数理解析研究所講究録
第2011巻 2016年 147-153

ALMOST EVERYWHERE CONVERGENCE OF ERGODIC AVERAGES OF CERTAIN ORDER-PRESERVING OPERATORS ON L^1

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1. INTRODUCTION

Let $(\Omega, \mathcal{A}, m)$ be a positive measure space. For $1 \leq p < \infty$, L^p denotes the space of measurable functions f on Ω for which $\int_{\Omega} |f|^p \, dm < \infty$ and L^{∞} denotes the space of essentially-bounded measurable functions on Ω. A lot of effort has been devoted to the study of pointwise ergodic theorems for linear operators on L^1. The classical pointwise ergodic theorem is due to Hopf: If m is finite and T is a positive linear operator on L^1 with $\|T\| \leq 1$ and $T1 = 1$, then for each $f \in L^1$, the limit $\lim_{n\to\infty} n^{-1} \sum_{k=0}^{n-1} (T^k f)(w)$ exists almost everywhere on Ω. Meanwhile, the problem concerning non-linear operators in L^1 has been ignored. It is, however, known that in [11], Krengel constructed an order-preserving non-linear operator T in L^1 with $T0 = 0$ which is nonexpansive in L^p for $1 \leq p \leq \infty$ and a bounded function $f^* \geq 0$ such that the Cesàro means $n^{-1} \sum_{k=0}^{n-1} (T^k f^*)(w)$ diverge on a set of positive measure, concluding that the theorem of Hopf fails to extend to non-linear operators in L^1, although the possibility of positive results for specific classes of non-linear operators remains.

In this paper, we summarize the arguments presented in [17] about almost everywhere convergence of ergodic averages of order-preserving operators on L^1. We first give an elementary proof for the theorem of Hopf in the sense that it receives a general treatment in the context of fixed point theory and the theory of means on semigroups. The argument allows us to show a pointwise ergodic theorem for certain order-preserving operators on L^1.

2. PRELIMINARIES

Throughout the paper, let $\mathbb{N}, \mathbb{N}_+$ and $\mathbb{R}$ denote the set of positive integers, the set of non-negative integers and the set of real numbers, respectively. Let $\langle E, F \rangle$ be the duality between vector spaces E and F over $\mathbb{R}$. For each $y \in F$, we define a linear form f_y on E by $f_y(x) = \langle x, y \rangle$ $(x \in E)$. Then, $\sigma(E, F)$ denotes the weak topology on E generated by the family $\{f_y : y \in F\}$ and $\tau(E, F)$ denotes the Mackey topology on E with respect to $\langle E, F \rangle$, that is, the topology of uniform convergence on the circled, convex, $\sigma(F, E)$-compact subsets of F. Let $(E, \mathfrak{T})$ be a locally convex space. Then, the topological dual of E is denoted by E'. The bilinear form $(x, f) \mapsto f(x)$ on $E \times E'$ defines a duality $\langle E, E' \rangle$. The weak topology $\sigma(E, E')$ on E generated by E' is called the weak topology of E (associated with $\mathfrak{T}$ if this distinction is necessary). The topological dual of E under the Mackey topology $\tau(E', E)$ is denoted by E'_{τ}.

The vector space of bounded sequences $x = (x_n)_{n \geq 0}$ of real numbers is denoted by l^{∞}; under the norm $x = (x_n) \mapsto \|x\| = \sup_{n \geq 0} |x_n|$, l^{∞} is a Banach space. A linear functional μ on l^{∞} is said to be a mean on $\mathbb{N}_+$ if $\|\mu\| = \mu(e) = 1$, where $e_n = 1$ for each $n \in \mathbb{N}_+$. For each $k \in \mathbb{N}_+$, we define a point evaluation $\delta(k)$ by $\delta(k)x = x_k$ for each $x \in l^{\infty}$. A convex combination of point evaluations is called a finite mean on $\mathbb{N}_+$. As is well known, a linear functional μ on l^∞ is a mean on $\mathbb{N}_+$ if and only if $\inf_{n\geq 0} x_n \leq \mu(x) \leq \sup_{n\geq 0} x_n$ for each $x \in l^\infty$. If μ is a mean on $\mathbb{N}_+$ and $x \in l^\infty$, then we often write $\mu_n(x_n)$ for the value $\mu(x)$. A mean μ on $\mathbb{N}_+$ is said to be a Banach limit if $\mu_n(x_n) = \mu_n(x_{n+1})$ for each $x \in l^\infty$; for more details, see [2, 24].

