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Convolution identities for Cauchy numbers of the first kind and of the second kind

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1 Introduction

The Cauchy numbers c_n $(n \geq 0)$ are defined by

$$c_n = \int_0^1 x(x-1)\dots(x-n+1)dx$$

and the generating function of c_n is given by

$$\frac{x}{\ln(1+x)} = \sum_{n=0}^{\infty} c_n \frac{x^n}{n!} \quad (|x|<1)$$

([4, 11]). Several initial values are

$$c_0 = 1,\ c_1 = \frac{1}{2},\ c_2 = -\frac{1}{6},\ c_3 = \frac{1}{4},\ c_4 = -\frac{19}{30},\ c_5 = \frac{9}{4},\ c_6 = -\frac{863}{84},\ c_7 = \frac{1375}{24}.$$

2 Preliminaries

$c(x) = x/\ln(1+x)$ satisfies the identity

$$c(x)^2 = (1+x)c(x) - (1+x)xc'(x)\,. \tag{1}$$

Since for $i, \nu \geq 0$ it holds that

$$x^i c^{(\nu)}(x) = \sum_{n=0}^{\infty} \frac{n!}{(n-i)!} c_{n+\nu-i} \frac{x^n}{n!}\,, \tag{2}$$

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the identity (1) immediately leads to the formula

$$\sum_{k=0}^{n} \binom{n}{k} c_k c_{n-k} = -n(n-2)c_{n-1} - (n-1)c_n \quad (n \ge 0)\,. \tag{3}$$

Differentiating both sides of (1) by x and dividing them by 2, we obtain

$$c(x)c'(x) = -\frac{1}{2}x(x+1)c''(x) - \frac{1}{2}xc'(x) + \frac{1}{2}c(x)\,. \tag{4}$$

Proposition 1.

$$c(x)^3 = \frac{1}{2}(x+1)(x+2)c(x) - \frac{1}{2}x(x+1)(x+2)c'(x) + \frac{1}{2}x^2(x+1)^2c''(x)\,. \tag{5}$$

Proof. By (1) and (4),

$$\begin{aligned} c(x)^3 &= (1+x)\big((1+x)c(x) - (1+x)xc'(x)\big) \\ &\quad - (1+x)x\left(-\frac{1}{2}x(x+1)c''(x) - \frac{1}{2}xc'(x) + \frac{1}{2}c(x)\right) \\ &= \frac{1}{2}(x+1)(x+2)c(x) - \frac{1}{2}x(x+1)(x+2)c'(x) + \frac{1}{2}x^2(x+1)^2c''(x)\,. \end{aligned}$$

□

Theorem 1. *For $n \ge 2$ we have*

$$\begin{aligned} &\sum_{\substack{k_1+k_2+k_3=n \\ k_1,k_2,k_3\ge 0}} \frac{n!}{k_1!k_2!k_3!} c_{k_1}c_{k_2}c_{k_3} \\ &= \frac{(n-1)(n-2)}{2}c_n + \frac{n(n-2)(2n-5)}{2}c_{n-1} + \frac{n(n-1)(n-3)^2}{2}c_{n-2}\,. \end{aligned}$$

Remark. In [2, Corollary 3]

$$\sum_{\substack{k_1+k_2+k_3=n \\ k_1,k_2,k_3\ge 0}} \frac{n!}{k_1!k_2!k_3!} B_{k_1}B_{k_2}B_{k_3} = \frac{(n-1)(n-2)}{2}B_n + \frac{3n(n-2)}{2}B_{n-1} + n(n-1)B_{n-2}\,.$$

