数理解析研究所講究録
第2013巻 2016年 68-80

Recent developments in the theory of Stirling numbers

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February 12, 2015

Abstract

The theory of Stirling numbers is constantly developing. In this survey we will focus on some recent results connected to the research of the present author and we mention some interesting open problems.

1 Introduction

The theory of Stirling numbers is dated back to James Stirling (1692-1770). For historical details the reader may refer to [17]. The (unsigned) Stirling numbers of the first kind $s(n,k)$ gives the number of permutations on n elements with k cycles. The $S(n,k)$ Stirling numbers of the second kind count partitions of an n element set with k blocks. Then, as it is obvious,

$$\sum_{k=1}^{n}s(n,k)=n! \quad (n\geq 1),$$

and

$$\sum_{k=1}^{n}S(n,k)=B_{n} \quad (n\geq 1).$$

B_{n} is the nth Bell number, which is the total number of the possible partitions of an n element set. For more details on these numbers see [1, 6, 11, 17, 41].

There are many generalizations of these numbers and they have many connections to other combinatorial number sequences. Here we study some of these generalizations.

2 The r-Whitney numbers and Bernoulli polynomials

The r-Whitney numbers mentioned in the section title are two parameter generalizations of the Stirling numbers. First we describe the two particular cases belonging to the individual parameters.

2.1 The r-Stirling numbers

There are two generalizations of Stirling numbers which can be unified easily. One generalization is the notion of r-Stirling numbers.

The r-Stirling numbers $S_{r}(n,k)$ and (the unsigned) $s_{r}(n,k)$ count similar combinatorial configurations as the usual Stirlings do but here there is an additional restriction: the first r elements are restricted to be in different blocks/cycles. Note that for the r-Stirlings $S_{r}(n,k)$ and $s_{r}(n,k)$ there is a shift in the indices: $S_{r}(n,k)$ gives the number of restricted partitions of $n+r$ elements into $k+r$ blocks, and the same for $s_{r}(n,k)$, mutatis mutandis.

For example, if $r=2$ (and always when $r>1$), then

$$(\underline{12}5)(347)(6)$$

*The research of István Mező was supported by the Scientific Research Foundation of Nanjing University of Information Science & Technology, and The Startup Foundation for Introducing Talent of NUIST. Project no.: S8113062001

is a tilted permutation, because 1 and 2 share the same cycle.

It is obvious that if $r = 0$, we get back the classical Stirling numbers. The r-Stirlings (under different names) are known since a century [9], but they became more popular with the paper of Broder [7]. A number of properties of these numbers and other references can be found in a recent survey of Villamizar [41].

2.2 The Whitney numbers

The other generalization we referred to before Section 2.1 is the class of Whitney numbers. These originally were defined by Dowling [16] in a lattice theoretical way but it turned out that they have simple partition theoretical interpretation, too [27].

The first kind Whitney numbers are denoted by $w_m(n,k)$ and the second kind numbers re denoted by $W_m(n,k)$. $W_m(n,k)$ is the number of partitions on n elements with k blocks such that the elements of the blocks are coloured with m colors, but the first element in the blocks is not coloured [31]. (The first kind Whitney numbers have a similar interpretation.)

A typical partition counted by the Whitney numbers with $m = 2$ is like

$$\{1,3,7\} \cup \{2,4\} \cup \{5,\overset{r}{9}\} \cup \{6,\overset{g}{8},\overset{r}{10}\}.$$

Here and later the letters r, g and b stand for red, green and blue, respectively.

2.3 The r-Whitney numbers and their connections to Bernoulli polynomials and power sums

2.3.1 Combinatorial definition of the r-Whitney numbers

The r-Stirling numbers (with the additional parameter r) and Whitney numbers (with parameter m) can be unified. These four-parameter sequences are called r-Whitney numbers of the first- and second kind. Their combinatorial interpretation was given by Mihoubi and Rahmani and before by Corcino et al [12] as follows: $W_{m,r}(n,k)$ is the number of partitions on $n+r$ elements with $k+r$ blocks such that

- the first r elements (which we call distinguished) are in different blocks (as dictated by the r-Stirling part)
- the elements of the blocks not containing distinguished elements are coloured with m colors, but the first element in the blocks is not coloured (as dictated by the Whitney part).

