u\|$ imply $x_n \to u$; see [9].

The duality mapping J from E into 2^{E^*} is defined by

$$Jx = \{x^* \in E^* : \langle x, x^*\rangle = \|x\|^2 = \|x^*\|^2\}$$

for every $x \in E$. Let $U = \{x \in E : \|x\| = 1\}$. The norm of E is said to be Gâteaux differentiable if for each $x, y \in U$, the limit

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t}$$

exists. In the case, E is called smooth. We know that E is smooth if and only if J is a single-valued mapping of E into E^*. We also know that E is reflexive if and only if J is surjective, and E is strictly convex if and only if J is one-to-one. Therefore, if E is a smooth, strictly convex and reflexive Banach space, then J is a single-valued bijection and in this case, the inverse mapping J^{-1} coincides with the duality mapping J_* on E^*. For more details, see [22] and [23]. We know the following result:

Lemma 2.1. *Let E be a smooth Banach space and let J be the duality mapping on E. Then, $\langle x-y, Jx-Jy\rangle \geq 0$ for all $x, y \in E$. Furthermore, if E is strictly convex and $\langle x-y, Jx-Jy\rangle = 0$, then $x = y$.*

Let C be a nonempty, closed and convex subset of a strictly convex and reflexive Banach space E. Then we know that for any $x \in E$, there exists a unique element $z \in C$ such that $\|x - z\| \leq \|x - y\|$ for all $y \in C$. Putting $z = P_C x$, we call P_C the metric projection of E onto C.

Lemma 2.2 ([22]). *Let E be a smooth, strictly convex and reflexive Banach space. Let C be a nonempty, closed and convex subset of E and let $x_1 \in E$ and $z \in C$. Then, the following conditions are equivalent*

(1) $z = P_C x_1$;
(2) $\langle z - y, J(x_1 - z)\rangle \geq 0, \quad \forall y \in C$.

Let E be a Banach space and let A be a mapping of E into 2^{E^*}. The effective domain of A is denoted by $\mathrm{dom}(A)$, that is, $\mathrm{dom}(A) = \{x \in E : Ax \neq \emptyset\}$. A multi-valued mapping A on E is said to be monotone if $\langle x - y, u^* - v^*\rangle \geq 0$ for all $x, y \in \mathrm{dom}(A)$, $u^* \in Ax$, and $v^* \in Ay$. A monotone operator A on E is said to be maximal if its graph is not properly contained in the graph of any other monotone operator on E. The following theorem is due to Browder [4]; see also [23, Theorem 3.5.4].

Theorem 2.3 ([4]). *Let E be a uniformly convex and smooth Banach space and let J be the duality mapping of E into E^*. Let A be a monotone operator of E into 2^{E^*}. Then A is maximal if and only if for any $r > 0$,*

$$R(J + rA) = E^*,$$

where $R(J + rA)$ is the range of $J + rA$.

Let E be a uniformly convex Banach space with a Gâteaux differentiable norm and let A be a maximal monotone operator of E into 2^{E^*}. For all $x \in E$ and $r > 0$, we consider the following equation

$$0 \in J(x_r - x) + rAx_r.$$

This equation has a unique solution x_r. We define J_r by $x_r = J_r x$. Such $J_r, r > 0$ are called the metric resolvents of A. The set of null points of A is defined by $A^{-1}0 = \{z \in E : 0 \in Az\}$. We know that $A^{-1}0$ is closed and convex; see [23].

Let E be a smooth, strictly convex and reflexive Banach space and let η be a real number with $\eta \in (-\infty, 1)$. Then a mapping $U : E \to E$ with $F(U) \neq \emptyset$ is called η-demimetric [27] if, for any $x \in E$ and $q \in F(U)$,

$$\langle x - q, J(x - Ux)\rangle \geq \frac{1-\eta}{2}\|x - Ux\|^2,$$

where $F(U)$ is the set of fixed points of U.

Examples. We know examples of η-demimetric mappings from [27].

(1) Let H be a Hilbert space and let k be a real number with $0 \leq k < 1$. Let U be a strict pseud-contraction [5] of H into itself such that $F(U) \neq \emptyset$. Then U is k-demimetric.

(2) Let E be a strictly convex, reflexive and smooth Banach space and let C be a nonempty, closed and convex subset of E. Let P_C be the metric projection of E onto C. Then P_C is (-1)-demimetric.