Proof of Theorem 1. By (2) in Proposition 1

$$
\begin{aligned}
&\frac{1}{2}(x+1)(x+2)c(x) - \frac{1}{2}x(x+1)(x+2)c'(x) + \frac{1}{2}x^2(x+1)^2c''(x)\\
&= \sum_{n=0}^{\infty}\left(c_n + \frac{3}{2}nc_{n-1} + \frac{1}{2}n(n-1)c_{n-2}\right)\frac{x^n}{n!}\\
&\quad - \sum_{n=0}^{\infty}\left(nc_n + \frac{3}{2}n(n-1)c_{n-1} + \frac{1}{2}n(n-1)(n-2)c_{n-2}\right)\frac{x^n}{n!}\\
&\quad + \sum_{n=0}^{\infty}\left(\frac{1}{2}n(n-1)c_n + n(n-1)(n-2)c_{n-1} + \frac{1}{2}n(n-1)(n-2)(n-3)c_{n-2}\right)\frac{x^n}{n!}\\
&= \sum_{n=0}^{\infty}\left(\frac{(n-1)(n-2)}{2}c_n + \frac{n(n-2)(2n-5)}{2}c_{n-1} + \frac{n(n-1)(n-3)^2}{2}c_{n-2}\right)\frac{x^n}{n!}\,.
\end{aligned}
$$

□

The fundamental result of the third order is given by the following.

Theorem 2. *For $\mu, n \ge 0$, we have*

$$
\begin{aligned}
\sum_{\substack{\kappa_1+\kappa_2+\kappa_3=\mu\\ \kappa_1,\kappa_2,\kappa_3\ge 0}} \frac{\mu!}{\kappa_1!\kappa_2!\kappa_3!}(c_{\kappa_1}+c_{\kappa_2}+c_{\kappa_3})^n &= \frac{(n+\mu-1)(n+\mu-2)}{2}c_{n+\mu}\\
&+ \frac{(n+\mu)(n+\mu-2)(2n+2\mu-5)}{2}c_{n+\mu-1}\\
&+ \frac{(n+\mu)(n+\mu-1)(n+\mu-3)^2}{2}c_{n+\mu-2}\,.
\end{aligned}
$$

Remark. If we put $\mu = 0$, we have the identity in Theorem (1). If we put $\mu = 1$, we have

$$
\begin{aligned}
&(c_0+c_0+c_1)^n\\
&= \frac{n(n-1)}{6}c_{n+1} + \frac{(n+1)(n-1)(2n-3)}{6}c_n + \frac{n(n+1)(n-2)^2}{6}c_{n-1}\,.
\end{aligned}
$$

If we put $\mu = 2$, we have

$$
\begin{aligned}
&(c_0+c_0+c_2)^n + 2(c_0+c_1+c_1)^n\\
&= \frac{n(n+1)}{6}c_{n+2} + \frac{n(n+2)(2n-1)}{6}c_{n+1} + \frac{(n+1)(n+2)(n-1)^2}{6}c_n\,.
\end{aligned}
$$

If we put $\mu = 3$, we have

$$
\begin{aligned}
&(c_0 + c_0 + c_3)^n + 6(c_0 + c_1 + c_2)^n + 2(c_1 + c_1 + c_1)^n \\
&= \frac{(n+1)(n+2)}{6} c_{n+3} + \frac{(n+1)(n+3)(2n+1)}{6} c_{n+2} + \frac{n^2(n+2)(n+3)}{6} c_{n+1} \,.
\end{aligned}
$$

To prove Theorem 2 is based upon a relation about the function $c(x)$.

Proposition 2. *For $\mu \geq 0$, we have*

$$
\begin{aligned}
&\sum_{\substack{\kappa_1+\kappa_2+\kappa_3=\mu \\ \kappa_1,\kappa_2,\kappa_3 \geq 0}} \frac{\mu!}{\kappa_1!\kappa_2!\kappa_3!} c^{(\kappa_1)}(x) c^{(\kappa_2)}(x) c^{(\kappa_3)}(x) \\
&= \frac{1}{2} x^2 (x+1)^2 c^{(\mu+2)}(x) \\
&\quad + \frac{1}{2} x(x+1)\big((4\mu-1)x + (2\mu-2)\big) c^{(\mu+1)}(x) \\
&\quad + \frac{1}{2}\big((6\mu^2 - 9\mu + 1)x^2 + 3(2\mu^2 - 4\mu + 1)x + (\mu-1)(\mu-2)\big) c^{(\mu)}(x) \\
&\quad + \frac{\mu}{2}\big((4\mu^2 - 15\mu + 13)x + (2\mu-5)(\mu-2)\big) c^{(\mu-1)}(x) \\
&\quad + \frac{1}{2}\mu(\mu-1)(\mu-3)^2 c^{(\mu-2)}(x) \,.
\end{aligned}
$$