A typical partition counted by the 2-Whitney numbers with $m = 2$ is like

$$\{\underline{1},3,7\} \cup \{\underline{2},4\} \cup \{5,\overset{r}{9}\} \cup \{6,\overset{g}{8},\overset{r}{10}\}.$$

(The first kind r-Whitney numbers have a similar interpretation.)

2.3.2 A formula with the Bernoulli polynomials

The present author started to study these numbers [26], because they can be connected to the famous Bernoulli polynomials $\mathcal{B}_n(x)$ [17]. This connection reads as

$$\binom{n+1}{l}\mathcal{B}_{n-l+1}\left(\frac{r}{m}\right) = \frac{n+1}{m^n}\sum_{k=0}^{n} m^k W_{m,r}(n,k)\frac{S^1(k+1,l)}{k+1}$$

for any $r, n, l \geq 0$ and $m > 0$ integers. This is the polynomial generalization of the classical formula

$$\binom{n+1}{l}\mathcal{B}_{n-l+1} = (n+1)\sum_{k=0}^{n} S^2(n,k)\frac{S^1(k+1,l)}{k+1}$$

between the Bernoulli numbers $\mathcal{B}_n = \mathcal{B}_n(1)$ and Stirling numbers of both kinds.

A different study appeared recently [35] in which more relations are given between the Bernoulli polynomials and r-Whitney numbers.

2.3.3 Relation to power sums

It is well known that the power sums

$$S_n(\ell) = 1^n + 2^n + \cdots + (\ell - 1)^n$$

are polynomials of ℓ. This is Faulhaber's theorem [17]. In modern form

$$S_n(\ell) = \frac{1}{n+1}(\mathcal{B}_{n+1}(\ell) - \mathcal{B}_{n+1}).$$

These polynomials have coefficients expressible directly by the Stirling numbers:

$$S_n(\ell) = \sum_{i=0}^{n+1} \ell^i \left(\sum_{k=0}^{n} S_2(n,k) S_1(k+1,i) \frac{1}{k+1} \right).$$

If, in place of the first natural numbers, we take the power sum of the members of an arithmetic progression, then we have an interesting generalization of the above results. More precisely, let

$$S^n_{m,r}(\ell) = r^n + (m+r)^n + (2m+r)^n + \cdots + ((\ell-1)m+r)^n$$

be a generalized power sum of terms of an arithmetic progression where $m \neq 0, r$ are coprime integers. Then $S^n_{m,r}(\ell)$ is still a polynomial expressible by the Bernoulli polynomials:

$$S^n_{m,r}(\ell) = \frac{m^n}{n+1}\left(\mathcal{B}_{n+1}\left(\ell + \frac{r}{m}\right) - \mathcal{B}_{n+1}\left(\frac{r}{m}\right)\right).$$

This is a nice result of Bazsó et al. [2]. What are the explicit coefficients of these polynomials? This question was answered by Bazsó and Mező [3]: for all parameters $\ell > 1, n, m > 0, r \geq 0$ we have that, as a polynomial in ℓ, $S^n_{m,r}(\ell)$ has the following coefficients:

$$S^n_{m,r}(\ell) = \sum_{i=0}^{n+1} \ell^i \left(\sum_{k=0}^{n} \frac{m^k W_{m,r}(n,k)}{k+1} S_1(k+1,i) \right).$$

These examples show that the r-Whitney numbers are useful generalizations of the classical Stirling numbers.

We mention that the r-Whitney numbers were known before the work of Mező under the name of (r,β)-Stirling numbers [12] and as special cases of the Stirling number pairs [20]. They also appeared in a work of Rucinski and Voigt [37].

3 Some open problems

3.1 The maximizing indices

Having known the Stirling, r-Stirling, Whitney and r-Whitney numbers, we can turn to some interesting open problems. These problems are related to the so-called log-concavity property and the zeros of some specific polynomials connected to these numbers.

