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An explicit upper bound of the argument of Dirichlet L-functions on the generalized Riemann hypothesis

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Abstract

We prove an explicit upper bound of the function $S(t,\chi)$, defined by the argument of Dirichlet L-functions attached to a primitive Dirichlet character $\chi \pmod{q>1}$. An explicit upper bound of the function $S(t)$, defined by the argument of the Riemann zeta-function, have been obtained by A. Fujii [1]. Our result is obtained by applying the idea of Fujii's result on $S(t)$. The constant part of the explicit upper bound of $S(t,\chi)$ in this paper does not depend on χ. Our proof does not covor the case $q=1$ and indeed gives a better bound than the one of Fujii that covers the case $q=1$.

1 Introduction

The argument of the Riemann zeta-function on the clitical line defined by

$$S(t)=\frac{1}{\pi}\arg\zeta\left(\frac{1}{2}+it\right)$$

when t is not the ordinate of a zero of $\zeta(s)$. This is obtained by continuous variation along the straight lines connecting 2, $2+it$, and $\frac{1}{2}+it$, starting with the value zero. Also, when t is the ordinate of a zero of $\zeta(s)$, we define

$$S(t)=\frac{1}{2}\{S(t+0)+S(t-0)\}.$$

Now, we consider the argument of Dirichlet L-functions. Let $L(s,\chi)$ be the Dirichlet L-function, where $s=\sigma+it$ is a complex variable, assosiated with a primitive Dirichlet character $\chi \pmod{q>1}$. Here, we denote the non-trivial zeros of $L(s,\chi)$ by $\rho(\chi)=\beta(\chi)+i\gamma(\chi)$, where $\beta(\chi)$ and $\gamma(\chi)$ are real numbers. Then, when t is not the ordinate of a zero of $L(s,\chi)$, we define

$$S(t,\chi)=\frac{1}{\pi}\arg L\left(\frac{1}{2}+it,\chi\right).$$

This is given by continuous variation along the straight line $s=\sigma+it$, as σ varies from $+\infty$ to $\frac{1}{2}$, starting with the value zero. Also, when t is the ordinate of a zero of $L(s,\chi)$, we define

$$S(t,\chi)=\frac{1}{2}\{S(t+0,\chi)+S(t-0,\chi)\}.$$

Selberg proved

$$S(t,\chi)=O(\log q(t+1))$$

and under the generalized Riemann hypothesis (GRH)

$$S(t,\chi)=O\left(\frac{\log q(t+1)}{\log\log q(t+3)}\right).$$

in Selberg [2]. The purpose of the present article is to prove the following result.

Theorem 1. *Assuming GRH. Then, for $q > 1$*

$$|S(t,\chi)| < 0.804 \cdot \frac{\log q(t+1)}{\log\log q(t+3)} + O\left(\frac{\log q(t+3)}{(\log\log q(t+3))^2}\right).$$

The constant 0.804 obviously does not depend on χ. And we don't know anything concerning the optimality. Also, the implied constant of the error term does not depend on q. Our result does not include the case of the function $S(t)$ since we assume $q > 1$. An explicit upper bound of the function $S(t)$ is obtained by A. Fujii [1], where the value is 0.83.

The basic policy of the proof of this theorem is based on A. Fujii [1]. In the proof, $S(t,\chi)$ is seperated by three parts M_1, M_2 and M_3. Fujii's idea of [1] is applied to all parts. But we need Lemma 1, which is an explicit formula for $\frac{L'}{L}(s,\chi)$. This lemma is an analogue of Selberg's result.

2 Some notations and a lemma

Here we introduce the following notations.

Let $s = \sigma + it$. We suppose that $\sigma \geq \frac{1}{2}$ and $t \geq 2$. Let x be a positive number satisfying $4 \leq x \leq t^2$. Also, we put

$$\sigma_1 = \frac{1}{2} + \frac{1}{\log x}$$

and

$$\Lambda_x(n) = \begin{cases} \Lambda(n) & \text{for } 1 \leq n \leq x, \\ \Lambda(n)\frac{\log \frac{x^2}{n}}{\log x} & \text{for } x \leq n \leq x^2, \end{cases}$$

with

$$\Lambda(n) = \begin{cases} \log p & \text{if } n = p^k \text{ with a prime } p \text{ and an integer } k \geq 1, \\ 0 & \text{otherwise.} \end{cases}$$

Using these notations, we prove the following lemma.