Proof. By differentiating both sides of (5) μ times with respect to x, we have the desired result. The left-hand side is due to the General Leibniz's rule. The right-hand side can be proved by induction. □

Proof of Theorem 2. By (2) in Proposition 2, we have

$$
\begin{aligned}
&\frac{1}{2} x^2 (x+1)^2 c^{(\mu+2)}(x) \\
&= \sum_{n=0}^{\infty} \left(\frac{1}{2} n(n-1) c_{n+\mu} + n(n-1)(n-2) c_{n+\mu-1} + \frac{1}{2} n(n-1)(n-2)(n-3) c_{n+\mu-2} \right) \frac{x^n}{n!} \,,
\end{aligned}
$$

$$
\begin{aligned}
&\frac{1}{2} x(x+1)\big((4\mu-1)x + (2\mu-2)\big) c^{(\mu+1)}(x) \\
&= \sum_{n=0}^{\infty} \left((\mu-1) n c_{n+\mu} + \frac{6\mu-3}{2} n(n-1) c_{n+\mu-1} + \frac{4\mu-1}{2} n(n-1)(n-2) c_{n+\mu-2} \right) \frac{x^n}{n!} \,,
\end{aligned}
$$

$$\frac{1}{2}\big((6\mu^2-9\mu+1)x^2+3(2\mu^2-4\mu+1)x+(\mu-1)(\mu-2)\big)c^{(\mu)}(x)$$
$$=\sum_{n=0}^{\infty}\left(\frac{(\mu-1)(\mu-2)}{2}c_{n+\mu}+\frac{3(2\mu^2-4\mu+1)}{2}nc_{n+\mu-1}+\frac{6\mu^2-9\mu+1}{2}n(n-1)c_{n+\mu-2}\right)\frac{x^n}{n!},$$

$$\frac{\mu}{2}\big((4\mu^2-15\mu+13)x+(2\mu-5)(\mu-2)\big)c^{(\mu-1)}(x)$$
$$=\sum_{n=0}^{\infty}\left(\frac{\mu(2\mu-5)(\mu-2)}{2}c_{n+\mu-1}+\frac{\mu(4\mu^2-15\mu+13)}{2}nc_{n+\mu-2}\right)\frac{x^n}{n!}$$

and

$$\frac{1}{2}\mu(\mu-1)(\mu-3)^2c^{(\mu-2)}(x)=\sum_{n=0}^{\infty}\frac{\mu(\mu-1)(\mu-3)^2}{2}c_{n+\mu-2}\frac{x^n}{n!}.$$

Combining all the relations together, we obtain the desired result. □

3 Higher powers

In similar manners to Proposition 1, we have the following.

$$c(x)^4=\frac{(1+x)(x^2+6x+6)}{6}c(x)-\frac{x(1+x)(x^2+6x+6)}{6}c'(x)$$
$$+\frac{x^2(1+x)^2}{2}c''(x)-\frac{x^3(1+x)^3}{6}c'''(x),$$
$$c(x)^5=\frac{(1+x)(x^3+14x^2+36x+24)}{24}c(x)-\frac{x(1+x)(x^3+14x^2+36x+24)}{24}c'(x)$$
$$+\frac{x^2(1+x)^2(x^2+6x+12)}{24}c''(x)+\frac{(x-2)x^3(1+x)^3}{12}c^{(3)}(x)$$
$$+\frac{x^4(1+x)^4}{24}c^{(4)}(x),$$

