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A TRANSCENDENCE CRITERION WITH p-ADIC CONTINUED FRACTIONS

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ABSTRACT. In this paper, we give a new transcendence criterion for p-adic continued fractions which are called Schneider continued fractions.

1. INTRODUCTION

In this paper, we study transcendental p-adic continued fractions. Each of Schneider [8] and Ruban [7] gave an algorithm of p-adic continued fraction expansions. We only deal with Schneider's continued fraction expansion.

The paper is organized as follows. In Section 2, we state a main theorem and give examples of transcendental Schneider continued fractions. Section 3 presents some preliminaries for a proof of a main theorem. We prove the main theorem in Section 4.

Let p be a prime. For $x \in p\mathbb{Z}_p$, a function T_p is defined by

$$T_p(x) = \begin{cases} p^{\mathrm{ord}_p(x)}/x - a & (x \neq 0), \\ 0 & (x = 0), \end{cases}$$

where $a \in \{1, 2, \ldots, p-1\}$ is uniquely chosen such that $|p^{\mathrm{ord}_p(x)}/x - a|_p < 1$. For $\xi \in \mathbb{Q}_p$, there exist a unique $\xi_1 \in p\mathbb{Z}_p$ and $a_0 \in \{1, \ldots, p-1\}$ such that $\xi = p^{b_0}(a_0 + \xi_1)$ where $b_0 = \mathrm{ord}_p(\xi)$. Applying the function T_p to ξ_1 repeatedly, we deduced that

$$\xi = p^{b_0} \left(a_0 + \cfrac{p^{b_1}}{a_1 + \cfrac{p^{b_2}}{a_2 + \cdots}} \right) \tag{1}$$

where $a_n \in \{1, \ldots, p-1\}, b_n \in \mathbb{Z}_{>0}$ for $n \geq 1$. (1) is called a Schneider continued fraction. To simplify notation, we write the continued fraction (1)

$$\xi = p^{b_0} \left(a_0 + \frac{p^{b_1}|}{|a_1} + \frac{p^{b_2}|}{|a_2} + \cdots \right).$$

The p-adic number ξ is called ultimately periodic if there exist an non-negative integer N and a positive integer ℓ such that $a_n = a_{n+\ell}, b_n = b_{n+\ell}$ for all $n \geq N$.

In the rest of this section, we introduce known results related to transcendental Schneider continued fraction.

For $P(X) \in \mathbb{Z}[X]$, a height of $P(X)$, denoted by $H(P)$, is defined to be the maximum absolute value of coefficients of $P(X)$. For an algebraic number $\alpha \in \mathbb{Q}_p$, the height of α, denoted by $H(\alpha)$, is defined to be a height of the minimal polynomial of α over $\mathbb{Z}$. We define Mahler's exponent w_2 (*resp.* Koksma's exponent w_2^*) at a

p-adic number $\xi \in \mathbb{Q}_p$ by the supremum of a real number w (*resp.* w^*) which satisfy

$$0 < |P(\xi)|_p \leq H(P)^{-w-1} \quad (\textit{resp. } 0 < |\xi - \alpha|_p \leq H(\alpha)^{-w^*-1})$$

for infinitely many $P(X) \in \mathbb{Z}[X]$ with $\deg P \leq 2$ (*resp.* algebraic number $\alpha \in \mathbb{Q}_p$ with $\deg \alpha \leq 2$). It is known that for a p-adic number ξ, if $w_2^*(\xi) > 2$, then ξ is transcendental. The detail will appear in [2, Section 9.3]. Bugeaud and Pejković [3] constructed uncountably many p-adic numbers ξ for which $w_2(\xi) - w_2^*(\xi) = 1$.