Lemma 1. *Assume the GRH. Let $t \geq 2$ and $x > 0$ such that $4 \leq x \leq t^2$. Then for $\sigma \geq \sigma_1 = \frac{1}{2} + \frac{1}{\log x}$, there exist ω and ω' such that $|\omega| \leq 1$ and $-1 \leq \omega' \leq 1$, we have*

$$\begin{aligned} \frac{L'}{L}(\sigma+it,\chi) = &- \sum_{n<x^2} \frac{\Lambda_x(n)}{n^{\sigma+it}}\chi(n) - \frac{x^{\frac{1}{2}-\sigma}\left(1+x^{\frac{1}{2}-\sigma}\right)\omega}{1-\frac{1}{e}\left(1+\frac{1}{e}\right)\omega'} \Re\left(\sum_{n<x^2}\frac{\Lambda_x(n)}{n^{\sigma_1+it}}\chi(n)\right) \\ &+ \frac{x^{\frac{1}{2}-\sigma}\left(1+x^{\frac{1}{2}-\sigma}\right)\omega}{1-\frac{1}{e}\left(1+\frac{1}{e}\right)\omega'} \cdot \frac{1}{2}\log q(t+1) + O(x^{\frac{1}{2}-\sigma}). \end{aligned}$$

This is an analogue of Lemma 2 of A. Fujii [1].

Lemma 2. *Let $a = 0$ if $\chi(-1) = 1$, and $a = 1$ if $\chi(-1) = -1$. Then, for $x > 1$, $s \neq -2q - a$ $(q = 0, 1, 2, \cdots)$ and $s \neq \rho(\chi)$, we have*

$$\begin{aligned} \frac{L'}{L}(s,\chi) = &- \sum_{n<x^2} \frac{\Lambda_x(n)}{n^s}\chi(n) + \frac{1}{\log x}\sum_{q=0}^{\infty} \frac{x^{-2q-a-s} - x^{-2(2q+a+s)}}{(2q+a+s)^2} \\ &+ \frac{1}{\log x}\sum_{\rho} \frac{x^{\rho-s} - x^{2(\rho-s)}}{(s-\rho)^2}. \end{aligned}$$

Lemma 2 is similar to Lemma 15 of Selberg [2]. We write here only a sketch of the proof of Lemma 2. If $a = \max(1, \sigma)$, we have

$$\sum_{n<x^2} \frac{\Lambda_x(n)}{n^s}\chi(n) = \frac{1}{2\pi i \log x}\int_{a-\infty i}^{a+\infty i} \frac{x^{z-s} - x^{2(z-s)}}{(z-s)^2} \cdot \frac{L'}{L}(z,\chi)dz.$$

We consider residues which we encounter when we move the path of integration to the left. At the point $z = s$, the residue is $-(\log x)\frac{L'}{L}(s,\chi)$. At the zeros $-2q-a$ $(q = 0, 1, 2, \cdots)$, the residues are $\frac{x^{-2q-a-s} - x^{-2(2q+a+s)}}{(2q+a+s)^2}$. At the zeros $s = \rho$ of $L(s,\chi)$, the residues are $\frac{x^{\rho-s} - x^{2(\rho-s)}}{(s-\rho)^2}$. Thus, we obtain Lemma 2. *Proof of Lemma 1.* Assume the GRH. In Lemma 2, since for $\sigma \geq \sigma_1 = \frac{1}{2} + \frac{1}{\log x}$

$$\left|\frac{1}{\log x}\sum_{\rho} \frac{x^{\rho-s} - x^{2(\rho-s)}}{(s-\rho)^2}\right| \leq x^{\frac{1}{2}-\sigma}\left(1 + x^{\frac{1}{2}-\sigma}\right)\sum_{\gamma} \frac{\sigma_1 - \frac{1}{2}}{\left(\sigma_1 - \frac{1}{2}\right)^2 + (t-\gamma)^2},$$

we have

$$\frac{1}{\log x}\sum_{\rho} \frac{x^{\rho-s} - x^{2(\rho-s)}}{(s-\rho)^2} = x^{\frac{1}{2}-\sigma}\left(1 + x^{\frac{1}{2}-\sigma}\right)\omega\sum_{\gamma} \frac{\sigma_1 - \frac{1}{2}}{\left(\sigma_1 - \frac{1}{2}\right)^2 + (t-\gamma)^2},$$