Therefore,

$$
\begin{aligned}
&(c_0+c_0+c_0+c_0)^n\\
&=-\frac{(n-1)(n-2)(n-3)}{6}c_n-\frac{n(n-2)(n-3)^2}{2}c_{n-1}\\
&\quad-\frac{n(n-1)(n-3)(3n^2-21n+37)}{6}c_{n-2}-\frac{n(n-1)(n-2)(n-4)^3}{6}c_{n-3}\,,\\
&(c_0+c_0+c_0+c_0+c_0)^n\\
&=\frac{(n-1)(n-2)(n-3)(n-4)}{24}c_n+\frac{n(n-2)(n-3)(n-4)(2n-7)}{12}c_{n-1}\\
&\quad+\frac{n(n-1)(n-3)(n-4)(6n^2-48n+97)}{24}c_{n-2}\\
&\quad+\frac{n(n-1)(n-2)(n-4)(2n-9)(2n^2-18n+41)}{24}c_{n-3}\\
&\quad+\frac{n(n-1)(n-2)(n-3)(n-5)^4}{24}c_{n-4}\,.
\end{aligned}
$$

In general, we have the following.

Theorem 3. *For any integers $n\ge 1$ and $m\ge 2$, we have*

$$
\begin{aligned}
&(\underbrace{c_0+\cdots+c_0}_{m})^n\\
&=\frac{n!}{(m-1)!}\sum_{i=0}^{m-1}\left(\sum_{l=0}^{m-1}\sum_{k=0}^{m-l-1}\frac{(-1)^{l+k}(m-k-1)!}{l!(n-l-i)!}\left\{ {m-1 \atop m-k-1} \right\}\binom{m-k-1}{i-k+1}\right)c_{n-i}\,.
\end{aligned}
$$

By applying Theorem 3, we have the following result.

Theorem 4. *For any integers $n\ge 1$ and $m\ge 2$, we have*

$$
\begin{aligned}
&\sum_{\substack{\kappa_1+\cdots+\kappa_m=\mu\\ \kappa_1,\dots,\kappa_m\ge 0}}\frac{\mu!}{\kappa_1!\cdots\kappa_m!}(c_{\kappa_1}+\cdots+c_{\kappa_m})^n\\
&=\frac{(n+\mu)!}{(m-1)!}\sum_{i=0}^{m-1}\left(\sum_{l=0}^{m-1}\sum_{k=0}^{m-l-1}\frac{(-1)^{l+k}(m-k-1)!}{l!(n-l-i)!}\left\{ {m-1 \atop m-k-1} \right\}\binom{m-k-1}{i-k+1}\right)c_{n+\mu-i}\,.
\end{aligned}
$$

When $\mu = 1$ in Theorem 4, we have for any integers $n \ge 1$ and $m \ge 2$,

$$(\underbrace{c_0 + \cdots + c_0}_{m-1} + c_1)^n$$
$$= \frac{(n+1)!}{m!} \sum_{i=0}^{m-1} \left(\sum_{l=0}^{m-1} \sum_{k=0}^{m-l-1} \frac{(-1)^{l+k}(m-k-1)!}{l!(n-l-i)!} \left\{ \begin{matrix} m-1 \\ m-k-1 \end{matrix} \right\} \binom{m-k-1}{i-k+1} \right) c_{n-i+1} \,.$$

When $\mu = 2$ in Theorem 4, we have for any integers $n \ge 1$ and $m \ge 2$,

$$(\underbrace{c_0 + \cdots + c_0}_{m-1} + c_2)^n + (m-1)(\underbrace{c_0 + \cdots + c_0}_{m-2} + c_1 + c_1)^n$$
$$= \frac{(n+2)!}{m!} \sum_{i=0}^{m-1} \left(\sum_{l=0}^{m-1} \sum_{k=0}^{m-l-1} \frac{(-1)^{l+k}(m-k-1)!}{l!(n-l-i)!} \left\{ \begin{matrix} m-1 \\ m-k-1 \end{matrix} \right\} \binom{m-k-1}{i-k+1} \right) c_{n-i+2} \,.$$

4 Cauchy numbers of the second kind

The Cauchy numbers of the second kind $\widehat{c}_n$ ($n \ge 0$) are defined by

$$\widehat{c}_n = \int_0^1 (-x)(-x-1)\dots(-x-n+1)dx$$

and the generating function of $\widehat{c}_n$ is given by

$$\frac{x}{(1+x)\ln(1+x)} = \sum_{n=0}^{\infty} \widehat{c}_n \frac{x^n}{n!} \quad (|x| < 1)$$