Let E be a locally convex space and let μ be a mean on $\mathbb{N}_+$. We denote by $l_c^\infty(E)$ the vector space of sequences $x = (x_n)_{n\geq 0}$ of elements of E for which the closure of convex hull of $\{x_n : n = 0, 1, \dots\}$ is weakly compact; for each $x \in l_c^\infty(E)$, the closed, convex circled hull of $\{x_n : n = 0, 1, \dots\}$ is also weakly compact. For each $x \in l_c^\infty(E)$, we define a continuous linear functional $\tau(\mu)x$ on E'_τ by $\tau(\mu)x : x' \mapsto \mu_n\langle x_n, x'\rangle$ for each $x' \in E'$. Then, it follows from the separation theorem that $\tau(\mu)x$ is an element of E and is contained in the closure of convex hull of $\{x_n : n = 0, 1, \dots\}$. We denote by $\tau(\mu)$ the linear operator of $l_c^\infty(E)$ into E that assigns to each $x \in l_c^\infty(E)$ a unique element $\tau(\mu)x$ of E such that $\mu_n\langle x_n, x'\rangle = \langle \tau(\mu)x, x'\rangle$ for each $x' \in E'$. The operator $\tau(\mu)$ is called the vector-valued mean on $\mathbb{N}_+$ (generated by μ if explicit reference to μ is needed); for details, see [23, 24, 8, 16]. Note that it is also a vector-valued mean in the sense of Goldberg and Irwin [6]. An $x \in l_c^\infty(E)$ is said to have the mean value if there exists an element p of E such that $p = \tau(\mu)x$ for each Banach limit μ. The element p is called the mean value of x; see [19, 13, 2].

Let C be a closed convex subset of a locally convex space E, let T be a mapping of C into itself and let μ be a mean on $\mathbb{N}_+$. If $x \in C$, then $\mathcal{O}(x)$ denotes the orbit of x under T, that is, the set $\{T^n x : n = 0, 1, \dots\}$. We suppose that for some $x \in C$, the closure of convex hull of $\mathcal{O}(x)$ is weakly compact. Putting $\phi(x) = (T^n x)_{n\geq 0}$, we simply write $T(\mu)x$ in place of $\tau(\mu)(\phi(x))$. An element p of E is said to be the mean value of x under T if p is the mean value of $\phi(x)$, that is, $p = T(\mu)x$ for each Banach limit μ. Whenever the closure of convex hull of $\mathcal{O}(x)$ is weakly compact for each $x \in C$, we denote by $T(\mu)$ the mapping of C into itself that assigns to each $x \in C$ a unique element $T(\mu)x$ of C such that $\mu_n\langle T^n x, x'\rangle = \langle T(\mu)x, x'\rangle$ for each $x' \in E'$, and T is said to have the mean values on C if for each $x \in C$, there exists the mean value of x under T; see [21, 14].

Throughout the paper, let $(\Omega, \mathcal{A}, m)$ denote a positive measure space with σ-algebra $\mathcal{A}$ and measure m, and let $\mathcal{F}$ denote the family of measurable subsets of Ω with finite measure. Then, $\mathcal{F}$ is ordered by set inclusion in the sense that E is less than F $(E, F \in \mathcal{F})$ if and only if $E \subset F$, so that each finite subset of $\mathcal{F}$ has the least upper bound. For $1 \leq p < \infty$, let L^p denote the space of measurable functions f on Ω for which $\|f\|_p = (\int_\Omega |f|^p \, dm)^{\frac{1}{p}} < \infty$ and let L^∞ denote the space of measurable functions f on Ω for which $\|f\|_\infty = \inf_N \sup_{w\in\Omega\setminus N} |f(w)| < \infty$, where N ranges over the null subsets of Ω. For $1 \leq p \leq \infty$, the topology on L^p generated by the metric $(f, g) \mapsto \|f - g\|_p$ is sometimes called the L^p-norm topology and L^p is ordered by defining $f \leq g$ $(f, g \in L^p)$ to mean that $f(x) \leq g(x)$ almost everywhere on Ω, so that L^p becomes a Banach lattice. Let T be an operator on L^1. Then, T is said to be order-preserving if $f \leq g$ $(f, g \in L^1)$ implies $Tf \leq Tg$. Whenever T is linear, T is order-preserving if and only if T is positive. For each $f \in L^1$, a measurable function $M_\infty f$ on Ω is defined by $(M_\infty f)(s) = \sup_{n\geq 1} n^{-1} \sum_{k=0}^{n-1} (T^k f)(s)$ $(s \in \Omega)$.