where $|\omega| \leq 1$. Hence by Lemma 2, we have for $\sigma \geq \sigma_1$

$$\begin{aligned}\frac{L'}{L}(\sigma + it, \chi) = &-\sum_{n<x^2} \frac{\Lambda_x(n)}{n^{\sigma+it}}\chi(n) + O\left(\frac{x^{\frac{1}{2}-\sigma}}{\log x}\right) \\ &+ x^{\frac{1}{2}-\sigma}\left(1 + x^{\frac{1}{2}-\sigma}\right)\omega\sum_{\gamma} \frac{\sigma_1 - \frac{1}{2}}{\left(\sigma_1 - \frac{1}{2}\right)^2 + (t-\gamma)^2}. \qquad (1)\end{aligned}$$

In particular, since $x^{\frac{1}{2}-\sigma} \leq x^{-\frac{1}{\log x}} = \frac{1}{e}$ for $\sigma \geq \sigma_1$, we get

$$\begin{aligned}\Re\frac{L'}{L}(\sigma_1 + it, \chi) = &-\Re\left(\sum_{n<x^2} \frac{\Lambda_x(n)}{n^{\sigma_1+it}}\chi(n)\right) + O\left(\frac{1}{\log x}\right) \\ &+ \frac{1}{e}\left(1 + \frac{1}{e}\right)\omega'\sum_{\gamma} \frac{\sigma_1 - \frac{1}{2}}{\left(\sigma_1 - \frac{1}{2}\right)^2 + (t-\gamma)^2}, \qquad (2)\end{aligned}$$

where $-1 \leq \omega' \leq 1$.

Here, since by p. 46 of Selberg [2]

$$\Re\frac{L'}{L}(s,\chi) = \Re\left(-\frac{1}{2}\log\frac{q}{\pi} - \frac{1}{2}\log\left(\frac{s+a}{2}\right)\right) + \sum_{\gamma} \frac{\sigma - \frac{1}{2}}{\left(\sigma - \frac{1}{2}\right)^2 + (t-\gamma)^2} + O(1),$$

we get for $t \geq 2$

$$\Re\frac{L'}{L}(\sigma_1 + it, \chi) = -\frac{1}{2}\log q(t+1) + \sum_{\gamma} \frac{\sigma_1 - \frac{1}{2}}{\left(\sigma_1 - \frac{1}{2}\right)^2 + (t-\gamma)^2} + O(1). \qquad (3)$$

By (2) and (3) we have

$$\begin{aligned}&\left(1 - \frac{1}{e}\left(1 + \frac{1}{e}\right)\omega'\right)\sum_{\gamma} \frac{\sigma_1 - \frac{1}{2}}{\left(\sigma_1 - \frac{1}{2}\right)^2 + (t-\gamma)^2} \\ &\quad = -\Re\left(\sum_{n<x^2} \frac{\Lambda_x(n)}{n^{\sigma_1+it}}\chi(n)\right) + \frac{1}{2}\log q(t+1) + O\left(\frac{1}{\log x}\right) + O(1).\end{aligned}$$

Inserting the above inequality to (1), we obtain Lemma 1.

□

3 Proof of Theorem 1

The quantity $S(t,\chi)$ is separated into the following three parts.

$$\begin{aligned} S(t,\chi) &= -\frac{1}{\pi}\Bigg\{\Im\int_{\sigma_1}^{\infty}\frac{L'}{L}(\sigma+it,\chi)d\sigma+\Im\left\{\left(\sigma_1-\frac{1}{2}\right)\frac{L'}{L}(\sigma_1+it,\chi)\right\} \\ &\quad -\Im\int_{\frac{1}{2}}^{\sigma_1}\left\{\frac{L'}{L}(\sigma_1+it,\chi)-\frac{L'}{L}(\sigma+it,\chi)\right\}d\sigma\Bigg\} \\ &= -\frac{1}{\pi}\Im\left(M_1+M_2+M_3\right), \end{aligned}$$

say.