(3) Let E be a uniformly convex and smooth Banach space and let B be a maximal monotone operator with $B^{-1}0 \neq \emptyset$. Let $\lambda > 0$. Then the metric resolvent J_λ is (-1)-demimetric.

Furthermore, we know an important result for demimetric mappings in a smooth, strictly convex and reflexive Banach space.

Lemma 2.4 ([27]). *Let E be a smooth, strictly convex and reflexive Banach space and let η be a real number with $\eta \in (-\infty, 1)$. Let U be an η-demimetric mapping of E into itself. Then $F(U)$ is closed and convex.*

3 Main result and its Applications

In this section, using the demimetric operators, we prove a strong convergence theorem for finding a solution of the split common fixed point problem in Banach spaces. Let E be a Banach space and let C be a nonempty, closed and convex subset of E. A mapping $U : C \to E$ is called demiclosed if, for a sequence $\{x_n\}$ in C such that $x_n \rightharpoonup p$ and $x_n - Ux_n \to 0$, $p = Up$ holds. The following theorem was proved by Hojo and Takahashi [11].

Theorem 3.1 ([11]). *Let H be a Hilbert space and let F be a smooth, strictly convex and reflexive Banach space. Let J_F be the duality mapping on F and let η be a real number with $\eta \in (-\infty, 1)$. Let $T : H \to H$ be a nonexpansive mapping and let $U : F \to F$ be an η-demimetric and demiclosed mapping with $F(U) \neq \emptyset$. Let $A : H \to F$ be a bounded linear operator such that $A \neq 0$ and let A^* be the adjoint operator of A. Suppose that $F(T) \cap A^{-1}F(U) \neq \emptyset$. Let $x_1 \in H$ and let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} z_n = T\Big(x_n - \lambda_n A^* J_F(Ax_n - UAx_n)\Big), \\ y_n = \alpha_n x_n + (1-\alpha_n)z_n, \\ C_n = \{z \in H : \|y_n - z\| \leq \|x_n - z\|\}, \\ D_n = \{z \in H : \langle x_n - z, x_1 - x_n\rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap D_n} x_1, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $\{\alpha_n\} \subset [0,1]$ and $\{\lambda_n\} \subset (0,\infty)$ satisfy the conditions such that

$$0 \leq \alpha_n \leq a < 1, \quad \textit{and} \quad 0 < b \leq \lambda_n \|A\|^2 \leq c < (1-\eta)$$

for some $a, b, c \in \mathbb{R}$. *Then* $\{x_n\}$ *converges strongly to a point* $z_1 \in F(T) \cap A^{-1}F(U)$, *where* $z_1 = P_{F(T)\cap A^{-1}F(U)}x_1$.

Using Theorem 3.1, we get well-known and new strong convergence theorems which are connected with the split common fixed point problems in Banach spaces. We know the following result obtained by Marino and Xu [15]; see also [30].

Lemma 3.2 ([15]). *Let* H *be a Hilbert space and let* C *be a nonempty, closed and convex subset of* H. *Let* k *be a real number with* $0 \le k < 1$ *and* $U : C \to H$ *be a* k*-strict pseudo-contraction. If* $x_n \rightharpoonup z$ *and* $x_n - Ux_n \to 0$, *then* $z \in F(U)$.

Theorem 3.3. *Let* H_1 *and* H_2 *be Hilbert spaces. Let* k *be a real number with* $k \in [0, 1)$. *Let* $T : H_1 \to H_1$ *be a nonexpansive mapping and let* $U : H_2 \to H_2$ *be a* k*-strict pseud-contraction such that* $F(U) \neq \emptyset$. *Let* $A : H_1 \to H_2$ *be a bounded linear operator such that* $A \neq 0$ *and let* A^* *be the adjoint operator of* A. *Suppose that* $F(T) \cap A^{-1}F(U) \neq \emptyset$. *Let* $x_1 \in H$ *and let* $\{x_n\}$ *be a sequence generated by*