Let $\mathcal{L}_{loc}^1$ be the vector space of measurable functions f on Ω for which $\|f\|_{E,1} = \int_E |f| \, dm < \infty$ for each $E \in \mathcal{F}$ and let $\mathcal{N}_{loc}$ be the vector subspace of $\mathcal{L}_{loc}^1$ consisting of measurable functions f on Ω for which $\|f\|_{E,1} = 0$ for each $E \in \mathcal{F}$. If $[f]$ denotes the equivalence class of an $f \in \mathcal{L}_{loc}^1$ mod $\mathcal{N}_{loc}$, then $[f] = [g]$ $(f, g \in \mathcal{L}_{loc}^1)$ means that $f(x) = g(x)$ almost everywhere on E for each $E \in \mathcal{F}$. Whenever m is σ-finite, $[f] = [g]$ $(f, g \in \mathcal{L}_{loc}^1)$ if and only if $f(x) = g(x)$ almost everywhere on Ω. *In the sequel, we shall assume that the measure space* $(\Omega, \mathcal{A}, m)$ *is* σ*-finite* unless explicitly specified. For each $E \in \mathcal{F}$, $[f] \mapsto \|f\|_{E,1}$ is a semi-norm on the quotient space $\mathcal{L}^1_{loc}/\mathcal{N}_{loc}$, which becomes a locally convex space, denoted by L^1_{loc}, under the separated locally convex topology generated by the semi-norms $[f] \mapsto \|f\|_{E,1}$ ($E \in \mathcal{F}$). Every element of L^1_{loc} is considered as a measurable function f on Ω for which $\|f\|_{E,1} < \infty$ for each $E \in \mathcal{F}$, if no confusion will occur. If m is finite, then L^1_{loc} equals L^1. Note that L^1 is a dense subspace of L^1_{loc} and hence L^1 itself is a locally convex space in which the family of sets of the form $U(E;r) = \{f \in L^1 : \|f\|_{E,1} < r\}$ ($E \in \mathcal{F}, r > 0$) is a neighborhood base at 0 for the separated locally convex topology τ on L^1. If a subset C of L^1 is uniformly integrable and bounded relative to the L^1-norm topology, then C is relatively (sequentially) compact relative to the weak topology of L^1 associated with τ; for details, see also [16, 18]. In the sequel, L^1 is assumed to be a Banach space unless explicitly specified.

Let C be a closed convex subset of a Banach space and let T be a mapping of C into itself. If $x \in C$, then $\omega(x)$ denotes the set of weak cluster points of $\{T^n x : n = 0, 1, \dots\}$. An element p of C is said to be a fixed point of T if $Tp = p$. If $\|Tx - Ty\| \leq \|x - y\|$ for each $x, y \in C$, then T is said to be nonexpansive. Let D be a subset of C. If for each $x, y \in D$, the limit $\lim_{n\to\infty} \|T^{n+h}x - T^{n+k}y\|$ exists uniformly in $h, k \in \mathbb{N}_+$, then T is called asymptotically isometric on D. Let T be a nonexpansive mapping of C into itself. Then, T is said to be almost periodic at an $x \in C$ if the orbit $\mathcal{O}(x)$ of x under T is relatively compact. If for each $x \in C$, T is almost periodic at x, then T is called almost periodic; for details, see [14]. Let $\{x_n\}$ be a sequence of E and let $x \in E$. Then, x_n is said to converge strongly to x if $\|x_n - x\| \to 0$ as $n \to \infty$. If A is a subset of E, then $\overline{\text{co}}A$ denotes the closure of convex hull of A.

3. Almost everywhere convergence of ergodic averages for linear operators

In this section, we give an alternative proof for the classical pointwise ergodic theorem of Hopf in the context of fixed point theory and the theory of means on semigroups by means of the maximal ergodic theorem of Hopf.

The following result is well known as the mean ergodic theorem for linear operators on Banach spaces; see also [24].