First, we estimate M_1. By Lemma 1 we have

$$\begin{aligned} M_1 &= \int_{\sigma_1}^{\infty}\Bigg\{-\sum_{n<x^2}\frac{\Lambda_n(x)}{n^{\sigma+it}}\chi(n)-\frac{x^{\frac{1}{2}-\sigma}\left(1+x^{\frac{1}{2}-\sigma}\right)\omega}{1-\frac{1}{e}\left(1+\frac{1}{e}\right)\omega'}\Re\left(\sum_{n<x^2}\frac{\Lambda_n(x)}{n^{\sigma_1+it}}\chi(n)\right) \\ &\quad +\frac{x^{\frac{1}{2}-\sigma}\left(1+x^{\frac{1}{2}-\sigma}\right)\omega}{1-\frac{1}{e}\left(1+\frac{1}{e}\right)\omega'}\cdot\frac{1}{2}\log q(t+1)+O\left(x^{\frac{1}{2}-\sigma}\right)\Bigg\}d\sigma \\ &= -\sum_{n<x^2}\frac{\Lambda_n(x)}{n^{\sigma_1+it}\log n}\chi(n)+\eta_1(t), \end{aligned} \tag{4}$$

say. Here,

$$\begin{aligned} |\eta_1(t)| &\le \frac{1}{1-\frac{1}{e}\left(1+\frac{1}{e}\right)}\left|\Re\left(\sum_{n<x^2}\frac{\Lambda_n(x)}{n^{\sigma_1+it}}\chi(n)\right)-\frac{1}{2}\log q(t+1)\right| \\ &\quad\times\int_{\sigma_1}^{\infty}x^{\frac{1}{2}-\sigma}\left(1+x^{\frac{1}{2}-\sigma}\right)d\sigma+O\left(\int_{\sigma_1}^{\infty}x^{\frac{1}{2}-\sigma}d\sigma\right) \\ &\le \frac{\left(\frac{1}{e}+\frac{1}{2e^2}\right)}{1-\frac{1}{e}\left(1+\frac{1}{e}\right)}\cdot\frac{1}{2}\cdot\frac{\log q(t+1)}{\log x}+O\left(\frac{1}{\log x}\left|\sum_{n<x^2}\frac{\Lambda_n(x)}{n^{\sigma_1+it}}\chi(n)\right|\right), \end{aligned} \tag{5}$$

say.

Next, applying Lemma 1 to M_2, we get

$$|M_2| \le \frac{\left(\frac{1}{e}+\frac{1}{e^2}\right)}{1-\frac{1}{e}\left(1+\frac{1}{e}\right)}\cdot\frac{1}{2}\cdot\frac{\log q(t+1)}{\log x}+O\left(\frac{1}{\log x}\left|\sum_{n<x^2}\frac{\Lambda_n(x)}{n^{\sigma_1+it}}\chi(n)\right|\right), \tag{6}$$

say.

Next we estimate M_3. By Lemma 16 of Selberg [2] we get

$$\begin{aligned} |\Im(M_3)| &\le \left|\int_{\frac{1}{2}}^{\sigma_1}\sum_{\gamma}\frac{(t-\gamma)\left\{\left(\sigma-\frac{1}{2}\right)^2-\left(\sigma_1-\frac{1}{2}\right)^2\right\}}{\left\{\left(\sigma_1-\frac{1}{2}\right)^2+(t-\gamma)^2\right\}\left\{\left(\sigma-\frac{1}{2}\right)^2+(t-\gamma)^2\right\}}d\sigma\right|+O\left(\frac{1}{\log x}\right) \\ &< \int_{\frac{1}{2}}^{\sigma_1}\sum_{\gamma}N(\gamma,\sigma)d\sigma+O\left(\frac{1}{\log x}\right), \end{aligned}$$

say.

Here, we put $\mathfrak{N}=\int_{\frac{1}{2}}^{\sigma_1}\sum_{\gamma}N(\gamma,\sigma)d\sigma$. Then, we have

$$\begin{aligned} \mathfrak{N} &< \sum_{\gamma}\frac{\left(\sigma_1-\frac{1}{2}\right)^2}{\left(\sigma_1-\frac{1}{2}\right)^2+(t-\gamma)^2}\int_{\frac{1}{2}}^{\infty}\frac{|t-\gamma|}{\left(\sigma-\frac{1}{2}\right)^2+(t-\gamma)^2}d\sigma \\ &\le \frac{\pi}{2\log x}\sum_{\gamma}\frac{\sigma_1-\frac{1}{2}}{\left(\sigma_1-\frac{1}{2}\right)^2+(t-\gamma)^2} \end{aligned}$$

since $\sigma < \sigma_1$ for M_3.