([4, 11]). Several initial values are

$$\widehat{c}_0 = 1,\ \widehat{c}_1 = -\frac{1}{2},\ \widehat{c}_2 = \frac{5}{6},\ \widehat{c}_3 = -\frac{9}{4},\ \widehat{c}_4 = \frac{251}{30},\ \widehat{c}_5 = -\frac{475}{12},\ \widehat{c}_6 = \frac{19087}{84},\ \widehat{c}_7 = -\frac{36799}{24}.$$

In [10], an explicit expression of $(\widehat{c}_l + \widehat{c}_m)^n$ for $l, m, n \ge 0$ was determined, where with the classical umbral calculus notation (see, e.g., [12]), $(\widehat{c}_l + \widehat{c}_m)^n$ is defined by

$$(\widehat{c}_l + \widehat{c}_m)^n := \sum_{j=0}^{n} \binom{n}{j} \widehat{c}_{l+j} \widehat{c}_{m+n-j} \,.$$

As special cases, we obtained

$$(\widehat{c}_0+\widehat{c}_0)^n=n!\sum_{k=0}^n(-1)^{n-k}\frac{\widehat{c}_k}{k!}-n\widehat{c}_n\,, \tag{6}$$

$$(\widehat{c}_0+\widehat{c}_1)^n=-\frac{(n+1)!}{2}\sum_{k=0}^n(-1)^{n-k}\frac{\widehat{c}_k}{k!}-\frac{1}{2}n\widehat{c}_{n+1}\,, \tag{7}$$

$$\begin{aligned}(\widehat{c}_0+\widehat{c}_2)^n&=\frac{n!}{12}\sum_{k=0}^n\frac{(-1)^{n-k}}{k!}\bigl(2k(n+2k-2)+5(n-k+2)(n-k+1)\bigr)\widehat{c}_k\\&\quad-\frac{n}{2}\widehat{c}_{n+1}-\frac{n}{3}\widehat{c}_{n+2}\,,\end{aligned} \tag{8}$$

$$(\widehat{c}_1+\widehat{c}_1)^n=\frac{n!}{12}\sum_{k=0}^n\frac{(-1)^{n-k}}{k!}\bigl((n+1)(n+2)+k(8n-9k+19)\bigr)\widehat{c}_k-\widehat{c}_{n+1}-\frac{n+3}{6}\widehat{c}_{n+2} \tag{9}$$

(see [10]).

We shall consider the higher order recurrences for Cauchy numbers of the second kind:

$$(\widehat{c}_{l_1}+\cdots+\widehat{c}_{l_m})^n:=\sum_{\substack{k_1+\cdots+k_m=n\\k_1,\,..,k_m\ge 0}}\frac{n!}{k_1!\cdots k_m!}\widehat{c}_{k_1+l_1}\cdots\widehat{c}_{k_m+l_m}\,.$$

As special cases, we shall have

$$\begin{aligned}&(\widehat{c}_0+\widehat{c}_0+\widehat{c}_0)^n\\&=\frac{n^2}{2}\widehat{c}_n+\frac{n!}{2}\sum_{i=0}^n\frac{(-1)^{n-i}(n-4i+2)}{i!}\widehat{c}_i\\&=\frac{(n-1)(n-2)}{2}\widehat{c}_n+\frac{n!}{2}\sum_{i=0}^{n-1}\frac{(-1)^{n-i}(n-4i+2)}{i!}\widehat{c}_i\,,\end{aligned} \tag{10}$$

$$\begin{aligned}&(\widehat{c}_0+\widehat{c}_0+\widehat{c}_1)^n\\&=\frac{n(n-1)}{6}\widehat{c}_{n+1}-\frac{(n+1)!}{6}\sum_{i=0}^n\frac{(-1)^{n-i}(n-4i+3)}{i!}\widehat{c}_i\,.\end{aligned} \tag{11}$$

