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Arithmetical properties of p-adic numbers related to numerical systems

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Abstract

In this paper we investigate lower bounds for the numbers of nonzero digits of p-adic algebraic numbers. As a consequence, we introduce criteria for the transcendence of certain power series and we give new examples of transcendental numbers.

1 Introduction

Many mathematicians have studied the transcendence of the values of analytic functions $f(z)$ at algebraic points. In this paper we consider the case where

$$f(z)=\sum_{n=0}^{\infty} s_n z^n, \tag{1.1}$$

where $(s_n)_{n=0,1,\ldots}$ is a bounded sequence of nonnegative integers. We denote by $\mathcal{M}(\mathbb{Q})=\{\infty\}\cup\{p \mid p \text{ is prime}\}$ the set of the places of $\mathbb{Q}$. Let $v\in\mathcal{M}(\mathbb{Q})$. We denote by $\mathbb{C}_v$ a completion of an algebraic closure of $\mathbb{Q}_v$. If $v=\infty$, then let $|\cdot|_\infty$ be the usual Euclidean norm on $\mathbb{C}$. In the case where $v=p$ is a prime, $|\cdot|_p$ denotes the p-adic norm on $\mathbb{C}_p$ normalized so that $|p|_p=p^{-1}$.

Let $v\in\mathcal{M}(\mathbb{Q})$ be fixed. Let $(v_m)_{m=0,1,\ldots}$ be a sequence of nonnegative integers satisfying

$$v_{m+1}>v_m \tag{1.2}$$

for any sufficiently large m. Then the power series $\sum_{m=0}^{\infty} z^{v_m}$ is rewritten as (1.1), where $s_n\in\{0,1\}$ for any sufficiently large n. Corvaja and Zannier [9] proved for any algebraic number α with $0<|\alpha|_v<1$ that if

$$\liminf_{m\to\infty}\frac{v_{m+1}}{v_m}>1, \tag{1.3}$$

then $\sum_{m=0}^{\infty}\alpha^{v_m}$ is transcendental. However, the transcendence of $\sum_{m=0}^{\infty}\alpha^{v_m}$ is generally unknown if (1.3) does not hold. If $v=\infty$ and $\alpha=1/2$, then Bailey, Borwein, Crandall, and Pomerance [4] improved the criteria by Corvaja and Zannier as follows: If

$$\limsup_{m\to\infty}\frac{v_m}{m^R}=\infty \tag{1.4}$$

for any positive real number R, then $\sum_{m=0}^{\infty} 2^{-v_m}$ is transcendental. The result above is an application of the results on the binary expansions of algebraic irrational numbers.

In Section 2 we review known results on the β-expansions of algebraic numbers in the case where β is a Pisot or Salem number. In particular, we introduce criteria for the transcendence of power series in [11]. In Section 3 we investigate the digits of p-adic algebraic numbers. Consequently, we deduce criteria for the transcendence of the p-adic numbers, which gives new examples of transcendental numbers.

2 Criteria for transcendence related to β-expansion

In this section we denote the integral and fractional parts of a real number x by $\{x\}$ and $\lfloor x \rfloor$, respectively. Let β be a real number greater than 1 and $T_\beta : [0,1] \to [0,1)$ the β-transformation defined by

$$T_\beta(x) = \{\beta x\}.$$

Then the β-expansion of a real number $\xi \in [0,1]$ is given by

$$\xi = \sum_{n=1}^{\infty} t_n(\beta;\xi)\beta^{-n},$$

where $t_n(\beta;\xi) = \lfloor \beta T_\beta^{n-1}(\xi) \rfloor$ for any positive integer n.