Theorem 1.1. *Let $w > (5+\sqrt{17})/2$ be a real number, b be a positive integer and $(\varepsilon_i)_{i\geq 0}$ be a sequence taking its value in the set $\{0,1\}$. The sequence $(b_{n,w})_{n\geq 1}$ is defined by*

$$b_{n,w} = \begin{cases} b+3i+2 & \text{if } n = \lfloor w^i \rfloor \text{ for some } i \in \mathbb{Z}_{\geq 0}, \\ b+3i+\varepsilon_i & \text{if } \lfloor w^i \rfloor < n < \lfloor w^{i+1} \rfloor \text{ for some } i \in \mathbb{Z}_{\geq 0}. \end{cases}$$

Set

$$\xi_w = 1 + \frac{p^{b_{1,w}}|}{|1} + \frac{p^{b_{2,w}}|}{|1} + \cdots .$$

Then, we have

$$w_2^*(\xi_w) = w - 1, \quad w_2(\xi_w) = w.$$

In particular, ξ_w is transcendental.

Laohakosol and Ubolsri [5] studied an algebraic independence of Schneider continued fractions.

Theorem 1.2. *Let $n \geq 2$ be an integer. Consider Schneider continued fractions*

$$\xi_i = \frac{p^{b_{1,i}}|}{|a_{1,i}} + \frac{p^{b_{2,i}}|}{|a_{2,i}} + \cdots \quad (1 \leq i \leq n).$$

Assume that there exist real numbers $\tau, r > 1$, a function $g(j)$ for $j \in \mathbb{Z}_{>0}$ with $g(j) \to \infty$ $(j \to \infty)$ and a subsequence of positive integers $N_1 < N_2 < \cdots$ such that for $2 \leq i \leq n, N \in \mathbb{Z}_{\geq 0}, j \in \mathbb{Z}_{>0}$,

$$p^{b_{N,1}} \geq p^{\tau^k b_{N-k,1}} \ (1 \leq k \leq N),$$
$$p^{b_{N,i-1}} \geq r p^{b_{N,i}},$$
$$p^{b_{N_j,i}} \geq p^{g(j) b_{N_j-1,1}}.$$

Then, $\xi_1, \ldots, \xi_n$ are algebraically independent. In particular, $\xi_1, \ldots, \xi_n$ are transcendental.

2. Main result

Let $\mathbf{n} = (n_i)_{i\geq 0}, \boldsymbol{\lambda} = (\lambda_i)_{i\geq 0}$ and $\mathbf{k} = (k_i)_{i\geq 0}$ be sequences of positive integers. Assume that for all $i \geq 0$,

$$n_{i+1} \geq n_i + \lambda_i k_i.$$

We call a p-adic number $\xi \in \mathbb{Q}_p$ of the following form quasi-periodic Schneider continued fraction with respect to $(\mathbf{n}, \boldsymbol{\lambda}, \mathbf{k})$:

$$\begin{aligned}\xi =& p^{b_0}\Bigg(a_0 + \frac{p^{b_1}}{a_1} + \cdots + \frac{p^{b_{n_0-1}}}{a_{n_0-1}} + \overline{\frac{p^{b_{n_0}}}{a_{n_0}} + \cdots + \frac{p^{b_{n_0+k_0-1}}}{a_{n_0+k_0-1}}}^{\lambda_0} \\ &+ \frac{p^{b_{n_0+\lambda_0 k_0}}}{a_{n_0+\lambda_0 k_0}} + \cdots + \frac{p^{b_{n_1-1}}}{a_{n_1-1}} + \overline{\frac{p^{b_{n_1}}}{a_{n_1}} + \cdots + \frac{p^{b_{n_1+k_1-1}}}{a_{n_1+k_1-1}}}^{\lambda_1} + \cdots\Bigg)\end{aligned}$$

where the λ's indicate the number of times a block of partial quotients and partial numerators is repeated.

