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TRANSFORMATION FORMULAE AND ASYMPTOTIC EXPANSIONS FOR DOUBLE HOLOMORPHIC EISENSTEIN SERIES OF TWO COMPLEX VARIABLES (SUMMARIZED VERSION)

MASANORI KATSURADA, DEPT. MATH., FAC. ECON., KEIO UNIV.
(慶應義塾大学・経済学部・桂田 昌紀)
AND
TAKUMI NODA, DEPT. MATH., FAC. ENG., NIHON UNIV.
(日本大学・工学部・野田 工)

ABSTRACT. This is a summarized version of the forthcoming paper [10].

The main object of study in [10] is the double holomorphic Eisenstein series $\widetilde{\zeta_{\mathbb{Z}^2}}(\boldsymbol{s};\boldsymbol{z})$, defined by (1.10) with (1.9) below, having two complex variables $\boldsymbol{s}=(s_1,s_2)$ and two parameters $\boldsymbol{z}=(z_1,z_2)$ satisfying either $\boldsymbol{z}\in(\mathfrak{H}^+)^2$ or $\boldsymbol{z}\in(\mathfrak{H}^-)^2$, for which its transformation properties and asymptotic aspects are studied when the distance $|z_2-z_1|$ becomes small and large under certain natural settings on the movement of $\boldsymbol{z}\in(\mathfrak{H}^{\pm})^2$. Let $\varepsilon(w)$ be the signature defined by (1.2). We establish in [10] complete asymptotic expansions of $\widetilde{\zeta_{\mathbb{Z}^2}}(\boldsymbol{s};\boldsymbol{z})$ when $\boldsymbol{z}$ moves within either the poly-sector $(\mathfrak{H}^+)^2$ or $(\mathfrak{H}^-)^2$, so as that the 'pivotal' parameter η, defined by (1.7) with (1.2) and (1.5), tends to 0 through $|\arg\eta|<\pi/2$ in the *ascending* order of η (Theorem 1); this further leads us to show that 'counterpart' expansions exist for $\widetilde{\zeta_{\mathbb{Z}^2}}(\boldsymbol{s};\boldsymbol{z})$ in the *descending* order of η as $\eta\to\infty$ through $|\arg\eta|<\pi/2$ (Theorem 2). Our second main formula in Theorem 2 naturally reduces to various expressions of $\widetilde{\zeta_{\mathbb{Z}^2}}(\boldsymbol{s};\boldsymbol{z})$ (in finitely closed forms) at any integer lattice points $\boldsymbol{s}\in\mathbb{Z}^2$ (Corollaries 2.1–2.14). A major portion of these results reveals that specific values of $\widetilde{\zeta_{\mathbb{Z}^2}}(\boldsymbol{s};\boldsymbol{z})$ at $\boldsymbol{s}\in\mathbb{Z}^2$ are closely linked to Weierstraß' elliptic functions $\wp(w\mid 2\pi(1,\varepsilon(z_j)z_j))$ $(j=1,2)$, the classical Eisenstein series $\mathscr{S}_r(q_j)$ in (5.9), reformulated by Ramanujan [18], as well as the Jordan-Kronecker type functions $\phi_1(w;q_j)$ and $\phi_2(w;q_j)$ in (5.23), each defined with the distinct bases $q_j=e(\varepsilon(z_j)z_j)$ $(j=1,2)$, the latter two of which were extensively utilized by Ramanujan in the course of developing his theory of Eisenstein series, elliptic functions and theta functions (cf. [1][2][21]). As for the methods used in [10], crucial rôles are played by a class of Mellin-Barnes type integrals, manipulated with several properties of confluent hypergeometric functions.

1. INTRODUCTION

Throughout the present article, s and $\boldsymbol{s}=(s_1,s_2)$ are complex variables with $s=\sigma+it$ and $s_j=\sigma_j+it_j$ $(j=1,2)$, z and $\boldsymbol{z}=(z_1,z_2)$ complex parameters with $z=x+iy$ and $z_j=x_j+iy_j$ $(j=1,2)$, and $\widetilde{\mathbb{C}^{\times}}$ denotes the universal covering of $\mathbb{C}^{\times}=\mathbb{C}\setminus\{0\}$; note that $\arg w\in(-\infty,+\infty)$ is uniquely determined for any $w\in\widetilde{\mathbb{C}^{\times}}$. The principal leaves of the upper and lower half-planes are denoted respectively by

$$\mathfrak{H}^+=\{z\in\widetilde{\mathbb{C}^{\times}}\mid 0<\arg z<\pi\}\qquad\text{and}\qquad\mathfrak{H}^-=\{z\in\widetilde{\mathbb{C}^{\times}}\mid -\pi<\arg z<0\}.$$