If $\beta = b$ is a rational integer, then the β-expansion of ξ coincides with the usual base-b expansion of ξ. In particular, the sequence $(s_n(b;\xi))_{n=1,2,\ldots}$ is ultimately periodic for any rational number $\xi \in [0,1]$. Now we recall the definition of Pisot and Salem numbers. Let $\beta > 1$ be an algebraic integer with conjugates $\beta_1 = \beta, \beta_2, \ldots, \beta_d$. Then β is a Pisot number if $|\beta_i| < 1$ for any i with $i \geq 2$. In particular, any rational integer $b \geq 2$ is a Pisot number. On the other hand, β is a Salem number if $|\beta_i| \leq 1$ for any i with $2 \leq i \leq d$ and if there exists j with $2 \leq j \leq d$ such that $|\beta_j| = 1$. Schmidt [13] proved that if the β-expansion of each rational number $\xi \in [0,1)$ is ultimately periodic, then β is a Pisot or Salem number. In the rest of this section, we assume that β is a Pisot or Salem number. Bertrand [5] and Schmidt [13] proved for any Pisot number ξ that any number $\xi \in \mathbb{Q}(\beta) \cap [0,1]$ has the ultimately periodic β-expansion. However, the periodicity of the β-expansion of rational numbers of ξ is generally not known in the case where β is a Salem number.

In this section we study the numbers of nonzero digits in the β-expansions of algebraic numbers $\xi \leq 1$. Let

$$\nu(\beta,\xi;N) := \mathrm{Card}\{n \in \mathbb{Z} \mid 1 \leq n \leq N, t_n(\beta;\xi) \neq 0\}$$

for any positive integer N. It is generally difficult to give the asymptotic behavior of the function $\nu(\beta,\xi;N)$ for fixed $\beta > 1$ and $\xi \leq 1$. For instance, consider the case where $\beta = b$ is a rational integer. Borel [6] conjectured that any algebraic irrational number is normal. If Borel's conjecture is true, then we have

$$\lim_{N\to\infty} \frac{\nu(b,\xi;N)}{N} = \frac{b-1}{b} (> 0).$$

However, we know no example of a base $b \geq 2$ and an algebraic irrational number $\xi \leq 1$ such that the inequality

$$\limsup_{N\to\infty} \frac{\nu(b,\xi;N)}{N} > 0$$

has been proved.

In what follows, we introduce known results on the lower bounds for $\nu(\beta,\xi;N)$. Unless otherwise specified, "$\nu(\beta,\xi;N) \gg f(N)$ for any sufficiently large N" implies that there exist effectively computable positive constants $C(\beta,\xi)$ and $C'(\beta,\xi)$ depending only on β and ξ satisfying

$$\nu(\beta,\xi;N) \geq C(\beta,\xi)f(N)$$

for any integer $N \geq C'(\beta,\xi)$. Bugeaud [8] gave lower bounds for the digit changes in the β-expansions of algebraic numbers β and ξ. In the case where β is a Pisot or Salem number, using his results, we get the following: Let $\xi \in [0,1]$ be an algebraic number. Suppose that $t_n(\beta;\xi) \neq t_{n+1}(\beta;\xi)$ for infinitely many n's. Then

$$\nu(\beta,\xi;N) \gg \frac{(\log N)^{3/2}}{(\log\log N)^{1/2}} \tag{2.1}$$

for any sufficiently large N.

In the case where $\beta = 2$, then Bailey, Borwein, Crandall and Pomerance [4] gave better lower bounds as follows: For any algebraic irrational number ξ of degree D, we have

$$\nu(2,\xi;N) \geq C_1(\xi)N^{1/D} \tag{2.2}$$

for any integer $N \geq C_2(\xi)$, where $C_1(\xi)$ is an effective positive constant and C_2 is an ineffective positive constant. $C_1(\xi)$ and $C_2(\xi)$ depend only on ξ.