Theorem 2.1. *Let $\mathbf{n} = (n_i)_{i\geq 0}, \boldsymbol{\lambda} = (\lambda_i)_{i\geq 0}$, and $\mathbf{k} = (k_i)_{i\geq 0}$ be as in the above. Let ξ be an irrational quasi-periodic Schneider continued fraction with respect to $(\mathbf{n}, \boldsymbol{\lambda}, \mathbf{k})$ and b be a positive integer. Assume that $b_i \leq b$ for all $i \geq 0$ and $a_{n_i} = \cdots = a_{n_i+k_i-1} = p-1, b_{n_i} = \cdots = b_{n_i+k_i-1} = 1$ for infinitely many $i \geq 0$. If*

$$\liminf_{i\to\infty} \frac{\lambda_i}{n_i} > \frac{2\log\left(\frac{p-1+\sqrt{(p-1)^2+4p^b}}{2}\right)}{\log p} - 1,$$

then ξ is transcendental.

Theorem 2.1 is a p-adic analogue of transcendental criterion of Baker [1, Theorem 2]. The author [9, Theorem 1.1] proved an analogue of Theorem 2.1 for Ruban continued fractions.

For example, the following p-adic numbers are transcendental:

$$1 + \overline{\frac{p}{p-1}}^{4^1} + \overline{\frac{p}{1}}^{4^2} + \overline{\frac{p}{p-1}}^{4^3} + \overline{\frac{p}{1}}^{4^4} + \cdots + \overline{\frac{p}{p-1}}^{4^{2m+1}} + \overline{\frac{p}{1}}^{4^{2m}} + \cdots,$$

$$\begin{aligned}1 &+ \overline{\frac{p}{p-1} + \frac{p}{p-1}}^{8\cdot 17^0} + \overline{\frac{p}{1} + \frac{p^2}{1}}^{8\cdot 17^1} + \overline{\frac{p}{p-1} + \frac{p}{p-1}}^{8\cdot 17^2} \\ &+ \overline{\frac{p}{1} + \frac{p^2}{1}}^{8\cdot 17^3} + \cdots + \overline{\frac{p}{p-1} + \frac{p}{p-1}}^{8\cdot 17^{2m}} + \overline{\frac{p}{1} + \frac{p^2}{1}}^{8\cdot 17^{2m+1}} + \cdots.\end{aligned}$$

The first number is the case that $\lambda_i = 4^{i+1}, k_i = 1, n_i = (4^{i+1}-1)/3, a_0 = 1, b_0 = 0, a_{n_{2i+1}} = b_{n_{2i}} = 1, b_{n_{2i+1}} = 2, a_{n_{2i}} = p-1$ for $i \geq 0$, and the second number is the case that $\lambda_i = 8 \cdot 17^i, k_i = 2, n_i = 17^i, a_0 = 1, b_0 = 0, a_{n_{2i+1}} = a_{n_{2i+1}+1} = b_{n_{2i}} = b_{n_{2i}+1} = b_{n_{2i+1}} = 1, b_{n_{2i+1}+1} = 2, a_{n_{2i}} = a_{n_{2i}+1} = p-1$ for $i \geq 0$ in Theorem 2.1. It seems that above numbers are the first examples of transcendental Schneider continued fractions with bounded partial numerators.

3. Preliminaries

Let $(a_n)_{n\geq 0}$ be a sequence with $a_n \in \{1, \ldots, p-1\}$ for all $n \geq 0$ and $(b_n)_{n\geq 0}$ be an integer sequence with $b_n \geq 1$ for all $n \geq 1$. We define sequences $(p_n)_{n\geq -1}, (q_n)_{n\geq -1}$ by

$$\begin{cases} p_{-1} = p^{b_0}, p_0 = p^{b_0}a_0, p_n = a_n p_{n-1} + p^{b_n}p_{n-2}, & n \geq 1, \\ q_{-1} = 0, q_0 = 1, q_n = a_n q_{n-1} + p^{b_n}q_{n-2}, & n \geq 1. \end{cases}$$

Set

$$\xi = p^{b_0}\left(a_0 + \frac{p^{b_1}|}{|a_1} + \frac{p^{b_2}|}{|a_2} + \cdots\right).$$

We call p_n/q_n the n-th convergent to ξ.