We suppose throughout the article that $\boldsymbol{z}=(z_1,z_2)$ belongs to either $(\mathfrak{H}^+)^2$ or $(\mathfrak{H}^-)^2$, and that

$$\operatorname{Im}z_1\neq\operatorname{Im}z_2. \tag{1.1}$$

It is the principal aim of the paper [10] (announced partially in [8][17]) to study transformation properties and asymptotic aspects of the double holomorphic Eisenstein series, defined by (1.10) with (1.9) below, having two variables $\boldsymbol{s}=(s_1,s_2)\in\mathbb{C}^2$ and two parameters $\boldsymbol{z}=(z_1,z_2)\in(\mathfrak{H}^{\pm})^2$; this further leads us to establish its complete asymptotic expansions both

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in the *ascending* (Theorem 1) and *descending* (Theorem 2) orders when z moves within the poly-sectors $(\mathfrak{H}^{\pm})^2$ so as that the distance $|z_2 - z_1|$ becomes small and large respectively. Our main formulae in Theorem 2 naturally reduce to various expressions (in finitely closed forms) for evaluating specific values of $\widetilde{\zeta_{\mathbb{Z}^2}}(s;z)$ at any integer lattice pints $s \in \mathbb{Z}^2$ (Corollaries 2.1–2.14), as well as its certain 'central values' (Corollary 2.15).

We prepare here several notations necessary for describing our results. The symbol $\varepsilon(w)$ is defined for any $w \in \widetilde{\mathbb{C}^{\times}}$ (except when $\arg w = 0$) by

$$\varepsilon(w) = \operatorname{sgn}(\arg w) = \begin{cases} +1 & \text{if } \arg w > 0, \\ -1 & \text{if } \arg w < 0, \end{cases} \tag{1.2}$$

and the (vector-like) notations

$$z_{12} = z_2 - z_1 \qquad \text{and} \qquad z_{21} = z_1 - z_2 \tag{1.3}$$

are frequently used; their arguments are to be restricted as

$$0 < |\arg z_{12}| < \pi \qquad \text{and} \qquad 0 < |\arg z_{21}| < \pi, \tag{1.4}$$

where the first equalities come from (1.1). It is readily seen under (1.4) that

$$z_{21} = e^{-\varepsilon(z_{12})\pi i} z_{12} \qquad \text{and} \qquad z_{12} = e^{-\varepsilon(z_{21})\pi i} z_{21}, \tag{1.5}$$

and also that

$$\varepsilon(z_{21}) = -\varepsilon(z_{12}). \tag{1.6}$$

We next introduce a new parameter η defined by

$$\eta = \frac{1}{2} e^{-\varepsilon(z_{12})\pi i/2} z_{12} = \frac{1}{2} e^{-\varepsilon(z_{21})\pi i/2} z_{21}, \tag{1.7}$$

or equivalently by

$$z_{12} = 2e^{\varepsilon(z_{12})\pi i/2} \eta \qquad \text{and} \qquad z_{21} = 2e^{\varepsilon(z_{21})\pi i/2} \eta,$$

which plays pivotal rôles in describing our results; its argument satisfies under (1.4) that

$$|\arg \eta| = \left| \arg z_{12} - \frac{1}{2} \varepsilon(z_{12})\pi \right| = \left| \arg z_{21} - \frac{1}{2} \varepsilon(z_{21})\pi \right| < \frac{\pi}{2}. \tag{1.8}$$

We remark that the introduction of η above is made with the intention to the forthcoming study [11] (partially announced in [9]), where the *non-holomorphic* case $z = (z, \bar{z}) \in \mathfrak{H}^+ \times \mathfrak{H}^-$ or $z = (\bar{z}, z) \in \mathfrak{H}^- \times \mathfrak{H}^+$ is to be treated, where $z_{21} = z - \bar{z} = 2e^{\pi i/2} y$ or $z_{21} = \bar{z} - z = 2e^{-\pi i/2} y$ holds; this in comparison with (1.7) suggests that η is a 'complexification' of the real parameter y.