Consequently, we deduce the criteria for transcendence we mentioned in Section 1. In fact, let $(v_m)_{m=0,1,\ldots}$ be a sequence of nonnegative integers satisfying (1.2) for any sufficiently large integer m and (1.4) for any positive real number R. Put $\xi_1 := \sum_{m=0}^{\infty} 2^{-v_m}$. Then we have, for an arbitrary positive real number ε,

$$\liminf_{N\to\infty} \frac{\nu(2,\xi_1;N)}{N^{\varepsilon}} = 0.$$

In particular, (2.2) does not hold for any positive integer D. Hence, ξ_1 is transcendental.

Adamczewski, Faverjon [1] and Bugeaud [8] independently gave effective versions of (2.2) for any integral base as follows: Let $\beta = b \geq 2$ be any integer and $\xi \in [0,1]$ an irrational number of degree D. Then

$$\nu(b,\xi;N) \gg N^{1/D} \tag{2.3}$$

for any sufficiently large N.

In the case where β is a general Pisot or Salem number, then an analogy of (2.2) and (2.3) is obtained by the following criteria in [11]: Let B be a positive integer and $(s_n)_{n=0,1,\ldots}$ a sequence of integers such that

$$0 \leq s_n \leq B$$

for any nonnegative integer n. Put

$$\xi := \sum_{n=0}^{\infty} s_n \beta^{-n} \tag{2.4}$$

and

$$\widetilde{\nu}(N) := \mathrm{Card}\{n \in \mathbb{Z} \mid 0 \le n \le N, s_n \ne 0\}.$$

Note that (2.4) is not generally the β-expansion of ξ. Suppose that ξ is an algebraic number satisfying $[\mathbb{Q}(\beta,\xi) : \mathbb{Q}(\beta)] = D$, where $[L : K]$ denotes the degree of a field extension K/L. Then there exist effectively computable positive constants $C_3(\beta,\xi,B)$ and $C_4(\beta,\xi,B)$ depending only on β,ξ, and B such that

$$\widetilde{\nu}(N) \ge C_3(\beta,\xi,B)\frac{N^{1/(2D-1)}}{(\log N)^{1/(2D-1)}} \tag{2.5}$$

for any integer N with $N \ge C_4(\beta,\xi,B)$. In particular, applying the case where (2.4) is the β-expansion of ξ, we have

$$\nu(\beta,\xi;N) \gg \frac{N^{1/(2D-1)}}{(\log N)^{1/(2D-1)}}$$

for any sufficiently large N.

Moreover, we obtain criteria for the transcendence of power series as follows: Let again $(v_m)_{m=0,1,\ldots}$ be a sequence of nonnegative integers fulfilling (1.2) and (1.4). Then

$$\xi_2 := \sum_{m=0}^{\infty} \beta^{-v_m} \tag{2.6}$$

is transcendental. Note that (2.6) is not also generally the β-expansion of ξ_2. In the case where $\beta = b$ is a rational integer, then the criteria above were essentially obtained by Bailey, Borwein, Crandall, and Pomerance [4]. The main purpose of this paper is to give an analogy of (2.5) for p-adic algebraic numbers.

We give examples of transcendental numbers. Put

$$w_m := \lfloor m^{\log m} \rfloor = \lfloor \exp\left((\log m)^2\right) \rfloor \tag{2.7}$$

for any $m \ge 1$ and

$$x_m := \lfloor m^{\log\log m} \rfloor = \lfloor \exp\left((\log m)(\log\log m)\right) \rfloor \tag{2.8}$$

for any $m \ge 3$. Then we have

$$\lim_{m\to\infty} \frac{w_m}{m^R} = \infty, \lim_{m\to\infty} \frac{x_m}{m^R} = \infty$$

for any positive real number R. Hence,

$$y_1 := \sum_{m=1}^{\infty} \beta^{-w_m}, y_2 := \sum_{m=3}^{\infty} \beta^{-x_m}$$

are transcendental for any Pisot or Salem number β. Note that $(w_m)_{m=1,2,\dots}$ and $(x_m)_{m=3,4,\dots}$ do not satisfy (1.3) because

$$\lim_{m\to\infty}\frac{w_{m+1}}{w_m}=1,\ \lim_{m\to\infty}\frac{x_{m+1}}{x_m}=1.$$

Therefore, the transcendence of y_1 and y_2 is not obtained by the criteria by Corvaja and Zannier in Section 1.

