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Distribution modulo one of certain sequences

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This is a summary of our talk at the workshop in RIMS. After that time some results are refined.

For $x\in\mathbb{R}$ let $\lfloor x\rfloor$ denote the integral part of x; let $\{x\}=x-\lfloor x\rfloor$ be the residue of x modulo 1. Let $\chi_{[\alpha,\beta)}(x)$ be the characteristic function of the interval $[\alpha, \beta)\subset[0,1)$, that is, $\chi_{[\alpha,\beta)}(x)=1$ if $x\in[\alpha,\beta)$; $\chi_{[\alpha,\beta)}(x)=0$ otherwise.

Let $b\geq 2$ be an integer considered as a base for the development of a real number $x>0$ and $M_{b}(x)$ be the mantissa of x defined by $x=M_{b}(x)\times b^{n(x)}$ such that $1\leq M_{b}(x)<b$ holds, where $n(x)$ is a uniquely determined integer. Let $K=k_{1}k_{2}\cdots k_{r}$ be a positive integer expressed in the base b , that is

$$K=k_{1}b^{r-1}+k_{2}b^{r-2}+\cdots+k_{r-1}b+k_{r},$$

where $k_{1}\neq 0$ and at the same time $K=k_{1}k_{2}\cdots k_{r}$ is considered as an r-consecutive block of digits in the base b. Note that for x of the type $x=0.00\cdots 0k_{1}k_{2}\cdots k_{r}\cdots$, $k_{1}>0$, we have $M_{b}(x)=k_{1}.k_{2}\cdots k_{r}\cdots$ and the first zero digits is omitted. Thus arbitrary $x>0$ has the first r-digits, starting a non-zero digit, equal to $k_{1}k_{2}\cdots k_{r}$ if and only if

$$k_{1}.k_{2}\cdots k_{r}\leq M_{b}(x)<k_{1}.k_{2}\cdots(k_{r}+1). \tag{1}$$

Since $\log_{b}M_{b}(x)=\log_{b}x \bmod 1$ the inequality (1) is equivalent to

$$\log_{b}\left(\frac{K}{b^{r-1}}\right)\leq\log_{b}x \bmod 1<\log_{b}\left(\frac{K+1}{b^{r-1}}\right).$$

Definition 1 (P. Diaconis [1]). A sequence x_{n}, $n=1,2,\ldots$, of positive real numbers satisfies *Benford's law* (abbreviated to B.L.) in base b, if for every $r=1,2,\ldots$ and every r-digits integer $K=k_{1}k_{2}\cdots k_{r}$ we have

$$\lim_{N\to\infty}\frac{\#\{n\leq N;\ \text{first } r \text{ digits (starting a non-zero digit) of } x_{n}=K\}}{N}$$
$$=\log_{b}\left(\frac{K+1}{b^{r-1}}\right)-\log_{b}\left(\frac{K}{b^{r-1}}\right).$$

It is well known that:

Theorem 1 (P. Diaconis [1]). *A sequence x_{n}, $n=1,2,\ldots$, of positive real numbers satisfies B.L. in base b if and only if the sequence $\log_{b}x_{n}$ mod 1 is uniformly distributed (abbreviating u.d.) in $[0,1)$.*

Definition 2. A function $g : [0,1] \to [0,1]$ will be called distribution function if the following two conditions are satisfied

(i) $g(0) = 0$, $g(1) = 1$

(ii) g is non-decreasing.

Definition 3. Let x_n, $n = 1, 2, \dots$, be a sequence of real numbers and define the step distribution function of x_n mod 1

$$F_N(x) = \frac{1}{N} \sum_{n=1}^{N} \chi_{[0,x)}(\{x_n\})$$

for $x \in [0,1]$. The limit $g(x)$ of a subsequence $F_{N_k}(x)$ of $F_N(x)$

$$\lim_{k \to \infty} F_{N_k}(x) = g(x) \tag{2}$$

for every $x \in [0,1]$, is called a distribution function of x_n, where $N_1 < N_2 < \dots$ is related sequence of indices. Let $G(x_n \bmod 1)$ be the set of all possible limits (2).

Definition 4 (see [3]). Let $x_n, n = 1, 2, \dots$ be a sequence of real numbers and let $g(x)$ be distribution function. Then the discrepancies of $x_n \bmod 1$ with respect to $g(x)$ are defined by

$$D_N^*(x_n \bmod 1, g) = \sup_{0 \le x \le 1} |F_N(x) - g(x)|.$$

$$D_N(x_n \bmod 1, g) = \sup_{0 \le x < y \le 1} |(F_N(y) - F_N(x)) - (g(y) - g(x))|.$$

Definition 5. Let $u_n, n = 1, 2, \dots$ be a positive sequence and let $g(x)$ be a distribution function. Let $K = k_1 \cdots k_r = k_1 b^{r-1} + k_2 b^{r-2} + \cdots + k_{r-1} b + k_r$.

$$B_N(u_n, g) = \sup_{\substack{r \ge 1 \\ b^{r-1} \le K < b^r \\ (r, K \in \mathbb{Z})}} \left| \frac{\#\{1 \le n \le N : (\text{first } r \text{ digits starting a non-zero digit of } x_n) = K\}}{N} - \left(g\left(\log_b \frac{K+1}{b^{r-1}}\right) - g\left(\log_b \frac{K}{b^{r-1}}\right)\right) \right|.$$

From the definition, it follows that $B_N(u_n, g) = D_N(\log_b u_n \bmod 1, g)$.