Lemma 3.1. *Let x be a variable. Then, the following equalities hold:*

(2) $$\frac{p_n}{q_n} = p^{b_0}\left(a_0 + \frac{p^{b_1}|}{|a_1} + \frac{p^{b_2}|}{|a_2} + \cdots + \frac{p^{b_n}|}{|a_n}\right) \quad (n \geq 0),$$

(3) $$p_n q_{n-1} - p_{n-1} q_n = (-1)^{n+1} p^{\sum_{i=0}^{n} b_i} \quad (n \geq 0),$$

(4) $$p_n q_{n-2} - p_{n-2} q_n = (-1)^n p^{\sum_{i=0}^{n-1} b_i} a_n \quad (n \geq 1),$$

(5) $$|p_n|_p = p^{-b_0} \ (n \geq -1), \ |q_n|_p = 1 \quad (n \geq 0),$$

(6) $$p^{b_0}\left(a_0 + \frac{p^{b_1}|}{|a_1} + \frac{p^{b_2}|}{|a_2} + \cdots + \frac{p^{b_{n-1}}|}{|a_{n-1}} + \frac{p^{b_n}|}{|x}\right) = \frac{x p_{n-1} + p^{b_n} p_{n-2}}{x q_{n-1} + p^{b_n} q_{n-2}} \quad (n \geq 1),$$

(7) $$\left|\xi - \frac{p_n}{q_n}\right|_p = p^{-\sum_{i=0}^{n+1} b_i} \quad (n \geq 0).$$

Proof. A induction shows (2)-(6). It is clear that (7) for $n = 0$. Set

$$\xi_n = a_n + \frac{p^{b_{n+1}}|}{|a_{n+1}} + \frac{p^{b_{n+2}}|}{|a_{n+2}} + \cdots \ (n \geq 1).$$

For $n \geq 1$, we have

$$\begin{aligned}\left|\xi - \frac{p_n}{q_n}\right|_p &= \left|\frac{\xi_n(p_n q_{n-1} - p_{n-1} q_n) + p^{b_n}(p_n q_{n-2} - p_{n-2} q_n)}{q_n(\xi_n q_{n-1} + p^{b_n} q_{n-2})}\right|_p \\ &= p^{-\sum_{i=0}^{n} b_i} |\xi_n - a_n|_p = p^{-\sum_{i=0}^{n+1} b_i}.\end{aligned}$$

Hence, we obtain (7). □

Lemma 3.2. *Consider a Schneider continued fraction*

$$\xi' = p^{b_0'}\left(a_0' + \frac{p^{b_1'}|}{|a_1'} + \frac{p^{b_2'}|}{|a_2'} + \cdots\right).$$

If $a_i = a_i', b_i = b_i'$ for all $0 \leq i \leq n$, then we have

$$|\xi - \xi'|_p \leq p^{-\sum_{k=0}^{n} b_k - \min(b_{n+1}, b'_{n+1})}.$$

Proof. Since p_n/q_n is the n-th convergent to both ξ and ξ', we have

$$|\xi - \xi'|_p \leq \max\left(\left|\xi - \frac{p_n}{q_n}\right|_p, \left|\xi' - \frac{p_n}{q_n}\right|_p\right) \leq p^{-\sum_{k=0}^{n} b_k - \min(b_{n+1}, b'_{n+1})}$$

by Lemma 3.1 (7). □

Proposition 3.3. (Bundschuh [4]) *Let η be a p-adic number. Then, η is rational if and only if its Schneider continued fraction expansion is finite or ultimately periodic with the period $\frac{p|}{|p-1}$. In particular, we have*

$$p - 1 + \overline{\frac{p|}{|p-1}} = -1.$$

Throughout this section, we assume that $b_0 = 0$.