Theorem 1 (Kakutani and Yosida). *Let E be a Banach space and let T be a linear operator on E with $\|T\| \leq 1$. If for some $x \in E$, the orbit $\mathcal{O}(x)$ of x under T is relatively weakly compact, then the Cesàro means $n^{-1}\sum_{k=0}^{n-1} T^k x$ converge strongly to a fixed point p of T. In this case, $p = T(\mu)x$ for each Banach limit μ.*

Proposition 1. *Let T be an operator on L^1. If for some $f \in L^1$, the Cesàro means*

$$n^{-1}\sum_{k=0}^{n-1} T^k f$$

converge strongly or in the separated locally convex topology on L^1 generated by the semi-norms $f \mapsto \|f\|_{E,1}$ ($E \in \mathcal{F}$), then there exists a Banach limit μ such that $T(\mu)f$ is well-defined and

$$\liminf_{n\to\infty} n^{-1}\sum_{k=0}^{n-1}(T^k f)(s) \leq (T(\mu)f)(s) \leq \limsup_{n\to\infty} n^{-1}\sum_{k=0}^{n-1}(T^k f)(s)$$

a.e. on Ω.

Remark 1. It is suggested in Proposition 1 that the function $T(\mu)f$ is a candidate for the limit function in almost everywhere convergence of the Cesàro means

$n^{-1}\sum_{k=0}^{n-1} T^k f$. It suffices to show

$$\limsup_{n\to\infty}\left(n^{-1}\sum_{k=0}^{n-1} T^k f - T(\mu)f\right) \le 0 \text{ and } \limsup_{n\to\infty}\left(T(\mu)f - n^{-1}\sum_{k=0}^{n-1} T^k f\right) \le 0$$

a.e. on Ω in order to prove that $\lim_{n\to\infty} n^{-1}\sum_{k=0}^{n-1}(T^k f)(s) = (T(\mu)f)(s)$ a.e. on Ω under the assumption of Proposition 1.

Lemma 1. *Let T be a linear operator on L^1 with $\|Tf\|_\infty \le \|f\|_\infty$ $(f \in L^1)$. Then, for each $f \in L^1 \cap L^\infty$ and g in the convex hull of the orbit $\mathcal{O}(f)$ of f under T,*

$$\lim_{n\to\infty}\left\| n^{-1}\sum_{k=0}^{n-1} T^k f - n^{-1}\sum_{k=0}^{n-1} T^k g\right\|_\infty = 0.$$

The following result is a version of the maximal ergodic theorem of Hopf; see also [10].

Theorem 2. *Let T be an order-preserving operator on L^1 with $\|Tf\|_1 \le \|f\|_1$ $(f \in L^1)$ such that $T(f+g) \ge Tf + Tg$ and $\|Tf - Tg\|_\infty \le \|f-g\|_\infty$ $(f, g \in L^1)$. Then, for each $f \in L^1$ and $\alpha > 0$,*

$$m\{M_\infty f > \alpha\} \le \alpha^{-1}\|f\|_1,$$

where $(M_\infty f)(s) = \sup_{n\ge 1} n^{-1}\sum_{k=0}^{n-1}(T^k f)(s)$ $(s \in \Omega)$.

The next lemma follows from Lemma 1 and Theorem 2.

Lemma 2. *Let T be a positive linear operator on L^1 with $\|T\| \le 1$ and $\|Tf\|_\infty \le \|f\|_\infty$ $(f \in L^1)$. Then, for each $f \in L^1 \cap L^\infty$ and g in the closure $K(f)$ of convex hull of the orbit $\mathcal{O}(f)$ of f under T,*

$$n^{-1}\sum_{k=0}^{n-1}(T^k f)(s) - n^{-1}\sum_{k=0}^{n-1}(T^k g)(s) \to 0$$

a.e. on Ω as $n \to \infty$. Moreover, if m is finite, then

$$\lim_{n\to\infty} n^{-1}\sum_{k=0}^{n-1}(T^k f)(s) = (T(\mu)f)(s)$$

a.e. on Ω, where μ is a Banach limit. In this case, $T(\mu)f$ is a fixed point of T.