We have the following quantitative results on log-like sequences.

Theorem 2. *Let the real-valued function $f(t)$ be strictly increasing for $t \ge 1$. Assume that*

(i) $\lim_{t \to \infty} f(t) = \infty$,

(ii) $\psi(x) := \lim_{t \to \infty} \frac{f^{-1}(t+x)}{f^{-1}(t)}$ *for* $x \in [0,1]$.

Then

(a) *there exists* $\rho > 0$ *with* $\psi(x) = e^{\rho x}$,

(b) *it holds that*

$$\sup_{x \in [0,1]} \left| \frac{f^{-1}(k+x)}{f^{-1}(k)} - e^{\rho x} \right| \to 0 \quad (k \to \infty).$$

In addition, let $w \in [0,1]$, *let*

$$g_w(x) = \frac{1}{e^{\rho w}}\frac{e^{\rho x}-1}{e^{\rho}-1} + \frac{\min(e^{\rho x}, e^{\rho w})-1}{e^{\rho w}},$$

for $0 \le x \le 1$, *and let* $K_N = [f(N)]$, $w_N = \{f(N)\}$, *then*
(c) *it holds that*

$$D_N^*(f(n) \bmod 1, g_w) \le \frac{2}{N}\sum_{k=0}^{K_N-1} f^{-1}(k) \sup_{x\in[0,1]}\left|\frac{f^{-1}(k+x)}{f^{-1}(k)} - e^{\rho x}\right| +$$
$$+ (e^{\rho}+1)\sup_{x\in[0,1]}\left|\frac{f^{-1}(K_N+x)}{f^{-1}(K_N)} - e^{\rho x}\right| + (e^{\rho}+1)|e^{\rho w} - e^{\rho w_N}| + \frac{f(N)}{N} + \frac{2f^{-1}(0)}{N}.$$

Furthermore, assume that
(iii) $\lim_{t\to\infty} f'(t) = 0$
and set $N_i = \lfloor f^{-1}(i+w)\rfloor$ *for* $0 < w \le 1$, $N_i = \lceil f^{-1}(i)\rceil$ *for* $w = 0$, $w_{N_i} = \{f(N_i)\}$ *for* $i = 1, 2, \ldots$. *Then we have* $\lim_{i\to\infty} w_{N_i} = w$ *and*

$$\lim_{i\to\infty} D_{N_i}^*(f(n) \bmod 1, g_w) = 0. \tag{3}$$

Corollary 1. *For* $b \ge 2$ *be a positive integer and* $r > 0$, *let* $f(x) = \log_b x^r$,

$$g_w(x) = \frac{1}{b^{\frac{w}{r}}}\frac{b^{\frac{x}{r}}-1}{b^{\frac{1}{r}}-1} + \frac{\min(b^{\frac{x}{r}}, b^{\frac{w}{r}})-1}{b^{\frac{w}{r}}} \quad (w \in [0,1]).$$

Then we have $\lim_{i\to\infty}\{f(N_i)\} = 0$ *and*

$$D_{N_i}^*(f(n) \bmod 1, g_w) \le \frac{b^{\frac{1}{r}}(b^{\frac{1}{r}}+1)}{N_i} + \frac{2}{N_i} + \frac{r\log_b N_i}{N_i},$$

where $N_i = \lfloor b^{\frac{i+w}{r}}\rfloor$ *for* $0 < w \le 1$, $i = 1, 2, \ldots$.
Furthermore, if r *is positive integer, then* $\{f(N_{r^2-1})\} = \{r\log_b b^r\} = 0$ *for* $N_{r^2-1} = b^r$ *and*

$$D_{b^r}^*(\log_b n^r \bmod 1, g_0) = D_{b^r}^*(\log_b n^r \bmod 1, g_1) = O\left(\frac{r^2}{b^r}\right)$$

and

$$D_{b^r}^*(\log_b n^r \bmod 1) = O\left(\frac{1}{r}\right).$$

Remark 1. S. Eliahou-B. Massé-D. Schneider [2, Theorem 1] proved

$$D_{\phi(r)}^*(\log_{10} n^r \bmod 1) = O(r^{-1}), \tag{4}$$

where $\phi(r) = \lfloor e^r \rfloor$ by a different method (see [2]).

