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MANY TORIC IDEALS GENERATED BY QUADRATIC BINOMIALS POSSESS NO QUADRATIC GRÖBNER BASES (SUMMARY)

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ABSTRACT. This is a brief summary of Hibi–Nishiyama–Ohsugi–Shikama [6]. Let G be a finite connected simple graph and I_G the toric ideal of the edge ring of G. In the present paper, we study finite graphs G with the property that I_G is generated by quadratic binomials and I_G possesses no quadratic Gröbner basis. First, we give a nontrivial infinite series of finite graphs with the above property. Second, we implement a combinatorial characterization for I_G to be generated by quadratic binomials and, by means of the computer search, we classify the finite graphs G with the above property, up to 8 vertices.

INTRODUCTION

Let G be a finite connected simple graph on the vertex set $[n] = \{1, 2, \ldots, n\}$ with $E(G) = \{e_1, \ldots, e_d\}$ its edge set. (Recall that a finite graph is *simple* if it possesses no loop and no multiple edge.) Let K be a field and $K[\mathbf{t}] = K[t_1, \ldots, t_n]$ the polynomial ring in n variables over K. If $e = \{i, j\} \in E(G)$, then $\mathbf{t}^e$ stands for the quadratic monomial $t_i t_j \in K[\mathbf{t}]$. The *edge ring* ([15]) of G is the subring $K[G] = K[\mathbf{t}^{e_1}, \ldots, \mathbf{t}^{e_d}]$ of $K[\mathbf{t}]$. Let $K[\mathbf{x}] = K[x_1, \ldots, x_d]$ denote the polynomial ring in d variables over K with each $\deg x_i = 1$ and define the surjective ring homomorphism $\pi : K[\mathbf{x}] \to K[G]$ by setting $\pi(x_i) = \mathbf{t}^{e_i}$ for each $1 \leq i \leq d$. The *toric ideal* I_G of G is the kernel of π. It is known [17, Corollary 4.3] that I_G is generated by those binomials $u - v$, where u and v are monomials of $K[\mathbf{x}]$ with $\deg u = \deg v$, such that $\pi(u) = \pi(v)$. The distinguished properties on $K[G]$ and I_G in which commutative algebraists are especially interested are as follows:

(i) I_G is generated by quadratic binomials[1];
(ii) $K[G]$ is Koszul;
(iii) I_G possesses a quadratic Gröbner basis, i.e., a Gröbner basis consisting of quadratic binomials.

The hierarchy (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i) is true. However, (i) $\Rightarrow$ (ii) is false. We refer the reader to [15] for the quick information together with basic literature on these

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[1]Even if $I_G = (0)$, we say that "I_G is generated by quadratic binomials" and "I_G possesses a quadratic Gröbner basis."

properties. A Koszul toric ring whose toric ideal possesses no quadratic Gröbner basis is given in [15, Example 2.2]. Moreover, consult, e.g., to [5, Chapter 2] for fundamental materials on Gröbner bases.

We study finite connected simple graphs G satisfying the following condition:

$(*)$ I_G is generated by quadratic binomials and possesses no quadratic Gröbner basis.

We say that a finite connected simple graphs G is $(*)$-*minimal* if G satisfies the condition $(*)$ and if no induced subgraph H $(\neq G)$ satisfies the condition $(*)$. A $(*)$-minimal graph is given in [15, Example 2.1].

In the present paper, after summarizing known results on I_G in Section 1, a nontrivial infinite series of $(*)$-minimal finite graphs is given in Section 2. In Section 3, we implement a combinatorial characterization for I_G to be generated by quadratic binomials ([15, Theorem 1.2]) and, by means of the computer search, we classify the finite graphs G satisfying the condition $(*)$, up to 8 vertices.