Throughout the following, the notation $\langle s \rangle = s_1 + s_2$ for any $s = (s_1, s_2) \in \mathbb{C}^2$ is used, and $\widetilde{\zeta_{\mathbb{Z}^2}}^{\pm}(s;z)$ denotes the double holomorphic Eisenstein series of two variables $s \in \mathbb{C}^2$ and with two parameters $z \in (\mathfrak{H}^{\pm})^2$, defined by

$$\widetilde{\zeta_{\mathbb{Z}^2}}^{\pm}(s;z) = \sum_{\substack{m,n=-\infty \\ (m,n)\neq(0,0)}}^{\infty} (m + nz_1)^{-s_1} (m + nz_2)^{-s_2} \qquad (\operatorname{Re}\langle s \rangle > 1), \tag{1.9}$$

where the branch of each summand is to be chosen such that $\arg(m + nz_j)$ falls within the range $(-\pi, \pi]$ in $\widetilde{\zeta_{\mathbb{Z}^2}}^{+}(s;z)$, and within $[-\pi, \pi)$ in $\widetilde{\zeta_{\mathbb{Z}^2}}^{-}(s;z)$, for $j = 1, 2$. We now introduce the main object by taking (from a viewpoint of symmetry) the arithmetical mean of $\widetilde{\zeta_{\mathbb{Z}^2}}^{\pm}(s;z)$:

$$\widetilde{\zeta_{\mathbb{Z}^2}}(s;z) = \frac{1}{2} \left\{ \widetilde{\zeta_{\mathbb{Z}^2}}^{+}(s;z) + \widetilde{\zeta_{\mathbb{Z}^2}}^{-}(s;z) \right\}, \tag{1.10}$$

for which we establish at first in [10] complete asymptotic expansions when $z = (z_1, z_2)$ moves within either the poly-sector $(\mathfrak{H}^+)^2$ or $(\mathfrak{H}^-)^2$, so as that the 'pivotal' parameter η in (1.7) tends to 0 through the sector $|\arg \eta| < \pi/2$ (Theorem 1). The case $N = 0$ of our first main result (2.5) with (2.6) (resp. (2.7)) and (2.10) reduces to a transformation formulae for $\widetilde{\zeta_{\mathbb{Z}^2}}(s;z)$ in terms of Kummer's hypergeometric function ${}_1F_1\left({\lambda \atop \nu}; Z\right)$ (Corollary 1.1); this further leads us to show that 'counterpart' expansions exist for $\widetilde{\zeta_{\mathbb{Z}^2}}(\boldsymbol{s};\boldsymbol{z})$ when η tends to ∞ through $|\arg\eta| < \pi/2$ (Theorem 2).

The use of Mellin-Barnes type integrals in [10], manipulated with several properties of hypergeometric functions, is crucial throughout the proofs; the transference from Theorem 1 to Theorem 2 is for instance achieved by a classical connection formula relating Kummer's confluent hypergeometric functions of the first and second kind.

As for asymptotic aspects of relevant Eisenstein series (of one complex variable), Matsumoto [15] obtained complete asymptotic expansions (with respect z) of *holomorphic* Eisenstein series, while the second author [16] studied an asymptotic formula (as $t \longrightarrow +\infty$) for the *non-holomorphic* Eisenstein series $E_0(s;z)$ (of weight 0). Complete asymptotic expansions for the classical Epstein zeta-function $\zeta_{\mathbb{Z}^2}(s;z)$ as $y = \operatorname{Im} z \longrightarrow +\infty$ have been established by the first author [5], in which similar expansions were also derived for the Laplace-Mellin transform of $\zeta_{\mathbb{Z}^2}(s;z)$. The main formula in [5] for $\zeta_{\mathbb{Z}^2}(s;z)$ is readily switched to that for $E_0(s;z)$ by the relation $E_0(s;z) = y^s\zeta_{\mathbb{Z}^2}(s;z)/2\zeta(s)$, where $\zeta(s)$ denotes the Riemann zeta-function; this could further be transferred to complete asymptotic expansions as $y \longrightarrow +\infty$ for $E_k(s;z)$ (of any even weight k) by the authors [7] upon using Maaß' weight change operators. Furthermore, complete asymptotic expansions for a more general Epstein zeta-function $\psi_{\mathbb{Z}^2}(s;a,b;\mu,\nu;z)$ as $y \longrightarrow +\infty$ have recently been established by the first author [6], together with those for the Riemann-Liouville transform of $\zeta_{\mathbb{Z}^2}(s;z)$. We further mention that Eisenstein type series (of two complex variables) relevant to (1.10) have recently been treated in [12][13][14].