It is a classical result that the Stirling numbers of both kind form log-concave (LC) sequences, that is, there is some index K_n such that

$$S(n,1) < S(n,2) < \cdots < S(n,K_n) \leq S(n,K_n+1) > S(n,K_n+2) > \cdots > S(n,n),$$

and the same for $s(n,k)$ with some k_n. It was proven by P. Erdős in 1953 that the k_n maximum is *unique* for the first kind Stirlings $s(n,k)$ for any $n > 1$, but nobody knows whether the same is true for $S(n,k)$ and K_n. Wegner conjectured [42] that the maximizing index of K_n is unique (see also [8, 23] for two recent discussions about this problem). It is also known [15, 32] that

$$K_{n+1} \in \{K_n, K_n + 1\}. \qquad (1)$$

There exists bounds for the maximizing indices k_n and K_n for both kind of Stirling numbers (see [28] and [8, p. 3.] for references, and [40] for a nice estimation for K_n via probability theory).

What about these problems with respect to the r-Stirling and Whitney numbers? It is known that r-Whitney numbers of both kinds are SLC. In the first kind case this follows immediately from the identity [26]

$$m^n x(x+1)(x+2)\cdots(x+n-1) = \sum_{k=0}^{n} w_{m,r}(n,k)(mx+r)^k \tag{2}$$

via a classical result of Newton [6, p. 22.] (note that the $w_{m,r}(n,k)$ numbers are *signed* numbers, so we are talking about the log-concavity of the *absolute values* of $w_{m,r}(n,k)$). The second kind case is a result of Cheon and Jung [10, Theorem 5.3] (the particular case of Whitney numbers with $r = 0$ was proven by Benoumhani [4] and the r-Stirling case was proven by Mező [28]).

Corcino and Corcino found bounds for the maximizing index of the r-Whitney numbers of the second kind, namely, for the maximizing index $K_{n,r,m}$ of $W_{m,r}(n,k)$ we have that

$$\frac{n}{m\log n} - \frac{r}{m} < K_{n,m,r} < \frac{n}{\log n - \log\log n} \tag{3}$$

if n is sufficiently large [13].

Up to our knowledge, similar result for the first kind case r-Whitney numbers is not known. For the r-Stirlings the present author has an estimation [28] and for the Whitney numbers Benoumhani [4].

For the r-Stirling numbers it is also known [28] that for the $K_{n,r}$ maximizing index

$$K_{n+1,r} \in \{K_{n,r}, K_{n+1,r}\}.$$

It was not studied whether this property holds when we take the Whitney and r-Whitney numbers.

Open problem. Can we generalize the property

$$K_{n+1} \in \{K_n, K_{n+1}\}$$

to the Whitney- and r-Whitney numbers?

3.2 Some related polynomials

3.2.1 The real zero property

It is very common to prove the LC property of a positive real sequence $(a_k)_{k=1}^n$ via the real zero property of the attached polynomial

$$p(x) = \sum_{k=1}^{n} a_k x^k.$$

If this polynomial has only real zeros, then the coefficients satisfy the inequality

$$a_k^2 \geq a_{k-1}a_{k+1} \quad (k \geq 2)$$

(and an even stronger inequality) from which it follows that for some index k^*

$$a_1 < a_2 < \cdots < a_{k^*} \leq a_{k^*+1} > a_{k^*+2} > \cdots > a_n.$$

More about this can be find in [6, p. 22.] or the standard reference [38].

As we mentioned in the previous subsection, in the most general situation of the first kind r-Whitney numbers the attached polynomials have only real zeros, as it is trivially comes from (2). The second kind case is harder to handle. It was known by Harper [19] that the

$$B_n(x) = \sum_{k=1}^{n} S(n,k)x^k$$

Bell polynomials have only real zeros. Taking the r-Stirling numbers, the

$$B_{n,r}(x) = \sum_{k=1}^{n} S_r(n,k)x^k$$

r-Bell polynomials have only real zeros [28], too. (More on the r-Bell polynomials can be found in [25].) That the

$$D_m(n,x) = \sum_{k=1}^{n} W_m(n,k)x^k$$

polynomials attached to the Whitney numbers have only real zeros was proven by Benoumhani [4]. The most general theorem is due to Cheon and Jung [10]. They proved that the polynomials

$$D_{m,r}(n,x) = \sum_{k=1}^{n} W_{m,r}(n,k)x^k$$

have only real zeros, whence it follows that the $W_{m,r}(n,k)$ numbers are LC. The $D_{m,r}(n,x)$ polynomials are called r-Dowling polynomials, while $D_m(n,x)$ are the Dowling polynomials.