1. Known results on toric ideals of graphs

In this section, we introduce graph theoretical terminology and known results. Let G be a connected graph with the vertex set $V(G) = [n] = \{1, 2, \ldots, n\}$ and the edge set $E(G)$. We assume that G has no loops and no multiple edges. A walk of length q of G connecting $v_1 \in V(G)$ and $v_{q+1} \in V(G)$ is a finite sequence of the form

$$\Gamma = (\{v_1, v_2\}, \{v_2, v_3\}, \ldots, \{v_q, v_{q+1}\}) \tag{1}$$

with each $\{v_k, v_{k+1}\} \in E(G)$. An *even* (resp. *odd*) *walk* is a walk of even (resp. odd) length. A walk Γ of the form (1) is called *closed* if $v_{q+1} = v_1$. A *cycle* is a closed walk

$$C = (\{v_1, v_2\}, \{v_2, v_3\}, \ldots, \{v_q, v_1\}) \tag{2}$$

with $q \geq 3$ and $v_i \neq v_j$ for all $1 \leq i < j \leq q$. A *chord* of a cycle (2) is an edge $e \in E(G)$ of the form $e = \{v_i, v_j\}$ for some $1 \leq i < j \leq q$ with $e \notin E(C)$. If a cycle (2) is even, an *even-chord* (resp. *odd-chord*) of (2) is a chord $e = \{v_i, v_j\}$ with $1 \leq i < j \leq q$ such that $j - i$ is odd (resp. even). If $e = \{v_i, v_j\}$ and $e' = \{v_{i'}, v_{j'}\}$ are chords of a cycle (2) with $1 \leq i < j \leq q$ and $1 \leq i' < j' \leq q$, then we say that e and e' *cross* in C if the following conditions are satisfied:

(i) Either $i < i' < j < j'$ or $i' < i < j' < j$;
(ii) Either $\{\{v_i, v_{i'}\}, \{v_j, v_{j'}\}\} \subset E(C)$ or $\{\{v_i, v_{j'}\}, \{v_j, v_{i'}\}\} \subset E(C)$.

A *minimal* cycle of G is a cycle having no chords. If C_1 and C_2 are cycles of G having no common vertices, then a *bridge* between C_1 and C_2 is an edge $\{i, j\}$ of G with $i \in V(C_1)$ and $j \in V(C_2)$.

The toric ideal I_G is generated by the binomials associated with even closed walks. Given an even closed walk $\Gamma = (e_{i_1}, e_{i_2}, \ldots, e_{i_{2q}})$ of G, we write f_Γ for the binomial

$$f_\Gamma = \prod_{k=1}^{q} x_{i_{2k-1}} - \prod_{k=1}^{q} x_{i_{2k}} \in I_G.$$

It is known ([19, Proposition 3.1], [17, Chapter 9] and [15, Lemma 1.1]) that

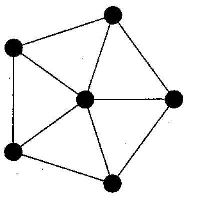

FIGURE 1. Wheel with 6 vertices.

Proposition 1.1. *Let G be a connected graph. Then, I_G is generated by all the binomials f_Γ, where Γ is an even closed walk of G. In particular, $I_G = (0)$ if and only if G has at most one cycle and the cycle is odd.*

Note that, for a binomial $f \in I_G$, $\deg(f) = 2$ if and only if there exists an even cycle C of G of length 4 such that $f = f_C$. On the other hand, a criterion for the existence of a quadratic binomial generators of I_G is given in [15, Theorem 1.2].

Proposition 1.2. *Let G be a finite connected graph. Then, I_G is generated by quadratic binomials if and only if the following conditions are satisfied:*

(i) *If C is an even cycle of G of length ≥ 6, then either C has an even-chord or C has three odd-chords e, e' and e'' such that e and e' cross in C;*
(ii) *If C_1 and C_2 are minimal odd cycles having exactly one common vertex, then there exists an edge $\{i,j\} \notin E(C_1) \cup E(C_2)$ with $i \in V(C_1)$ and $j \in V(C_2)$;*
(iii) *If C_1 and C_2 are minimal odd cycles having no common vertex, then there exist at least two bridges between C_1 and C_2.*

If G is bipartite, then the following is shown in [14]:

Proposition 1.3. *Let G be a bipartite graph. Then the following conditions are equivalent:*

(i) *Every cycle of G of length ≥ 6 has a chord;*
(ii) *I_G possesses a quadratic Gröbner basis;*
(iii) *$K[G]$ is Koszul;*
(iv) *I_G is generated by quadratic binomials.*

If G is not bipartite, then the conditions (iii) and (iv) are not equivalent.