3 Review of p-adic normal numbers

Let b be an integer greater than 1 and $\xi\le 1$ a real number. Recall that ξ is normal in base-b if the following property holds: Let L be any positive integer and v any word from the alphabet $\{0,1,\dots,b-1\}$ with length L. Then v appears in the base-b expansion of ξ with average frequency tending to b^{-L}, namely, we have

$$\lim_{N\to\infty}\frac{\mathrm{Card}\{n\mid 1\le n\le N, v=t_n(b;\xi)\dots t_{n+L-1}(b;\xi)\}}{N}=\frac{1}{b^L}.$$

Ferrero and Washington [10] introduced the notion of joint normality of p-adic integers, where p is a prime number. For any $\xi\in\mathbb{Z}_p$, we denote the p-adic expansion by

$$\xi=\sum_{n=0}^{\infty}t_n^{(p)}(\xi)p^n,$$

where $t_n^{(p)}(\xi)\in\{0,1,\dots,p-1\}$ for any nonnegative integer n. For simplicity, put

$$\boldsymbol{t}_{n,M}^{(p)}(\xi):=t_n^{(p)}(\xi)t_{n+1}^{(p)}(\xi)\dots t_{n+M-1}^{(p)}(\xi)$$

for any integers $n\ge 0$ and $M\ge 1$. In what follows, put

$$\mathcal{A}:=\{0,1,\dots,p-1\}.$$

Let $\xi_1,\dots,\xi_r\in\mathbb{Z}_p$. We say that $\xi_1,\dots,\xi_r$ are jointly normal if the following property holds: Let L be any positive integer and $v_1,\dots,v_r$ be any words from the alphabet $\mathcal{A}$ with length L. Then we have

$$\lim_{N\to\infty}\frac{\mathrm{Card}\{n\mid 1\le n\le N, v_i=\boldsymbol{t}_{n,L}^{(p)}(\xi_i)\text{ for any }i=1,\dots,r\}}{N}=\frac{1}{p^{rL}}.$$

In the case of $r=1$, if ξ is jointly normal, then we call ξ normal. It is remarkable that the notion of joint normality is applicable to Iwasawa theory. In fact, Ferrero and Washington [10] used certain jointly normal numbers to verify that the Iwasawa invariant $\mu_p(k)$ vanishes for any abelian number field k (see also [15]).

Moreover, Anglès [3] proposed a problem on the transcendence of the Iwasawa power series modulo certain uniformization of $\mathbb{Z}_p[\theta]$ over $\overline{\mathbb{F}_p}(T)$, where $\overline{\mathbb{F}_p}$ is an algebraic closure of the finite field with p elements. Sun [14] proved the transcendence on this problem under the assumption that Borel's conjecture on the normality of p-adic algebraic irrational number $\xi \in \mathbb{Z}_p$ holds.

Borel's conjecture of p-adic version implies for each prime number p that any algebraic irrational number $\xi \in \mathbb{Z}_p$ is normal. However, this conjecture has not also been proved. For any $\xi \in \mathbb{Z}_p$, we write the complexity function of the sequence $(t_n^{(p)}(\xi))_{n=0,1,\ldots}$ by

$$P(\xi; N) := \mathrm{Card}\{\boldsymbol{t}_{n,N}^{(p)}(\xi) \mid n \in \mathbb{Z}, n \geq 0\} \ (N = 1, 2, \ldots).$$