$$\begin{cases} z_n = T\Big(x_n - \lambda_n A^*(Ax_n - UAx_n)\Big), \\ y_n = \alpha_n x_n + (1 - \alpha_n)z_n, \\ C_n = \{z \in H : \|y_n - z\| \le \|x_n - z\|\}, \\ D_n = \{z \in H : \langle x_n - z, x_1 - x_n\rangle \ge 0\}, \\ x_{n+1} = P_{C_n \cap D_n}x_1, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $\{\alpha_n\} \subset [0, 1]$ *and* $\{\lambda_n\} \subset (0, \infty)$ *satisfy the conditions such that*

$$0 \le \alpha_n \le a < 1, \quad \textit{and} \quad 0 < b \le \lambda_n\|A\|^2 \le c < (1 - k)$$

for some $a, b, c \in \mathbb{R}$. *Then* $\{x_n\}$ *converges strongly to a point* $z_1 \in F(T) \cap A^{-1}F(U)$, *where* $z_1 = P_{F(T)\cap A^{-1}F(U)}x_1$.

Theorem 3.4. *Let* H *be a Hilbert space and let* F *be a smooth, strictly convex and reflexive Banach space. Let* J_F *be the duality mapping on* F. *Let* C *and* D *be nonempty, closed and convex subsets of* H *and* F, *respectively. Let* P_C *and* P_D *be the metric projections of* H *onto* C *and* F *onto* D, *respectively. Let* $T : H \to H$ *be a nonexpansive mapping, let* $A : H \to F$ *be a bounded linear operator such that* $A \neq 0$ *and let* A^* *be the adjoint operator of* A. *Suppose that* $C \cap A^{-1}D \neq \emptyset$. *Let* $x_1 \in H$ *and let* $\{x_n\}$ *be a sequence generated by*

$$\begin{cases} z_n = P_C\Big(x_n - \lambda_n A^* J_F(Ax_n - P_D Ax_n)\Big), \\ y_n = \alpha_n x_n + (1 - \alpha_n)z_n, \\ C_n = \{z \in H : \|y_n - z\| \le \|x_n - z\|\}, \\ D_n = \{z \in H : \langle x_n - z, x_1 - x_n\rangle \ge 0\}, \\ x_{n+1} = P_{C_n \cap D_n}x_1, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $\{\alpha_n\} \subset [0, 1]$ *and* $\{\lambda_n\} \subset (0, \infty)$ *satisfy the conditions such that*

$$0 \le \alpha_n \le a < 1, \quad \textit{and} \quad 0 < b \le \lambda_n\|A\|^2 \le c < 2$$

for some $a, b, c \in \mathbb{R}$. *Then the sequence* $\{x_n\}$ *converges strongly to a point* $z_0 \in C \cap A^{-1}D$, *where* $z_0 = P_{C\cap A^{-1}D}x_1$.

Theorem 3.5. *Let H be a Hilbert space and let F be a uniformly convex and smooth Banach space. Let J_F be the duality mapping on F. Let A and B be maximal monotone operators of H into H and F into F^*, respectively. Let J_λ be the resolvent of A for $\lambda > 0$ and let Q_μ be the metric resolvent of B for $\mu > 0$, respectively. Let $T : H \to F$ be a bounded linear operator such that $T \neq 0$ and let T^* be the adjoint operator of T. Suppose that $A^{-1}0 \cap T^{-1}(B^{-1}0) \neq \emptyset$. Let $x_1 \in H$ and let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} z_n = J_\lambda\Big(x_n - \lambda_n T^* J_F(Tx_n - Q_\mu Tx_n)\Big), \\ y_n = \alpha_n x_n + (1-\alpha_n) z_n, \\ C_n = \{z \in H : \|y_n - z\| \leq \|x_n - z\|\}, \\ D_n = \{z \in H : \langle x_n - z, x_1 - x_n \rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap D_n} x_1, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $\{\alpha_n\} \subset [0,1]$ and $\{\lambda_n\} \subset (0,\infty)$ satisfy the conditions such that

$$0 \leq \alpha_n \leq a < 1, \quad \text{and} \quad 0 < b \leq \lambda_n \|T\|^2 \leq c < 2$$

for some $a, b, c \in \mathbb{R}$. Then the sequence $\{x_n\}$ converges strongly to a point $z_0 \in A^{-1}0 \cap T^{-1}(B^{-1}0)$, where $z_0 = P_{A^{-1}0 \cap T^{-1}(B^{-1}0)} x_1$.

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