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Fixed point theorems for contractively widely more generalized hybrid mappings in metric spaces

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Abstract

In this paper we consider a broad class of mappings containing Kannan mappings and contratively generalized hybrid mappings. Then we deal with fixed point theorems for such a mapping. Using these results, we show directly well-known fixed point theorems in complete metric spaces.

1 Introduction

Let (X,d) be a metric space. A mapping T from X into itself is said to be contractive if there exists k wirh $k\in[0,1)$ such that

$$d(Tx,Ty)\leq kd(x,y)$$

for any $x,y\in X$. Such a mapping is called a k-contractive mapping. A mapping T from X into itself is said to be Kannan [5] if there exists k with $k\in\left[0,\frac{1}{2}\right)$ such that

$$d(Tx,Ty)\leq k(d(x,Tx)+d(y,Ty))$$

for any $x,y\in X$. A mapping T from X into itself is said to be contractively nonspreading [1,4,9] if there exists k with $k\in\left[0,\frac{1}{2}\right)$ such that

$$d(Tx,Ty)\leq k(d(x,Ty)+d(y,Tx))$$

for any $x,y\in X$. A mapping T from X into itself is said to be contractively hybrid [3] if there exists k with $k\in\left[0,\frac{1}{3}\right)$ such that

$$d(Tx,Ty)\leq k(d(Tx,y)+d(Ty,x)+d(x,y))$$

for any $x,y\in X$. Recently, Hasegawa, Komiya and Takahashi [3] introduced the concept of contratively generalized hybrid mappings on metric spaces and studied the fixed point theorems for such mappings on complete metric spaces. A mapping T from X into itself is said to be contratively generalized hybrid if there exist $\alpha,\beta,r\in\mathbb{R}$ with $r\in[0,1)$ such that

$$\alpha d(Tx,Ty)+(1-\alpha)d(x,Ty)\leq r(\beta d(Tx,y)+(1-\beta)d(x,y))$$

for any $x, y \in X$. Such a mapping is called an (α, β, r)-contratively generalized hybrid mapping; see also Kocourek, Takahashi and Yao [7] for such a mapping in Hilbert spaces. For instance, if $\alpha = 1$ and $\beta = 0$, then an (α, β, r)-contratively generalized hybrid mapping is contractive; if $\alpha = 1 + r$ and $\beta = 1$, then an (α, β, r)-contratively generalized hybrid mapping is contractively nonspreading; if $\alpha = 1 + \frac{r}{2}$ and $\beta = \frac{1}{2}$, then an (α, β, r)-contratively generalized hybrid mapping is contractively hybrid; see Hasegawa, Komiya and Takahashi [3].

In this paper, motivated by Hasegawa, Komiya and Takahashi [3], we consider a broad class of mappings containing Kannan mappings and contratively generalized hybrid mappings. Then we deal with fixed point theorems for such a mapping. Using these results, we show directly well-known fixed point theorems in complete metric spaces.

2 Preliminaries

We know the following Caristi's fixed point theorem which was generalized by Takahashi [8].

Theorem 2.1. *Let (X, d) be a complete metric space, let ψ be a proper, bounded below, and lower semicontinuous mapping from X into $(-\infty, \infty]$, and let T be a mapping from X into itself. Suppose that*

$$d(x, Tx) + \psi(Tx) \leq \psi(x)$$

for any $x \in X$. Then T has a fixed point.

Let ℓ^∞ be the Banach space of bounded sequences with supremum norm. Let μ be an element of $(\ell^\infty)^*$, which is the dual space of ℓ^∞. Then we denote by $\mu(x)$ the value of μ at $x = (x_1, x_2, \dots) \in \ell^\infty$. Sometimes we denote by $\mu_n(x_n)$ the value $\mu(x)$. A linear functional μ on ℓ^∞ is called a mean if $\mu(e) = \|\mu\| = 1$, where $e = (1, 1, \dots)$. A mean μ is called a Banach limit on ℓ^∞ if $\mu_n(x_{n+1}) = \mu_n(x_n)$. We know that there exists a Banach limit on ℓ^∞. If μ is a Banach limit on ℓ^∞, then

$$\liminf_{n\to\infty} x_n \leq \mu_n(x_n) \leq \limsup_{n\to\infty} x_n$$

holds for any $x = (x_1, x_2, \dots) \in \ell^\infty$. In particular, if $x = (x_1, x_2, \dots) \in \ell^\infty$ and $x_n \to a \in \mathbb{R}$, then we obtain $\mu_n(x_n) = a$. See [8] for the proof of existence of a Banach limit and its other elementary properties.

Moreover we use the following lemma and theorem showed by Hasegawa, Komiya and Takahashi [3].