Gathering the facts we have mentioned till now: the LC property is established both for the unsigned $w_{m,r}(n,k)$ and $W_{m,r}(n,k)$. Good estimations for the maximizing index exists for the second kind r-Whitney numbers (see (3) above), but in the first kind case we do not have estimations with respect to the r-Whitneys in its generality.

Open Problem. How does the maximizing index $k_{n,m,r}$ of the $|w_{m,r}(n,k)|$ numbers behaves asymptotically?

3.2.2 Asymptotic growth of the leftmost zeros

By (2) it follows easily that the attached polynomials have only real zeros, as well as that these zeros form an arithmetic progression. But what else can we say about the zeros of the $B_n(x)$ Bell polynomials and their generalizations $B_{n,r}(x)$, $D_m(n,x)$, and $D_{m,r}(n,x)$? After knowing that all of their zeros are real and negative, we can ask that how rapidly does the leftmost zero grow? Corcino and Mező proved that if $z^*_{n,r}$ is the leftmost zero of the nth r-Bell polynomial, then $|z^*_{n,r}|$ asymptotically no grows faster than

$$\frac{1}{2}\sqrt{\frac{5}{3}}n^{\frac{3}{2}}.$$

Note that this is independent of r. The details can be found in [30]. Suggested by numerical calculations we phrase the following.

Open problem. The leftmost zero $z^*_{n,r}$ of the r-Bell polynomials asymptotically grows linearly, that is,

$$|z^*_{n,r}| \sim c_r n \quad (n \to \infty).$$

These questions were not studied for Whitney and r-Whitney polynomials (which also have only real zeros), so we give another open problem.

Open problem. Let $z^*_{n,m,r}$ be the leftmost zero of the $D_{m,r}(n,x)$ polynomials. Give a bound for $|z^*_{n,m,r}|$, and find how rapidly does $|z^*_{n,m,r}|$ grow with n asymptotically.

4 The Fubini and Eulerian numbers and their generalizations

The F_n Fubini numbers (also known as preferential arrangement numbers or ordered Bell numbers) count the partitions of a set where the order of the blocks counts [18, 21, 39]. By this explication it follows that

$$F_n = \sum_{k=1}^{n} k!S(n,k).$$

In [21] James used these numbers to calculate the factorizations of square-free integers. Another interpretation can be found in the book of Comtet [11, p. 228.] and in the book of Wilf [43].

Another class of numbers is the (double) sequence of Eulerian numbers $E(n,k)$. $E(n,k)$ is the number of permutations on n elements with k ascents [6]. In a permutation $p_1 \cdots p_n$ the position i is an ascent if $p_i < p_{i+1}$. For example, in the permutation

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 4 & 1 & 2 & 5 \end{pmatrix}$$

there are 3 ascents: $3-4$, $1-2$ and $2-5$.

Frobenius proved an interesting formula [39]:

$$F_n = \sum_{k=0}^{n} E(n,k)2^k \quad (n \geq 1).$$

A nice combinatorial proof was found by Remmel and Wachs [36]. Our goal in this section is to describe the generalizations of this formula via the r-Stirling numbers and Whitney numbers.

4.1 The r-Fubini numbers and r-Eulerian numbers and their polynomials

4.1.1 The r-Fubini numbers and polynomials

The r-Fubini polynomials are defined similarly as the r-Bell polynomials:

$$F_{n,r}(x) = \sum_{k=0}^{n}(k+r)!S_r(n,k)x^k.$$

(Note that $S_r(n,k)$ refers to $n+r$ elements and $k+r$ partitions, so in place of $k!$ we have to write $(k+r)!$.) The r-Fubini numbers are defined as

$$F_{n,r} = F_{n,r}(1).$$

We shortly recollect the most important properties of these polynomials, as they appeared in the (Hungarian) PhD thesis of the author [29]. The proofs are omitted.