and

$$(\widehat{c}_0+\widehat{c}_0+\widehat{c}_0+\widehat{c}_0)^n$$
$$=-\frac{n^3}{6}\widehat{c}_n+\frac{n!}{12}\sum_{i=0}^{n}\frac{(-1)^{n-i}(n^2-16in+11n+27i^2-33i+12)}{i!}\widehat{c}_i\,, \tag{12}$$
$$(\widehat{c}_0+\widehat{c}_0+\widehat{c}_0+\widehat{c}_1)^n$$
$$=-\frac{(n+1)^3}{24}\widehat{c}_{n+1}-\frac{(n+1)!}{48}\sum_{i=0}^{n+1}\frac{(-1)^{n-i}\big(n^2-16in+13n+27i^2-49i+24\big)}{i!}\widehat{c}_i\,. \tag{13}$$

$\widehat{c}(x)=x/\big((1+x)\ln(1+x)\big)$ satisfies the identity

$$\widehat{c}(x)^2=-x\widehat{c}'(x)+\frac{1}{1+x}\widehat{c}(x)\,. \tag{14}$$

Since for $i,\nu\ge 0$ it holds that

$$x^i\widehat{c}^{(\nu)}(x)=\sum_{n=0}^{\infty}\frac{n!}{(n-i)!}\widehat{c}_{n+\nu-i}\frac{x^n}{n!}\,, \tag{15}$$

the identity (14) immediately leads to the formula

$$\sum_{k=0}^{n}\binom{n}{k}\widehat{c}_k\widehat{c}_{n-k}=-n\widehat{c}_n+n!\sum_{i=0}^{n}(-1)^{n-i}\frac{\widehat{c}_i}{i!}\quad(n\ge 0)\,. \tag{16}$$

Differentiating both sides of (14) by x and dividing them by 2, we obtain

$$\widehat{c}(x)\widehat{c}'(x)=-\frac{1}{2}x\widehat{c}''(x)-\frac{x}{2(1+x)}\widehat{c}'(x)-\frac{1}{2(1+x)^2}\widehat{c}(x)\,. \tag{17}$$

Proposition 3.

$$\widehat{c}(x)^3=\frac{1}{2}x^2\widehat{c}''(x)+\frac{x(x-2)}{2(1+x)}\widehat{c}'(x)+\frac{x+2}{2(1+x)^2}\widehat{c}(x)\,. \tag{18}$$

Proof. The proof is similat to that of Proposition 5. By (14) and (17), we get the result. □

Applying (15) and Proposition 3, we have the result of the third order.

Theorem 5. *For $n \geq 0$ we have*

$$(\widehat{c}_0+\widehat{c}_0+\widehat{c}_0)^n=\frac{n^2}{2}\widehat{c}_n+\frac{n!}{2}\sum_{i=0}^n\frac{(-1)^{n-i}(n-4i+2)}{i!}\widehat{c}_i\,.$$

Similarly, for $n \geq 0$ we have

$$(\widehat{c}_0+\widehat{c}_0+\widehat{c}_0+\widehat{c}_0)^n=-\frac{n^3}{6}\widehat{c}_n+\frac{n!}{12}\sum_{i=0}^n\frac{(-1)^{n-i}(n^2-16in+11n+27i^2-33i+12)}{i!}\widehat{c}_i\,.$$

$$\begin{aligned}&(\widehat{c}_0+\widehat{c}_0+\widehat{c}_0+\widehat{c}_0+\widehat{c}_0)^n\\&=\frac{n^4}{24}\widehat{c}_n+\frac{n!}{144}\sum_{i=0}^n\frac{(-1)^{n-i}}{i!}(n^3-48in^2+39n^2+243i^2n\\&\qquad-393in+176n-256i^3+564i^2-476i+144)\widehat{c}_i\,.\end{aligned}$$

In general, we can state the following.