Proof. Let $f \in L^1 \cap L^\infty$ and let $g \in K(f)$. For each $n \in \mathbb{N}$, put $\lambda_n = n^{-1}\sum_{k=0}^{n-1}\delta(k)$. Then, it follows from Lemma 1 and Theorem 2 that for each h in the convex hull of $\mathcal{O}(f)$ and $\alpha > 0$,

$$\begin{aligned} T(\lambda_n)f - T(\lambda_n)g &= T(\lambda_n)f - T(\lambda_n)h + T(\lambda_n)(h-g) \\ &\le T(\lambda_n)f - T(\lambda_n)h + M_\infty(h-g) \end{aligned}$$

and hence

$$m\{\limsup_{n\to\infty}(T(\lambda_n)f - T(\lambda_n)g) > \alpha\} \le m\{M_\infty(h-g) > \alpha\} \le \alpha^{-1}\|h-g\|_1.$$

So, we have $m\{\limsup_{n\to\infty}(T(\lambda_n)f - T(\lambda_n)g) > 0\} = 0$. Similarly, we also have $m\{\limsup_{n\to\infty}(T(\lambda_n)g - T(\lambda_n)f) > 0\} = 0$. This implies $\lim_{n\to\infty}(T(\lambda_n)f)(s) - (T(\lambda_n)g)(s) = 0$ a.e. on Ω for each $f \in L^1 \cap L^\infty$ and $g \in K(f)$. If m is finite and μ is a Banach limit, then it follows from Theorem 1 that a fixed point $T(\mu)f$ of T is contained in $K(f)$. Hence, we have $\lim_{n\to\infty}(T(\lambda_n)f)(s) = (T(\mu)f)(s)$ a.e. on Ω for each $f \in L^1 \cap L^\infty$. □

Remark 2. Under the assumption of Lemma 2, it is known that if m is infinite σ-finite, then for each $f \in L^1$, a fixed point $T(\mu)f$ of T is contained in the closure of convex hull of the orbit $\mathcal{O}(f)$ of f under T with respect to the separated locally convex topology on L^1 generated by the semi-norms $f \mapsto \|f\|_{E,1}$ $(E \in \mathcal{F})$; see also [16].

Theorem 3 (Hopf). *Let m be finite and let T be a positive linear operator on L^1 with $\|T\| \leq 1$ and $T1 = 1$. Then, for each $f \in L^1$,*

$$\lim_{n\to\infty} n^{-1} \sum_{k=0}^{n-1} (T^k f)(s) = (T(\mu)f)(s)$$

a.e. on Ω, where μ is a Banach limit. In this case, $T(\mu)f$ is a fixed point of T.

Proof. Let μ be a Banach limit and let $f \in L^1$. For each $n \in \mathbb{N}$, put $\lambda_n = n^{-1}\sum_{k=0}^{n-1}\delta(k)$. Then, $T(\mu)$ is a linear operator on L^1 with $\|T(\mu)\| \leq 1$. It follows from Lemma 2 and Theorem 2 that for each $g \in L^1 \cap L^\infty$ and $\alpha > 0$,

$$\begin{aligned} &T(\lambda_n)f - T(\mu)f \\ &\quad = T(\lambda_n)f - T(\lambda_n)g + T(\lambda_n)g - T(\mu)g + T(\mu)g - T(\mu)f \\ &\quad \leq M_\infty(f-g) + T(\lambda_n)g - T(\mu)g + T(\mu)(g-f) \end{aligned}$$

and hence

$$\begin{aligned} m\{\limsup_{n\to\infty}(T(\lambda_n)f - T(\mu)f) > 2\alpha\} &\leq m\{M_\infty(f-g) > \alpha\} + m\{T(\mu)(g-f) > \alpha\} \\ &\leq \alpha^{-1}\|f-g\|_1 + \alpha^{-1}\|T(\mu)(g-f)\|_1 \\ &\leq 2\alpha^{-1}\|f-g\|_1. \end{aligned}$$

So, we have $m\{\limsup_{n\to\infty}(T(\lambda_n)f - T(\mu)f) > 0\} = 0$. Similarly, we also have $m\{\limsup_{n\to\infty}(T(\mu)f - T(\lambda_n)f) > 0\} = 0$. This implies $\lim_{n\to\infty}(T(\lambda_n)f)(s) = (T(\mu)f)(s)$ a.e. on Ω for each $f \in L^1$. □

In the case of *σ-finite* measure spaces, we obtain the following result concerning the ergodicity of linear operators T on L^1 in the sense that for each $f \in L^1$, the Cesàro means $n^{-1}\sum_{k=0}^{n-1} T^k f$ converge to a fixed point of T in a separated locally convex topology on L^1; see also [16].