The first ordered r-Bell numbers

	$n=0$	$n=1$	$n=2$	$n=3$	$n=4$	$n=5$
$r=0$	1	1	3	13	75	541
$r=1$	1	3	13	75	541	4683
$r=2$	2	10	62	466	4142	42610
$r=3$	6	42	342	3210	34326	413322
$r=4$	24	216	2184	24696	310344	4304376
$r=5$	120	1320	15960	211560	3063000	48226920
$r=6$	720	9360	131760	2005200	32911920	580919760

The first ordered r-Bell polynomials

$$\begin{aligned}
F_{0,r}(x) &= r! \\
F_{1,r}(x) &= r![(1+r)x+r] \\
F_{2,r}(x) &= r![(2+3r+r^2)x^2+(1+3r+2r^2)x+r^2] \\
F_{3,r}(x) &= r![(6+11r+6r^2+r^3)x^3+(6+15r+12r^2+3r^3)x^2+ \\
&\quad (1+4r+6r^2+3r^3)x+r^3]
\end{aligned}$$

The exponential generating function of $F_{n,r}(x)$ is as follows.

Theorem 1 *We have*

$$\sum_{n=0}^{\infty} F_{n,r}(x)\frac{t^n}{n!} = \frac{r!e^{rt}}{(1-x(e^t-1))^{r+1}} = r!e^{rt}\left(\sum_{n=0}^{\infty} F_n(x)\frac{t^n}{n!}\right)^{r+1}.$$

Here $F_n(x) = F_{n,0}(x)$, the nth "ordinary" ordered Bell polynomial [33, 39].
In particular,

$$\sum_{n=0}^{\infty} F_{n,r}\frac{t^n}{n!} = \frac{r!e^{rt}}{(2-e^t)^{r+1}}.$$

We present a recursion.

Theorem 2 *The polynomial $F_{n,r+1}(x)$ can be determined by the following double sum:*

$$F_{n,r+1}(x) = (r+1)\sum_{k=0}^{n}\binom{n}{k}\sum_{l=0}^{k}\binom{k}{l}F_l(x)F_{k-l,r}(x).$$

Corollary 1 *The sequence $F_{n,r}$ is log-convex in n, because it is a convolution of such ones [24]. This means that the following inequality holds*

$$F_{n-1,r}F_{n+1,r} \geq F_{n,r}^2.$$

Theorem 3 *The following recursive formula holds for any $n > 0$:*

$$F_{n,r}(x) = x[(r+1)F_{n-1,r}(x) + (1+x)F'_{n-1,r}(x)] + rF_{n-1,r}(x). \tag{4}$$

with the initial value $F_{0,r}(x) = r!$.

Corollary 2 *By the theorem, it can easily be seen that*

$$x^{1-r}\left[(x^{r+1}+x^r)F_{n-1,r}(x)\right]' = F_{n,r}(x).$$

In particular,

$$F_n(x) = x((1+x)F_{n-1}(x))'.$$

Whence we can get the structure of the zeros of the r-Fubini polynomials $F_{n,r}(x)$: these polynomials have only real zeros and all of them are contained in the interval $]-1,0]$ for all $n > 0$.

Corollary 3 *The already mentioned Newton theorem yields that the sequence $((k+r)!S_r(n,k))_{k=0}^n$ is LC (for a fixed n), that is,*

$$S_r(n,k)^2 \geq (k+r+1)S_r(n,k+1)\cdot\frac{1}{k+r}S_r(n,k-1),$$

and this implies that $(S_r(n,k))_{k=0}^n$ is also LC – a fact we knew before.

We will find useful Darroch's theorem [14]. Let $a_0, a_1, \ldots, a_n$ be a sequence of positive real numbers[1] with attached polynomial $p(x) = a_0 + a_1x + \cdots + a_nx^n$. Then – as we know – the sequence $a_0, a_1, \cdots, a_n$ is LC, and for the (leftmost) maximum M

$$\left|M - \frac{p'(x)}{p(x)}\right| < 1.$$

Applying this estimation for the maximizing index $M_{n,r}$ of the sequence $((k+r)!S_r(n,k))_{k=0}^n$ we have that

$$\left|M_{n,r} - \frac{F'_{n,r}(1)}{F_{n,r}(1)}\right| = \left|M_{n,r} - \frac{F_{n+1,r} - (2r+1)F_{n,r}}{2F_{n,r}}\right| < 1.$$

The derivative is calculated from the recursion formula (4).