Lemma 2.1. *Let (X, d) be a metric space, let $\{x_n\}$ be a bounded sequence in X, let μ be a mean on ℓ^∞ and let g be a mapping from X into $\mathbb{R}$ defined by*

$$g(x) = \mu_n d(x_n, x)$$

for any $x \in X$. Then g is continuous.

Theorem 2.2. *Let (X, d) be a complete metric space, let μ be a mean on ℓ^∞ and let T be a mapping from X into itself. Suppose that there exist a real number r with $0 \leq r < 1$ and $z \in X$ such that $\{T^n z \mid n \in \mathbb{N} \cup \{0\}\}$ is bounded and*

$$\mu_n d(T^n z, Tx) \leq r \mu_n d(T^n z, x)$$

for any $x \in X$. Then the following hold:

(i) *T has a unique fixed point $u \in X$;*

(ii) *$u = \lim_{n\to\infty} T^n x$ for any $x \in X$.*

3 Fixed point theorems

In this section we consider an $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta)$-contractively widely more generalized hybrid mapping from a metric space X into itself; see also Kawasaki and Takahashi [6] for such a mapping in Hilbert spaces.

Definition 3.1. Let (X, d) be a metric space and let T be a mapping from X into itself. We say that T is contractively widely more generalized hybrid if T satisfies the following condition: there exist real numbers α, β, γ, δ, ε and ζ such that

$$\alpha d(Tx, Ty) + \beta d(x, Ty) + \gamma d(Tx, y) + \delta d(x, y) + \varepsilon d(x, Tx) + \zeta d(y, Ty) \leq 0$$

for any $x, y \in X$. Such a mapping T is called an $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta)$-contractively widely more generalized hybrid mapping.

Firstly we consider criteria for an $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta)$-contractively widely more generalized hybrid mapping T from a metric space X into itself such that $\{T^n x \mid n \in \mathbb{N} \cup \{0\}\}$ is a Cauchy sequence for any $x \in X$.

Lemma 3.1. *Let (X, d) be a metric space and let T be an $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta)$-contractively widely more generalized hybrid mapping from X into itself satisfying* (B1), (B2) *or* (B3)*:*

(B1) $\alpha + \beta + \zeta \geq 0$ *and* $\alpha + 2\min\{\beta, 0\} + \delta + \varepsilon + \zeta > 0$;

(B2) $\alpha + \gamma + \varepsilon \geq 0$ *and* $\alpha + 2\min\{\gamma, 0\} + \delta + \varepsilon + \zeta > 0$;

(B3) $2\alpha + \beta + \gamma + \varepsilon + \zeta \geq 0$ *and* $\alpha + \min\{\beta + \gamma, 0\} + \delta + \varepsilon + \zeta > 0$.

Then $\{T^n x \mid n \in \mathbb{N} \cup \{0\}\}$ is a Cauchy sequence for any $x \in X$.

Using Lemma 3.1, we obtain directly the following theorem.

Theorem 3.1. *Let (X, d) be a complete metric space and let T be an $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta)$-contractively widely more generalized hybrid mapping from X into itself satisfying* (B1), (B2) *or* (B3)*. Then for any $x \in X$ there exists $\lim_{n\to\infty} T^n x$.*

Remark 3.1. Let (X, d) be a metric space and let $\{x_n \mid n \in \mathbb{N} \cup \{0\}\}$ be a Cauchy sequence in X. Then $\{x_n \mid n \in \mathbb{N} \cup \{0\}\}$ is bounded. Indeed, since $\{x_n \mid n \in \mathbb{N} \cup \{0\}\}$ is a Cauchy sequence, for any positive number ρ there exists $N \in \mathbb{N}$ such that $d(x_m, x_n) < \rho$ for any $m, n \geq N$. Put $M = \max\{d(x_0, x_N), \ldots, d(x_{N-1}, x_N), \rho\}$. Then $d(x_n, x_N) \leq M$ for any $n \in \mathbb{N} \cup \{0\}$.

Using Theorem 2.1, we show the following fixed point theorem.

Theorem 3.2. *Let (X, d) be a complete metric space and let T be an $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta)$-contractively widely more generalized hybrid mapping from X into itself satisfying* (C1), (C2) *or* (C3)*:*

(C1) $\zeta > 0$, $\alpha + \beta \geq 0$ *and* $\alpha + \beta + \gamma + \delta + 2\min\{\varepsilon, 0\} \geq 0$*;*

(C2) $\varepsilon > 0$, $\alpha + \gamma \geq 0$ *and* $\alpha + \beta + \gamma + \delta + 2\min\{\zeta, 0\} \geq 0$*;*

(C3) $\varepsilon + \zeta > 0$, $2\alpha + \beta + \gamma \geq 0$ *and* $\alpha + \beta + \gamma + \delta \geq 0$.