Theorem 4. *Let T be a positive linear operator on L^1 with $\|T\| \leq 1$ and $\|Tf\|_\infty \leq \|f\|_\infty$ $(f \in L^1)$. Then, for each $f \in L^1$, the Cesàro means*

$$n^{-1}\sum_{k=0}^{n-1} T^k f$$

converge to a fixed point $T(\mu)f$ of T in the separated locally convex topology τ on L^1 generated by the semi-norms $f \mapsto \|f\|_{E,1}$ $(E \in \mathcal{F})$, where μ is a Banach limit.

Proof. Let μ be a Banach limit and let $f \in L^1 \cap L^\infty$. For each $n \in \mathbb{N}$, put $\lambda_n = n^{-1}\sum_{k=0}^{n-1}\delta(k)$. We have $\lim_{n\to\infty}\|T(\lambda_n)f - T(\mu)f\|_2 = 0$ from Theorem 1 and the convexity theorem of Riesz and Thorin. Let $\{\lambda_{n_k}\}$ be a subsequence of $\{\lambda_n\}$. Then, there exists a subsequence $\{\lambda_{n_{k_l}}\}$ of $\{\lambda_{n_k}\}$ such that $\lim_{l\to\infty}(T(\lambda_{n_{k_l}})f)(s) = (T(\mu)f)(s)$ a.e. on Ω. Since the orbit $\mathcal{O}(f)$ of f under T is uniformly integrable and bounded relative to the L^1-norm topology, $T(\lambda_{n_{k_l}})f$ converges in τ to $T(\mu)f$ as $l \to \infty$ by virtue of the convergence theorem of Vitali. Hence, $T(\lambda_n)f$ converge in τ to $T(\mu)f$. It is easy to show that for each $f \in L^1$, $T(\lambda_n)f$ converge in τ to $T(\mu)f$, for $T(\mu)$ is a linear operator on L^1 with $\|T(\mu)\| \leq 1$. □

4. Almost everywhere convergence of ergodic averages for certain order-preserving operators

In this section, we show a pointwise ergodic theorem for certain order-preserving operators on L^1, by applying similar arguments to those used in the previous section to such operators on L^1.

Theorem 5. *Let C be a closed convex subset of a Banach space and let T be a nonexpansive mapping of C into itself which is almost periodic at some $x \in C$. Then, T is asymptotically isometric on $\{x\}$ and the Cesàro means $n^{-1}\sum_{k=0}^{n-1} T^k x$ converge strongly to a point p of C. In this case, $p = T(\mu)x$ for each Banach limit μ.*

Remark 3. Let C be a closed convex subset of a Banach space. If T is a contractive mapping of C into itself, that is, there exists a non-negative number $r < 1$ such that $\|Tx - Ty\| \le r\|x - y\|$ for each $x, y \in C$, then T is almost periodic. For the details of Theorem 5, see also [14].

Lemma 3. *If a subset C of $L^1 \cap L^\infty$ is relatively compact in L^1 and bounded in L^∞, then C is relatively compact in L^2.*

Remark 4. In Lemma 3, let D be a subset of C. If D_1 and D_2 are the closures of D in L^1 and L^2 respectively, then $D_1 = D_2 \subset L^1 \cap L^2$, for D_1 is closed relative to the separated locally convex topology on L^1 generated by the semi-norms $f \mapsto \|f\|_{E,1}$ ($E \in \mathcal{F}$). For compactness in L^1, see also [5, 3, 18].

Proposition 2 (Bruck). *Let C be a bounded, closed convex subset of a uniformly convex Banach space and let T be a nonexpansive mapping of C into itself which is asymptotically isometric on $\{x\}$ for some $x \in C$. Then, $\overline{co}\,\omega(x)$ is invariant under T, that is, $T(\overline{co}\,\omega(x)) \subset \overline{co}\,\omega(x)$ and T is affine on $\overline{co}\,\omega(x)$, where $\omega(x)$ is the set of weak cluster points of $\{T^n x : n = 0, 1, \dots\}$.*