Lemma 3.4. *For $h \in \mathbb{Z}_{\geq 1}$, we define a rational number η_h by*

$$\eta_h = a_0 + \frac{p^{b_1}}{\lfloor a_1} + \cdots + \frac{p^{b_h}}{\lfloor a_h} + \frac{p}{\lfloor p-1}$$

Then, we have $H(\eta_h) \leq \max(p_h, pp_{h-1})$.

Proof. By Lemma 3.1 and Proposition 3.3 , we have

$$\eta_h = a_0 + \frac{p^{b_1}}{\lfloor a_1} + \cdots + \frac{p^{b_h}}{\lfloor a_h - p} = \frac{p_h - pp_{h-1}}{q_h - pq_{h-1}}.$$

It is seen that $q_n \leq p_n$ for $n \geq -1$ by induction on n. Hence, we get

$$H(\eta_h) \leq |p_h - pp_{h-1}| \leq \max(p_h, pp_{h-1}).$$

□

Lemma 3.5. *Let b be a positive integer. Assume that $b_n \leq b$ for $n \geq 1$. Then, we have*

$$p_n \leq \left(\frac{p-1+\sqrt{(p-1)^2+4p^b}}{2}\right)^{n+1} \quad \text{for } n \geq -1. \tag{8}$$

Proof. Put

$$A = \frac{p-1+\sqrt{(p-1)^2+4p^b}}{2}.$$

The proof is an induction on n. Clearly, (8) holds for $n = -1, 0$. We assume $n > 0$. Then, we have

$$p_n \leq (p-1)p_{n-1} + p^b p_{n-2} \leq A^{n-1}((p-1)A + p^b) = A^{n+1}.$$

□

The proof of Theorem 2.1 deeply depends on the following theorem.

Theorem 3.6. (Ridout [6]) *Let α be an algebraic irrational p-adic number and δ be a positive number. Then there are only finitely many $\eta \in \mathbb{Q}$ with the solution of the following inequality:*

$$|\alpha - \eta|_p \leq \frac{1}{H(\eta)^{2+\delta}}.$$

4. Proof of Main Theorem

Without loss of generality, we can assume that $b_0 = 0$. Put

$$A = \frac{p-1+\sqrt{(p-1)^2+4p^b}}{2}, \quad B = \frac{2\log A}{\log p} - 1.$$

Let Λ be an infinite set of positive integers $i \geq 1$ which satisfy

$$a_{n_i} = a_{n_i+1} = \cdots = a_{n_i+k_i-1} = p-1, \quad b_{n_i} = b_{n_i+1} = \cdots = b_{n_i+k_i-1} = 1.$$

Let $\eta^{(i)}$ be the Schneider continued fraction η_{n_i-1} of Lemma 3.4 for $i \in \Lambda$. We have $\eta^{(i)}$ is rational and $H(\eta^{(i)}) \leq A^{n_i}$ for $i \in \Lambda$ by Lemma 3.4 and 3.5. Assume that ξ is an algebraic irrational number. Put $\chi > 2$. Using Theorem 3.6, we can show that

$$|\xi - \eta^{(i)}|_p > A^{-\chi n_i}$$

for sufficiently large $i \in \Lambda$. By Lemma 3.2, we get

$$|\xi - \eta^{(i)}|_p \leq p^{-\sum_{j=1}^{n_i+\lambda_i k_i-1} b_j}$$

for $i \in \Lambda$. Therefore, we have

$$p^{\sum_{j=1}^{n_i+\lambda_i k_i-1} b_j} < A^{\chi n_i}$$

for sufficiently large $i \in \Lambda$. By the assumption, there exists $\delta > 0$ such that $\lambda_i > (B + \delta)n_i$ for sufficiently large i. A computation show that

$$2 + \frac{\log p}{\log A}\delta < \chi.$$

This completes the proof.

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