Then T has a fixed point if and only if there exists $z \in X$ such that $\{T^n z \mid n \in \mathbb{N} \cup \{0\}\}$ is bounded. In particular, if $\alpha + \beta + \gamma + \delta > 0$, then T has a unique fixed point.

Using Lemma 3.1, Remark 3.1 and Theorem 3.2, we obtain the following fixed point theorem.

Theorem 3.3. *Let (X, d) be a complete metric space and let T be an $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta)$-contractively widely more generalized hybrid mapping from X into itself satisfying the following:*

(B) *one of* (B1), (B2) *and* (B3) *holds;*

(C) *one of* (C1), (C2) *and* (C3) *holds.*

Then T has a fixed point. In particular, if $\alpha + \beta + \gamma + \delta > 0$, then T has a unique fixed point.

Using Theorem 2.2, we show the following fixed point theorem.

Theorem 3.4. *Let (X, d) be a complete metric space and let T be an $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta)$-contractively widely more generalized hybrid mapping from X into itself satisfying* (H1), (H2) *or* (H3)*:*

(H1) $\alpha + \beta + \zeta > 0$ *and* $\alpha + \beta + \gamma + \delta + 2\min\{\varepsilon, 0\} + 2\min\{\zeta, 0\} > 0$*;*

(H2) $\alpha + \gamma + \varepsilon > 0$ *and* $\alpha + \beta + \gamma + \delta + 2\min\{\varepsilon, 0\} + 2\min\{\zeta, 0\} > 0$*;*

(H3) $2\alpha + \beta + \gamma + \varepsilon + \zeta > 0$ *and* $\alpha + \beta + \gamma + \delta + 2\min\{\varepsilon + \zeta, 0\} > 0$.

Then T has a fixed point if and only if there exists $z \in X$ such that $\{T^n z \mid n \in \mathbb{N} \cup \{0\}\}$ is bounded. Moreover the following hold:

(i) *T has a unique fixed point* $u \in X$;

(ii) $u = \lim_{n\to\infty} T^n x$ *for any* $x \in X$.

Using Lemma 3.1, Remark 3.1 and Theorem 3.4, we obtain the following fixed point theorem.

Theorem 3.5. *Let* (X, d) *be a complete metric space and let* T *be an* $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta)$*-contractively widely more generalized hybrid mapping from* X *into itself satisfying the following:*

(B) *one of* (B1), (B2) *and* (B3) *holds;*

(H) *one of* (H1), (H2) *and* (H3) *holds.*

Then the following hold:

(i) *T has a unique fixed point* $u \in X$;

(ii) $u = \lim_{n\to\infty} T^n x$ *for any* $x \in X$.

Moreover, if (B) is satisfied, we also show the following fixed point theorem.

Theorem 3.6. *Let* (X, d) *be a complete metric space and let* T *be an* $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta)$*-contractively widely more generalized hybrid mapping from* X *into itself satisfying* (B), *and one of* (M1), (M2) *and* (M3):

(M1) $\alpha + \beta + \zeta > 0$;

(M2) $\alpha + \gamma + \varepsilon > 0$;

(M3) $2\alpha + \beta + \gamma + \varepsilon + \zeta > 0$.

Then T *has a fixed point. In particular, if* $\alpha + \beta + \gamma + \delta > 0$, *then the following hold:*

(i) *T has a unique fixed point* $u \in X$;

(ii) $u = \lim_{n\to\infty} T^n x$ *for any* $x \in X$.

4 Applications

Theorem 4.1. *Let* (X, d) *be a complete metric space and let* T *be a contractively generalized hybrid mapping form* X *into itself, that is, there exist* $\alpha, \beta, r \in \mathbb{R}$ *with* $0 \le r < 1$ *such that*

$$\alpha d(Tx, Ty) + (1-\alpha)d(x, Ty) \le r(\beta d(Tx, y) + (1-\beta)d(x, y))$$

for any $x, y \in X$. *Suppose that* $\alpha > r(1 + |\beta|)$. *Then the following hold:*

(i) *T has a unique fixed point* $u \in X$;

(ii) $u = \lim_{n\to\infty} T^n x$ *for any* $x \in X$.

Theorem 4.2. *Let* (X, d) *be a complete metric space and let* T *be a mapping form* X *into itself satisfying there exist* $\varepsilon, \zeta \in \mathbb{R}$ *such that* $\varepsilon + \zeta < 1$ *and*

$$d(Tx, Ty) \leq \varepsilon d(x, Tx) + \zeta d(y, Ty)$$

for any $x, y \in X$. *Then the following hold:*

(i) T *has a unique fixed point* $u \in X$;

(ii) $u = \lim_{n\to\infty} T^n x$ *for any* $x \in X$.

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