To close this section we present an interesting summation formula for the r-Fubini numbers, which may be considered as an analogue of the Dobiński-formula for the ordinary Bell numbers (cf. [25] and the references therein):

$$B_n = \frac{1}{e}\sum_{k=0}^{\infty}\frac{k^n}{k!}.$$

The analogue of this infinite sum representation formula for the r-Fubini numbers is contained in the below statement.

Theorem 4 *For any $n, r \geq 0$ we have that*

$$F_{n,r} = \sum_{k=0}^{\infty}\frac{(k+r)^n}{2^{k+r+1}}\frac{(k+r)!}{k!}.$$

In the case $r = 0$ [18, 21, 39, 43] this takes the shape

$$F_n = \sum_{k=0}^{\infty}\frac{k^n}{2^{k+1}}.$$

[1] That the sequence contains only positive real numbers can be weakened. What is necessary is that $p(1) > 0$.

Since $F_{0,r} = r!$, we get the simple but interesting fact that

$$\sum_{k=0}^{\infty} \frac{1}{2^k} \frac{(k+r)!}{k!} = 2^{r+1} r!.$$

This, in fact, can be deduced in another way by using the generating function identity

$$\sum_{k=0}^{\infty} \binom{n+k}{n} x^k = \frac{1}{(1-x)^{n+1}}.$$

4.1.2 The r-Eulerian numbers and polynomials

To be able to describe the formerly mentioned generalization of Frobenius' theorem we need one more ingredient. This is the notion of r-Eulerian numbers. The classical Eulerian numbers $E(n,k)$ count permutations on n elements with k ascents. The $E_r(n,k)$ r-Eulerian numbers are defined as follows.

Definition 1 *The $E_r(n,k)$ numbers count the partitions on $n+r$ elements with k so-called r-ascents. An r-ascent is a classical ascent in which* not both *elements belong to the set $\{1,\ldots,r\}$.*

For example, in the permutation

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 4 & 1 & 2 & 5 \end{pmatrix}$$

there are 3 *ordinary* ascents: $3-4$, $1-2$ and $2-5$.

But if, for instance, $r = 3$ then $1-2$ will no longer be an ascent, because $1,2 \in \{1,2,3\}$. Note that, however, $3-4$ is still an ascent, because $4 \notin \{1,2,3\}$.

Now we list the results with respect to the r-Eulerian numbers and the below defined r-Eulerian polynomials from our PhD thesis, omitting the proofs. (Some of these results were deduced jointly with Dr. Gábor Nyul. The original proofs are mainly combinatorial.)

Theorem 5 *The r-Eulerian numbers satisfy the following recursion:*

$$E_r(n,m) = (m+1)E_r(n-1,m) + (n-m+r)E_r(n-1,m-1) \tag{5}$$

with the initial values $E_r(0,0) = 1$ and $E_r(n,0) = r!$ if $n > 0$.

Theorem 6 *The r-Stirling numbers can be represented with the r-Eulerian numbers as follows:*

$$(k+r)!S_r(n,k) = \sum_{m=0}^{n} E_r(n,m)\binom{m}{n-k}. \tag{6}$$

Summing over k, we get the generalized Frobenius theorem we were looking for.

Theorem 7 *For any $n \geq 1$ and $r \geq 0$*

$$F_{n,r} = \sum_{k=0}^{n} E_r(n,k)2^k.$$

The combinatorial proof – which is an extension of the Remmel-Wachs proof [36] – is contained in [29].

Theorem 7 can be extended to the r-Fubini polynomials, too (the $r=0$ case appears in the paper of Tanny [39]):

Theorem 8

$$F_{n,r}(x) = \sum_{k=0}^{n} E_r(n,k)(x+1)^k x^{n-k}.$$

The particular case $x = 1$ gives back the previous theorem.

By the proof of this formula it turns out that (6) can be reversed, and we can express the r-Eulerian numbers via the r-Stirlings:

Theorem 9

$$E_r(n,m)=\sum_{k=0}^{n}(k+r)!S_r(n,k)\binom{n-k}{m}(-1)^{n-k-m}.$$

Hence

$$E_r(n,n)=r!r^n,$$
$$E_r(n,0)=r!$$

After the r-Eulerian numbers we turn to the polynomials.