If ξ is normal, then we have $P(\xi; N) = p^N$ for any positive integer N. Adamczewski and Bugeaud [2] verified for any algebraic irrational number $\xi \in \mathbb{Z}_p$ that

$$\lim_{N\to\infty} \frac{P(\xi; N)}{N} = \infty. \tag{3.1}$$

In the same paper, they also investigated certain criteria for the transcendence of p-adic integers. For any finite word $W = w_1 \ldots w_l$ and any positive real number x, set

$$|W| := l$$

and

$$W^x := \underbrace{WW \ldots W}_{\lfloor x \rfloor} W',$$

where W' is the prefix of W with length $\lceil \{x\}|W| \rceil$. We say for any $\xi \in \mathbb{Z}_p$ that ξ satisfies Condition A if $(t_n^{(p)}(\xi))_{n=0,1,\ldots}$ is not ultimately periodic and if there exists a real number $w > 1$ satisfying the following: there exist two sequences of finite words $(U_n)_{n=1,2,\ldots}$ and $(V_n)_{n=1,2,\ldots}$ from the alphabet $\mathcal{A}$ such that

1. $U_n V_n^w$ is a prefix of the sequence $(t_n^{(p)}(\xi))_{n=0,1,\ldots}$;
2. $(|U_n|/|V_n|)_{n=1,2,\ldots}$ is bounded;
3. $(|V_n|)_{n=1,2,\ldots}$ is strictly increasing.

Adamczewski and Bugeaud [2] proved for any $\xi \in \mathbb{Z}_p$ that if ξ satisfies Condition A, then ξ is transcendental. Finally, we note that (3.1) and the above criteria for transcendence are analogies of β-expansions of real numbers by a Pisot or Salem number β in [2].

4 Criteria for transcendence related to p-adic expansion

In this section, p and d denote a fixed prime number and a fixed positive integer, respectively. Put $L := \mathbb{Q}_p(p^{1/d})$. Then the extension $L/\mathbb{Q}_p$ is totally ramified of degree d and the ring of integers of L is $\mathcal{O} = \mathbb{Z}_p(p^{1/d})$. For any $\xi \in \mathcal{O}$, there exist a unique expansion

$$\xi = \sum_{n=0}^{\infty} t_n^{(p)}(\xi) p^{n/d},$$

where $t_n^{(p)}(\xi) \in \mathcal{A} = \{0, 1, \ldots, p-1\}$ for any $n \geq 0$. Set

$$\nu^{(p)}(\xi; N) := \mathrm{Card}\{n \in \mathbb{Z} \mid 0 \leq n \leq N, t_n^{(p)}(\xi) \neq 0\}.$$

We introduce an analogy of the lower bounds for $\nu(\beta, \xi; N)$ in Section 2 as follows:

THEOREM 4.1. *Let $\xi \in \mathcal{O}$ be an algebraic number of degree D. Assume that $t_n^{(p)}(\xi) \neq 0$ for infinitely many n's. Then there exist effectively computable positive constants $C_5(p, d, \xi)$ and $C_6(p, d, \xi)$ depending only on p, d and ξ such that*

$$\nu^{(p)}(\xi; N) \geq C_5(p, d, \xi) N^{1/D}$$

for any integer N with $N \geq C_6(p, d, \xi)$.

Applying Theorem 4.1, we obtain criteria for transcendence as follows: Let $\xi \in \mathcal{O}$. Suppose that $t_n^{(p)}(\xi) \neq 0$ for infinitely many n and that

$$\liminf_{N \to \infty} \frac{\nu^{(p)}(\xi; N)}{N^R} = 0$$

for any positive real number R. Then ξ is transcendental.

We give new examples of transcendental numbers in $\mathcal{O}$. Recall that $(w_m)_{m=1,2,\ldots}$ and $(x_m)_{m=3,4,\ldots}$ are defined by (2.7) and (2.8), respectively. Then

$$\sum_{m=1}^{\infty} p^{w_m/d}, \sum_{m=3}^{\infty} p^{x_m/d}$$

are transcendental.

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