Theorem 6. *For any integers $n \geq 1$ and $m \geq 2$, we have*

$$\begin{aligned}&(\underbrace{\widehat{c}_0+\cdots+\widehat{c}_0}_{m})^n\\&=\frac{n!}{(m-1)!}\sum_{i=0}^{n-1}\left(\sum_{l=0}^{i}\sum_{k=0}^{\min\{n-i,m-l-1\}}\frac{(-1)^{n-i+l}(m-k-1)!}{(i-l)!l!}\left\{{m-1\atop m-k-1}\right\}\binom{n-i}{k}\right.\\&\left.\sum_{i=0}^{n-1}\sum_{k=0}^{m-1}\left\{{m-1\atop k}\right\}\binom{n-i-1}{m-k-1}\frac{(-1)^{n-i+k-1}}{(i-k)!}\right)\widehat{c}_i\\&\quad+\sum_{l=0}^{\min\{n,m\}}(-1)^l\binom{n}{l}\widehat{c}_n\,.\end{aligned}$$

Theorem 7. *For any integers $n\ge 1$ and $m\ge 2$, we have*

$$\sum_{\substack{\kappa_1+\cdots+\kappa_m=\mu\\ \kappa_1,\dots,\kappa_m\ge 0}}\frac{\mu!}{\kappa_1!\cdots\kappa_m!}(\widehat{c}_{\kappa_1}+\cdots+\widehat{c}_{\kappa_m})^n$$
$$=\frac{(n+\mu)!}{(m-1)!}\sum_{i=0}^{n+\mu-1}\left(\sum_{l=0}^{i}\sum_{k=0}^{\min\{n+\mu-i,m-l-1\}}\frac{(-1)^{n+\mu-i+l}(m-k-1)!}{(i-l)!l!}\left\{{m-1\atop m-k-1}\right\}\binom{n+\mu-i}{k}\right.$$
$$\left.\sum_{i=0}^{n+\mu-1}\sum_{k=0}^{m-1}\left\{{m-1\atop k}\right\}\binom{n+\mu-i-1}{m-k-1}\frac{(-1)^{n+\mu-i+k-1}}{(i-k)!}\right)\widehat{c}_i$$
$$+\sum_{l=0}^{\min\{n+\mu,m\}}(-1)^l\binom{n+\mu}{l}\widehat{c}_{n+\mu}\,.$$

If $m=4$ and $\mu=2$ in Theorem 7, we have

$$12(\widehat{c}_0+\widehat{c}_0+\widehat{c}_1+\widehat{c}_1)^n+4(\widehat{c}_0+\widehat{c}_0+\widehat{c}_0+\widehat{c}_2)^n$$
$$=-\binom{n+1}{3}\widehat{c}_{n+2}+\frac{(n+2)!}{12}\sum_{l=0}^{n+1}\frac{(-1)^{n-l}}{l!}\big(27l^2-16nl-65l+n^2+15n+38\big)\widehat{c}_l\,.$$

If $m=4$ and $\mu=3$ in Theorem 7, we have

$$24(\widehat{c}_0+\widehat{c}_1+\widehat{c}_1+\widehat{c}_1)^n+36(\widehat{c}_0+\widehat{c}_0+\widehat{c}_1+\widehat{c}_2)^n+4(\widehat{c}_0+\widehat{c}_0+\widehat{c}_0+\widehat{c}_3)^n$$
$$=-\binom{n+2}{3}\widehat{c}_{n+3}+\frac{(n+3)!}{12}\sum_{l=0}^{n+2}\frac{(-1)^{n-l+1}}{l!}\big(27l^2-16nl-81l+n^2+17n+54\big)\widehat{c}_l\,.$$

If $m=5$ and $\mu=1$ in Theorem 7, we have

$$5(\widehat{c}_0+\widehat{c}_0+\widehat{c}_0+\widehat{c}_0+\widehat{c}_1)^n$$
$$=\binom{n}{4}\widehat{c}_{n+1}$$
$$-\frac{(n+1)!}{144}\sum_{l=0}^{n}\frac{(-1)^{n-l+1}}{l!}\big(256l^3-(243n+807)l^2$$
$$+(48n^2+489n+917)l-(n^3+42n^2+257n+360)\big)\widehat{c}_l\,.$$

References

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