Definition 2 *The r-Euler polinomials are defined as usual:*

$$E_{n,r}(x):=\sum_{k=0}^{n}E_r(n,k)x^k.$$

The r-Fubini and r-Eulerian numbers can be transformed into each other as the following statement shows. (The $r=0$ case was proven by Prodinger [33].)

Theorem 10 *We have that*

$$\begin{aligned} F_{n,r}(x) &= x^n E_{n,r}\left(\frac{x+1}{x}\right), \\ E_{n,r}(x) &= (x-1)^n F_{n,r}\left(\frac{1}{x-1}\right). \end{aligned} \tag{7}$$

Corollary 4 *All the zeros of the $E_{n,r}(x)$ polynomial are real and belong to the interval $]-\infty,0[$.*

Corollary 5 *The $(E_r(n,k))_{k=0}^{n+r}$ sequence is LC, that is,*

$$E_r(n,k)^2\geq E_r(n,k-1)E_r(n,k+1).$$

It is known that the classical Eulerian numbers $E(n,k)$ are symmetric – similarly to the binomial coefficients $\binom{n}{k}$ –, so the maximizing index is trivial to find. This symmetry is lost when $r>1$. Hence the following problem is interesting.

Open problem. Find the asymptotics of the maximizing index of $E_r(n,k)$.

Open problem. What is the asymptotics of the sequence $E_r(n,k)$ as n runs while k and r is kept to be fixed?

This is surely not an easy question as the case of the classical Eulerian numbers shows [22].

Theorem 11 *The exponential generating function of the r-Eulerian polynomials is as follows:*

$$\sum_{n=0}^{\infty}E_{n,r}(x)\frac{t^n}{n!}=r!e^{r(x-1)t}\left(\frac{x-1}{x-e^{(x-1)t}}\right)^{r+1}=r!e^{r(x-1)t}\left(\sum_{n=0}^{\infty}E_n(x)\frac{t^n}{n!}\right)^{r+1}.$$

Here $E_n(x)=E_{n,0}(x)$ is the classical Eulerian polynomial. An equivalent representation can be deduced:

$$\sum_{n=0}^{\infty}E_{n,r}(x)\frac{t^n}{n!}=r!e^{(1-x)t}\left(\frac{1-x}{1-xe^{(1-x)t}}\right)^{r+1}.$$

Letting

$$f(x,t)=\sum_{n=0}^{\infty}E_{n,r}(x)\frac{t^n}{n!}$$

we can deduce that $f(x,t)$ satisfies the partial differential equation

$$(x-x^2)\frac{\partial f}{\partial x}+(tx-1)\frac{\partial f}{\partial t}+(1+rx)f=0.$$

This is the generalization of the PDE known for the exponential generating function of the Eulerian polynomials:

$$(x-x^2)\frac{\partial f}{\partial x}+(tx-1)\frac{\partial f}{\partial t}+f=0.$$

Corollary 6 *For the r-Eulerian polynomials the following recursion holds:*

$$E_{n,r}(x) = (1 + (n + r - 1)x)E_{n-1,r}(x) + (x - x^2)E'_{n-1,r}(x).$$

Now we give an interesting application of the r-Eulerian numbers in the theory of infinite sums. Worpitzky's theorem [6, 17, 44] states that

$$\sum_{k=0}^{n} E(n,k)\binom{x+k}{n+r} = x^n.$$

This can be generalized to the $r > 0$ case.

Theorem 12 *We have that*

$$\sum_{k=0}^{n} E_r(n,k)\binom{x+k}{n+r} = x^n(x)_r.$$

Here $(x)_r = (x)(x+1)\cdots(x+r-1)$ *is the Pochhammer symbol.*

This interesting extension of Worpitzky's theorem can be used to find a generalization of a famous sum formula including the classical Eulerian numbers. This formula reads as

$$\sum_{i=0}^{\infty} i^n x^i = \frac{1}{(1-x)^{n+1}} \sum_{k=0}^{n} E(n,k)x^{n-k}$$

and follows directly from the formula of Worpitzky. Its generalization is contained in the following theorem, and the proof is similarly easy.