Theorem 6 (Krengel and Lin). *If T is an order-preserving nonexpansive operator on L^1 with $T0 = 0$ and $\|Tf - Tg\|_\infty \le \|f - g\|_\infty$ ($f, g \in L^1 \cap L^\infty$), then T is nonexpansive in L^p for $1 < p < \infty$.*

Theorem 7. *Let T be an order-preserving, almost periodic operator on L^1 with $T0 = 0$ such that $\|Tf - Tg\|_1 \le \|f - g\|_1$ and $\|Tf - Tg\|_\infty \le \|f - g\|_\infty$ ($f, g \in L^1$). If $T(f + g) \ge Tf + Tg$ ($f, g \in L^1$) and there exists an $r > 0$ such that $T^n f - T^n g \le r\,T^n(f - g)$ ($f, g \in L^1$) for each $n \in \mathbb{N}_+$, then for each $f \in L^1$,*

$$\lim_{n\to\infty} n^{-1} \sum_{k=0}^{n-1} (T^k f)(s) = (T(\mu)f)(s)$$

a.e. on Ω, where μ is a Banach limit. In this case, $T(\mu)f$ is a fixed point of T.

Sketch of proof. Let μ be a Banach limit and let $f \in L^1 \cap L^\infty$. For each $n \in \mathbb{N}$, put $\lambda_n = n^{-1}\sum_{k=0}^{n-1} \delta(k)$. Then, it follows from Theorem 2 that for each $\alpha > 0$, $k \in \mathbb{N}_+$ and g in the closure $C(f)$ of the orbit $\mathcal{O}(f)$ of f under T,

$$\begin{aligned} T(\lambda_n)f - T(\lambda_n)g &= T(\lambda_n)f - T(\lambda_n)(T^k f) + T(\lambda_n)(T^k f) - T(\lambda_n)g \\ &\le T(\lambda_n)f - T(\lambda_n)(T^k f) + r\,T(\lambda_n)(T^k f - g) \\ &\le T(\lambda_n)f - T(\lambda_n)(T^k f) + r\,M_\infty(T^k f - g) \end{aligned}$$

and hence

$$m\{\limsup_{n\to\infty}(T(\lambda_n)f - T(\lambda_n)g) > \alpha\} \le m\{M_\infty(T^k f - g) > r^{-1}\alpha\} \le r\alpha^{-1}\|T^k f - g\|_1$$

from $\lim_{n\to\infty}\|T(\lambda_n)f - T(\lambda_n)(T^k f)\|_\infty = 0$. So, we have $m\{\limsup_{n\to\infty}(T(\lambda_n)f - T(\lambda_n)g) > 0\} = 0$. This implies $\lim_{n\to\infty}(T(\lambda_n)f)(s) - (T(\lambda_n)g)(s) = 0$ a.e. on Ω for each $g \in C(f)$. It follows from Theorem 5, Lemma 3, Proposition 2 and Theorem 6 that since a closed ball $\{g \in L^2 : \|g\|_2 \leq \|f\|_2\}$ in L^2 is invariant under T, T is affine on the closure $K(f)$ of convex hull of $\omega(f)$. Let $g = \sum_{i=1}^N \alpha_i g_i$ with $g_i \in \omega(f) \subset C(f)$ and $\sum_{i=1}^N \alpha_i = 1$ $(0 < \alpha_i < 1)$. Then,

$$\lim_{n\to\infty}(T(\lambda_n)f)(s) - (T(\lambda_n)g)(s) = \sum_{i=1}^N \alpha_i \lim_{n\to\infty}(T(\lambda_n)f)(s) - (T(\lambda_n)g_i)(s) = 0$$

a.e. on Ω. From Theorem 2 and Theorem 5, we have $\lim_{n\to\infty}(T(\lambda_n)f)(s) - (T(\lambda_n)h)(s) = 0$ a.e. on Ω for each h in $K(f)$ and hence $\lim_{n\to\infty}(T(\lambda_n)f)(s) = (T(\mu)f)(s)$ a.e. on Ω, for $T(\mu)f$ is a fixed point of T and $T(\mu)f \in \bigcap_{n\geq 0}\overline{\mathrm{co}}\{T^k f : k \geq n\} = K(f)$. Since $T(\mu)$ is nonexpansive in L^1, it is verified as in proof of Theorem 3 that $\lim_{n\to\infty}(T(\lambda_n)f)(s) = (T(\mu)f)(s)$ a.e. on Ω for each $f \in L^1$. □

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