The sequence of all primes p_n do not satisfy B. L., i.e. the sequence $\log_b p_n$ is not u.d. mod 1, but $G(\log_b p_n \bmod 1) = G(\log_b n \bmod 1)$. In the following, we have quantitative results for the sequence $\log_b p_n$.

Theorem 3. *Let the real-valued function $f(t)$ be strictly increasing for $t \geq 1$ and let*
$B(x) := \frac{f^{-1}(x)}{\log f^{-1}(x)-1}$.
Assume that
(i) $\lim_{t\to\infty} f(t) = \infty$,
(ii) $\psi(x) := \lim_{t\to\infty} \frac{f^{-1}(t+x)}{f^{-1}(t)}$ *for* $x \in [0,1]$.
Then
(a) *there exists* $\rho > 0$ *such that*

$$\psi(x) = e^{\rho x},$$

(b) *it holds that*

$$\sup_{x\in[0,1]} \left| \frac{B(k+x)}{B(k)} - e^{\rho x} \right| \to 0 \quad (k \to \infty).$$

In addition, let $u \in [0,1]$, *let*

$$g_u(x) = \frac{1}{e^{\rho u}} \frac{e^{\rho x} - 1}{e^{\rho} - 1} + \frac{\min(e^{\rho x}, e^{\rho u}) - 1}{e^{\rho u}} \tag{5}$$

for $0 \leq x \leq 1$, *let* $\mathcal{K}_N = \lfloor f(p_N) \rfloor$, *let* $u_N = \{f(p_N)\}$, *and let* M *be an arbitrary positive integer with* $M \geq f(e^3)$. *Then*
(c) *it holds that for sufficiently large* N

$$\begin{aligned}
&D_N^*(f(p_n) \bmod 1, g_u) \leq \\
\leq &\frac{2}{N} \sum_{k=M}^{\mathcal{K}_N - 1} B(k) \sup_{x\in[0,1]} \left| \frac{B(k+x)}{B(k)} - e^{\rho x} \right| + 2(e^{\rho}+1) \sup_{x\in[0,1]} \left| \frac{B(\mathcal{K}_N + x)}{B(\mathcal{K}_N)} - e^{\rho x} \right| + \\
&+ 2e^{\rho}|e^{\rho u} - e^{\rho u_N}| + 2\frac{B(M)}{N} + O\left(\frac{1}{(\log f^{-1}(\mathcal{K}_N))^2}\right) + O\left(\frac{\log f^{-1}(\mathcal{K}_N)}{f^{-1}(\mathcal{K}_N)}\right) + \\
&+ O\left(\frac{1}{N} \sum_{k=M}^{\mathcal{K}_N+1} \frac{f^{-1}(k)}{(\log f^{-1}(k))^3}\right) + O\left(\frac{f^{-1}(0)}{N}\right) + O\left(\frac{f^{-1}(M)}{(\log f^{-1}(M))^3} \frac{1}{N}\right).
\end{aligned}$$

Corollary 2. *Let $f(x)$ be as in Theorem 3. In addition to the assumptions* (i)-(ii), assume that
(iii) $f'(x)$ is non-increasing and $f'(x) = O(x^{-1})$.
For $0 < u \leq 1$ let $N_i = \pi(f^{-1}(i+u))$. Then

$$\lim_{i\to\infty} \{f(p_{N_i})\} = u$$

and

$$\lim_{i\to\infty} D_{N_i}^*(f(p_n) \bmod 1, g_u) = 0,$$

where $g_u(x)$ is defined in (5).

Corollary 3. *Let* $\alpha > 0$, *let* $0 < u \leq 1$, *let* $N_i = \pi(e^{\frac{i+u}{\alpha}})$ *for* $i = 1, 2, \ldots$, *and let* $g_u(x)$ *be defined in* (5).
(I) *If* α *is a constant, then for sufficiently large* i

$$D_{N_i}^*(\alpha \log p_n \bmod 1, g_u) = O\left(\frac{1}{\log N_i}\right).$$

(II) *If α is a variable, then for sufficiently large i and α*

$$D^*_{N_i}(\alpha \log p_n \bmod 1, g_u) \ll \frac{1}{\log N_i} + \frac{\alpha}{(\log N_i)^2}. \tag{6}$$

Corollary 4. *Let $b \geq 2$ and r be positive integers, and let*

$$g_0(x) = \frac{b^{x/r} - 1}{b^{1/r} - 1} \quad (0 \leq x \leq 1).$$

Then for sufficiently large r

$$D^*_{\pi(b^r)}(\log_b p_n^r \bmod 1, g_0) = O\left(\frac{1}{r}\right),$$

$$D^*_{\pi(b^r)}(\log_b p_n^r \bmod 1) = O\left(\frac{1}{r}\right).$$

References

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