Example 1.4. ([15, Example 2.1]) Let G be the graph in Figure 1. Then, I_G is generated by quadratic binomials. On the other hand, $K[G]$ is not Koszul and hence I_G has no quadratic Gröbner bases.

If a graph G' on the vertex set $V(G') \subset V(G)$ satisfies $E(G') = \{\{i,j\} \in E(G) \mid i,j \in V(G')\}$, then G' is called an *induced subgraph* of G. The following proposition is a fundamental and important fact on the toric ideals of graphs.

Proposition 1.5 ([13]). *Let G' be an induced subgraph of a graph G. Then, $K[G']$ is a combinatorial pure subring of $K[G]$. In particular,*

(i) *If I_G possesses a quadratic Gröbner basis, then so does $I_{G'}$.*

(ii) *If $K[G]$ is Koszul, then so is $K[G']$;*
(iii) *If I_G is generated by quadratic binomials, then so is $I_{G'}$.*

2. Toric ideals of the suspension of graphs

In this section, we study the existence of quadratic Gröbner bases of toric ideals of the suspension of graphs. Let G be a graph with the vertex set $V(G) = [n] = \{1, 2, \ldots, n\}$ and the edge set $E(G)$. The suspension of the graph G is the new graph $\widehat{G}$ whose vertex set is $[n+1] = V(G) \cup \{n+1\}$ and whose edge set is $E(G) \cup \{\{i, n+1\} \mid i \in V(G)\}$. Note that, any graph G is an induced subgraph of its suspension $\widehat{G}$. We now characterize graphs G such that $I_{\widehat{G}}$ is generated by quadratic binomials. The *complementary* graph $\overline{G}$ of G is the graph whose vertex set is $[n]$ and whose edges are the non-edges of G. A graph G is said to be *chordal* if any cycle of length > 3 has a chord. Moreover, a graph G is said to be *co-chordal* if $\overline{G}$ is chordal. A graph G is called a *$2K_2$-free graph* if it is connected and does not contain two independent edges as an induced subgraph. For a connected graph G,

- G is $2K_2$-free $\Leftrightarrow$ any cycle of $\overline{G}$ of length 4 has a chord in $\overline{G}$;
- G is co-chordal $\Rightarrow$ G is $2K_2$-free,

hold in general. Moreover, it is known (e.g., [1]) that

Lemma 2.1. *Let G be a connected graph. Then,*

(i) *If G is co-chordal, then any cycle of G of length ≥ 5 has a chord;*
(ii) *If G is $2K_2$-free, then any cycle of G of length ≥ 6 has a chord.*

The toric ideals I_G of $2K_2$-free graphs G are studied in [16]. (In [16], $2K_2$-free graphs are called in a different name.) On the other hand, the edge ideals $I(G)$ of $2K_2$-free graphs G are studied by many researchers. See, e.g., [10] and [11] together with their references and comments. (In these papers, $2K_2$-free graphs are called "*C_4-free graphs.*") One can characterize the toric ideals $I_{\widehat{G}}$ of $\widehat{G}$ that are generated by quadratic binomials in terms of $2K_2$-free graphs.

Theorem 2.2. *Let G be a finite connected graph. Then the following conditions are equivalent:*

(i) *$I_{\widehat{G}}$ is generated by quadratic binomials;*
(ii) *G is $2K_2$-free and I_G is generated by quadratic binomials;*
(iii) *G is $2K_2$-free and satisfies the condition (i) in Proposition 1.2.*

Example 2.3. In general, there is no implication between the two conditions (1) I_G is generated by quadratic binomials and (2) G is $2K_2$-free. In fact,

(a) Let G be the graph in Figure 2. Then, I_G is not generated by quadratic binomials. On the other hand, G is co-chordal (and hence $2K_2$-free).
(b) If G is a bipartite graph consisting of a cycle C of length 6 and a chord of C, then I_G is generated by two quadratic binomials. On the other hand, G is not $2K_2$-free.

Thus, both (1) $\Rightarrow$ (2) and (2) $\Rightarrow$ (1) are false.