Here, by (2) and (3) we get

$$\sum_{\gamma} \frac{\sigma_1 - \frac{1}{2}}{\left(\sigma_1 - \frac{1}{2}\right)^2 + (t-\gamma)^2} = \frac{1}{1 - \frac{1}{e}\left(1 + \frac{1}{e}\right)\omega'} \cdot \frac{1}{2} \log q(t+1) + O\left(\left|\sum_{n<x^2} \frac{\Lambda_n(x)}{n^{\sigma_1+it}} \chi(n)\right|\right) + O\left(\frac{1}{(\log x)^2}\right).$$

So,

$$\mathfrak{N} = \frac{\pi}{4} \cdot \frac{1}{1 - \frac{1}{e}\left(1 + \frac{1}{e}\right)\omega'} \cdot \frac{1}{\log x} \cdot \log q(t+1) + O\left(\frac{1}{\log x}\left|\sum_{n<x^2} \frac{\Lambda_n(x)}{n^{\sigma_1+it}} \chi(n)\right|\right) + O\left(\frac{1}{(\log x)^3}\right).$$

Hence we have

$$\begin{aligned} |\Im(M_3)| &\leq \frac{\pi}{4} \cdot \frac{1}{1 - \frac{1}{e}\left(1 + \frac{1}{e}\right)\omega'} \cdot \frac{1}{\log x} \cdot \log q(t+1) \\ &\quad + O\left(\frac{1}{\log x}\left|\sum_{n<x^2} \frac{\Lambda_n(x)}{n^{\sigma_1+it}} \chi(n)\right|\right) + O\left(\frac{1}{\log x}\right) \\ &= \eta_4(t) + O\left(\frac{1}{\log x}\left|\sum_{n<x^2} \frac{\Lambda_n(x)}{n^{\sigma_1+it}} \chi(n)\right|\right) + O\left(\frac{1}{\log x}\right), \end{aligned} \tag{7}$$

say.

Finally, we estimate the sums on right-hand sides of (4), (5), (6) and (7). By definition of $\Lambda_x(n)$ we have

$$\left|\sum_{n<x^2} \frac{\Lambda_x(n)}{n^{\sigma_1+it}} \chi(n)\right| \leq \sum_{n<x} \frac{\Lambda(n)}{n^{\frac{1}{2}}} + \sum_{x \leq n \leq x^2} \frac{\Lambda(n) \log \frac{x^2}{n}}{n^{\frac{1}{2}}} \cdot \frac{1}{\log x} \ll \frac{x}{\log x}.$$

Similarly,

$$\left|\sum_{n<x^2} \frac{\Lambda_x(n)}{n^{\sigma_1+it} \log n} \chi(n)\right| \ll \frac{x}{(\log x)^2}.$$

So, we see

$$|M_1| \leq \frac{\left(\frac{1}{e} + \frac{1}{2e^2}\right)}{1 - \frac{1}{e}\left(1 + \frac{1}{e}\right)} \cdot \frac{1}{2} \cdot \frac{\log q(t+1)}{\log x} + O\left(\frac{x}{(\log x)^2}\right),$$

$$|M_2| \leq \frac{\left(\frac{1}{e} + \frac{1}{e^2}\right)}{1 - \frac{1}{e}\left(1 + \frac{1}{e}\right)} \cdot \frac{1}{2} \cdot \frac{\log q(t+1)}{\log x} + O\left(\frac{x}{(\log x)^2}\right),$$

and

$$|M_3| \leq \eta_4(t) + O\left(\frac{x}{(\log x)^2}\right).$$

For $\eta_1(t)$, $\eta_2(t)$, $\eta_3(t)$ and $\eta_4(t)$, taking $x = \log q(t+3)\sqrt{\log q(t+3)}$ we have

$$\begin{aligned} |S(t,\chi)| &< \frac{1}{\pi} \cdot \frac{1}{1 - \frac{1}{e}\left(1 + \frac{1}{e}\right)} \left\{ \frac{\left(\frac{1}{e} + \frac{1}{2e^2}\right)}{2} + \frac{\left(\frac{1}{e} + \frac{1}{e^2}\right)}{2} + \frac{\pi}{4} \right\} \frac{\log q(t+1)}{\log x} \\ &\quad + O\left(\frac{x}{(\log x)^2}\right) \\ &= 0.803986\cdots \frac{\log q(t+1)}{\log\log q(t+3)} + O\left(\frac{\log q(t+3)}{(\log\log q(t+3))^2}\right). \end{aligned}$$

Therefore we obtain the theorem. □

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