Theorem 13

$$\sum_{i=0}^{\infty} i^n (i)_r x^i = \frac{x^r}{(1-x)^{n+r+1}} \sum_{k=0}^{n} E_r(n,k)x^{n-k}.$$

At the end we add one more formula from our thesis.

Corollary 7

$$E_r(n, n-k-1+r) = \sum_{l=0}^{k} \binom{n+r+1}{l}(-1)^l(k+1-l)^n(k+1-l)_r.$$

This is the "r-case" of the known formula [17]

$$E(n, n-k-1) = \sum_{l=0}^{k} \binom{n+1}{l}(-1)^l(k+1-l)^n.$$

The first r-Eulerian polynomials

$$\begin{aligned}
E_{0,r}(x) &= r! \\
E_{1,r}(x) &= r!(1+rx) \\
E_{2,r}(x) &= r![1+(3r+1)x+r^2x^2] \\
E_{3,r}(x) &= r![1+(7r+4)x+(6r^2+4r+1)x^2+r^3x^3] \\
E_{4,r}(x) &= r![1+(15r+11)x+(25r^2+30r+11)x^2+ \\
&\quad (10r^3+10r^2+5r+1)x^3+r^4x^4].
\end{aligned}$$

4.2 Eulerian-like numbers connected to the Whitney numbers

The above studied r-Eulerian numbers are connected to the r-Stirling numbers. We know that there is another direction of generalization of the Stirlings: the Whitney numbers. Hence we can ask: can we define Fubini numbers and Eulerian numbers connected to the Whitney numbers? Can we connect them via a Frobenius-like formula as in Theorem 7?

The answer is twofold: there is an analogue, which is not a straight generalization (in the sense that it does not give back the Frobenius formula as a special case) but it has combinatorial interpretation. This is due to the present author [27]. On the other hand, one can find a direct generalization, too, but it has no known combinatorial meaning [34]. We describe the former shortly. For the details please refer to [27].

Let $p_1 \cdots p_{n-1}p_n$ be a permutation on n elements such that the last element is in the last position, i.e., $p_n = n$. Moreover, let the first $n-d$ elements be coloured with one of m colours, and the non-coloured elements are in increasing order at the end of the partition (followed by n, as we said before). Then we call this permutation an m-coloured Dowling d-permutation on n elements.

For example, the permutation

$$\underbrace{\overset{b\,r\,r\,g\,b}{37451}}_{n-d=5}\underbrace{2689}_{d=4}$$

is a 3-coloured Dowling 4-permutation on 9 elements.

We say that in the position i there is a *Dowling descent* if one of the next assumptions satisfy:

1. in position i there is a "classical" descent,
2. i separates the coloured and non-coloured elements,
3. i separates the non-coloured elements.

In the example above

$$\underbrace{\overset{b\,r\,r\,g\,b}{37451}}_{n-d=5}\underbrace{2689}_{d=4}$$

there are 6 Dowling descents:

$7-4$, $5-1$: these are descents in the classical sense, too.

$1-2$: is a descent by another reason: it separates coloured and non-coloured elements.

$2-6$, $6-8$, $8-9$: these separate non-coloured elements.

Moreover, a Dowling run is a maximal subsequence separated by Dowling descents. Let $A_{m,d}(n,k)$ denote the number of m-coloured Dowling d-permutations on n elements with k Dowling runs. Finally we sum on d:

$$A_m(n,k) = \sum_{d=1}^{k} A_{m,d}(n,k).$$

With the above definitions and notations we can present the analogue of the Frobenius formula.

Theorem 14 *It holds true that*

$$\sum_{k=1}^{n} W_m(n,k)m^k k! = \sum_{k=1}^{n} A_m(n+1,k+1)2^{n-k}.$$

The numbers on the left can be considered as the generalization of the Fubini numbers to the Whitney-setting. There is no "official" name for them, however they were investigated in the 1990's by M. Benoumhani [4, 5] and later by Rahmani [35].

We close the paper with a question.

Open problem. Having the original Frobenius formula, and its extensions towards the r-Stirling case and the Whitney number case, we ask the following: how to extend *combinatorially* the Frobenius formula to the r-Whitney case?

Acknowledgment

I am thankful to Professor Takao Komatsu for his invitation to take part in the Kyoto Workshop on Analytic Number Theory in 2014 October.

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