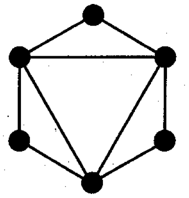

FIGURE 2. An even cycle with three odd chords.

By using the theory of the Rees ring of edge ideals (see, e.g., [5]), we have a necessary condition for $I_{\widehat{G}}$ to possess a quadratic Gröbner basis.

Proposition 2.4. *Let G be a connected graph. If $I_{\widehat{G}}$ possesses a quadratic Gröbner basis, then G is co-chordal.*

The converse of Proposition 2.4 is false in general. See, e.g., Example 2.9. However, if G is bipartite, then these conditions are equivalent:

Theorem 2.5. *Let G be a bipartite graph. Then the following conditions are equivalent:*

(i) *$I_{\widehat{G}}$ is generated by quadratic binomials;*
(ii) *$K[\widehat{G}]$ is Koszul;*
(iii) *$I_{\widehat{G}}$ possesses a quadratic Gröbner basis;*
(iv) *G is $2K_2$-free;*
(v) *G is co-chordal.*

Remark 2.6. Bipartite graphs satisfying one of the conditions in Theorem 2.5 are called *Ferrers graphs* (by relabeling the vertices). The edge ideal $I(G)$ of a Ferrers graph G is well-studied. See, e.g., [2] and [3].

If G is not bipartite, then the conditions (i) and (ii) in Theorem 2.5 are not equivalent. In fact,

Example 2.7. Let G be a cycle of length 5. Then $\overline{G}$ is also a cycle of length 5. Hence G is not co-chordal but $2K_2$-free. By Theorem 2.2 and Proposition 2.4, $I_{\widehat{G}}$ is generated by quadratic binomials and has no quadratic Gröbner bases. Note that $\widehat{G}$ is the graph in Example 1.4 and that $K[\widehat{G}]$ is not Koszul.

Recall that a finite connected simple graph G is called $(*)$-*minimal* if G satisfies

$(*)$ I_G is generated by quadratic binomials and possesses no quadratic Gröbner basis

and if no induced subgraph H $(\neq G)$ satisfies the condition $(*)$. The suspension graph $\widehat{G}$ given in Example 2.7 is a $(*)$-minimal graph. We generalize this example and give a nontrivial infinite series of $(*)$-minimal graphs:

Theorem 2.8. *Let G be the graph on the vertex set $[n]$ whose complement is a cycle of length n. If $n \geq 5$, then $\widehat{G}$ is $(*)$-minimal, i.e., $\widehat{G}$ satisfies the following:*

(i) $I_{\widehat{G}}$ *is generated by quadratic binomials;*
(ii) $I_{\widehat{G}}$ *has no quadratic Gröbner basis;*
(iii) *For any induced subgraph* H $(\neq \widehat{G})$ *of* $\widehat{G}$*, the toric ideal* I_H *of* H *possesses a quadratic Gröbner basis.*

Even if G is co-chordal, $\widehat{G}$ may be $(*)$-minimal:

Example 2.9. Let G be the graph whose complement is the chordal graph in Figure 2. Then, $I_{\widehat{G}}$ is generated by quadratic binomials since G is co-chordal (and hence $2K_2$-free) and $I_G = (0)$. On the other hand, computational experiments in Section 3 show that $\widehat{G}$ is $(*)$-minimal.

3. Computational experiments

In this section, we enumerate all finite connected simple graphs G satisfying the condition $(*)$ up to 8 vertices by utilizing various software. Proposition 1.2 gives an algorithm to determine if a toric ideal I_G is generated by quadratic binomials. Since the criteria in Proposition 1.2 are characterized by cycles of G, we need to enumerate all even cycles and minimal odd cycles of G in order to implement the algorithm. We implement the algorithm by utilizing CyPath [18] which is a cycles and paths enumeration program implemented by T. Uno. The algorithm is used at step (2) of the following procedure to search for the graphs satisfying $(*)$.

(1) (generating step) We use nauty [9] as a generator of all connected simple graphs with n vertices up to isomorphism.
(2) (criterion step) The criteria in Proposition 1.2 detect graphs G whose toric ideals I_G are generated by quadratic binomials. These are candidates for satisfying the condition $(*)$.
(3) (exclusion step) For each candidate G, we iterate the following computation.
    (a) A new weight vector w is chosen randomly on each iteration.
    (b) We compute a Gröbner basis of the toric ideal I_G with respect to the chosen weight vector w with Risa/Asir [12].
    (c) If the Gröbner basis is quadratic then G is excluded from candidates.
(4) (final check step) We check the Koszul property of $K[G]$ with Macaulay2 [4]. If it is not Koszul then I_G possesses no quadratic Gröbner basis. If it is indeterminable then we compute all Gröbner bases by using TiGERS [7] or CaTS [8].

In our experimentation, we take 10000 to be the number of iterations at step (3) in the case of 8 vertices. Then, there are 214 graphs as remaining candidates and we can check that 213 graphs of these are not Koszul with Macaulay2. The last one is indeterminable by computational methods in our environment. However, Theorem 2.8 tells us that it has no quadratic Gröbner basis, because it is the suspension of the complement graph of a cycle whose length is 7. Therefore, we complete classification of the finite graphs with 8 vertices. Table 1 shows numbers of (1) the connected simple graphs, (2) the graphs whose toric ideals I_G are generated by quadratic binomials (include number of zero ideals), (4) the graphs satisfying $(*)$ (include number of the graphs which have degree 1 vertices) respectively. We list

vertices	(1)	(2)		(4)	
3	2	2	(2)	0	
4	6	6	(3)	0	
5	21	20	(7)	0	
6	112	95	(14)	1	(0)
7	853	568	(34)	14	(2)
8	11117	4578	(78)	214	(51)

TABLE 1

the 14 graphs (Figures 3–16) satisfying $(*)$ with 7 vertices. Figure 15 belongs to the infinite series in Theorem 2.8 and Figure 5 is the $(*)$-minimal graph in Example 2.9.

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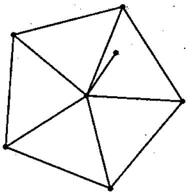

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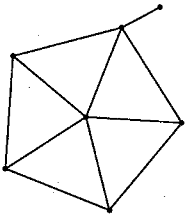

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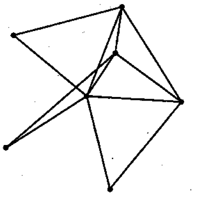

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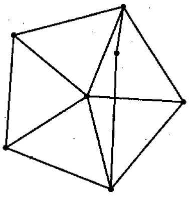

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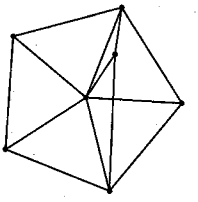

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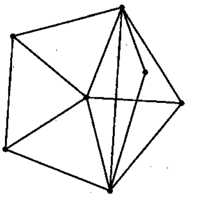

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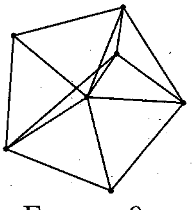

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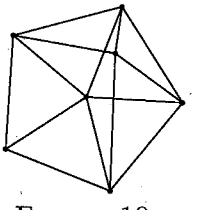

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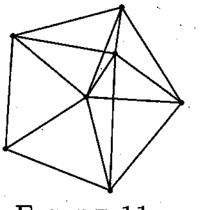

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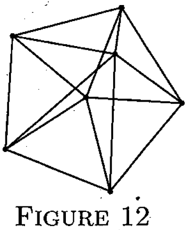

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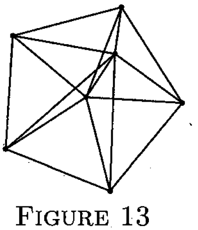

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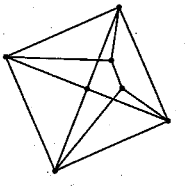

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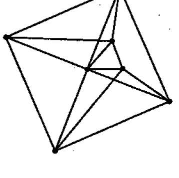

FIGURE 15

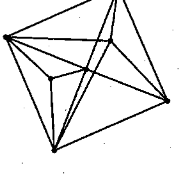

FIGURE 16