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2. Main results

Let $\mathscr{D}_j^{\pm}$ $(j = 1,2)$ be the domains in $\mathfrak{H}^{\pm}$, defined by

$$\mathscr{D}_1^{\pm} = \{\boldsymbol{z} \in (\mathfrak{H}^{\pm})^2 \mid |\operatorname{Im} z_1| < |\operatorname{Im} z_2|\},$$
$$\mathscr{D}_2^{\pm} = \{\boldsymbol{z} \in (\mathfrak{H}^{\pm})^2 \mid |\operatorname{Im} z_1| > |\operatorname{Im} z_2|\}.$$

We shall split the statements of our results into the following two cases (with the equivalent description in terms of (1.2)):

$$\begin{array}{lll} \text{Case i)} & \boldsymbol{z} \in \mathscr{D}_1^{+} \cup \mathscr{D}_1^{-} \iff \varepsilon(z_1) = \varepsilon(z_2) = \varepsilon(z_{12}); \\ \text{Case ii)} & \boldsymbol{z} \in \mathscr{D}_2^{+} \cup \mathscr{D}_2^{-} \iff \varepsilon(z_1) = \varepsilon(z_2) = \varepsilon(z_{21}). \end{array} \tag{2.1}$$

It is seen from the definition (1.10) with (1.9) of $\widetilde{\zeta_{\mathbb{Z}^2}}(\boldsymbol{s};\boldsymbol{z})$ that the assertions in Case i) can readily be switched to those in Case ii) by the replacements

$$\begin{cases} \mathbb{C}^2 \ni \boldsymbol{s} = (s_1,s_2) \longmapsto (s_2,s_1) = \widehat{\boldsymbol{s}} \in \mathbb{C}^2, \\ \mathscr{D}_1^{+} \cup \mathscr{D}_1^{-} \ni \boldsymbol{z} = (z_1,z_2) \longmapsto (z_2,z_1) = \widehat{\boldsymbol{z}} \in \mathscr{D}_2^{+} \cup \mathscr{D}_2^{-} \end{cases} \tag{2.2}$$

and vice versa; however, the statements are to be given in both the cases for clarifying symmetry of our results.

We frequently use the notation $e(w) = e^{2\pi i w}$ for any $w \in \widetilde{\mathbb{C}^{\times}}$. Let $\Gamma(s)$ be the gamma function, $(s)_n = \Gamma(s+n)/\Gamma(s)$ for any $n \in \mathbb{Z}$ the shifted factorial of s. Further let ${}_1F_1\binom{\lambda}{\nu};Z)$ and $U(\lambda;\nu;Z)$ denote respectively the first and second solutions of Kummer's hypergeometric differential equation, defined by

$${}_1F_1\left({\lambda \atop \nu};Z\right) = \sum_{k=0}^{\infty} \frac{(\lambda)_k}{(\nu)_k k!} Z^k$$

for $|Z| < +\infty$ (cf. [20]), and

$$U(\lambda;\nu;Z) = \frac{1}{\Gamma(\lambda)} \int_0^{\infty e^{i\varphi}} e^{-wZ} w^{\lambda-1}(1+w)^{\nu-\lambda-1} dw \tag{2.3}$$

for $\operatorname{Re}\lambda > 0$ and $|\varphi + \arg Z| < \pi/2$ with any fixed angle $\varphi \in (-\pi,\pi)$, where the path of integration is the ray from the origin to $\infty e^{i\varphi}$ (cf. [20]). Next let $\sigma_w(l) = \sum_{0<h|l} h^w$, and $\Phi_{r,s}(e(z))$

the function defined for any $z \in \mathfrak{H}^+$ by

$$\Phi_{r,s}(e(z)) = \sum_{h,k=1}^{\infty} e(hkz)h^r k^s = \sum_{l=1}^{\infty} \sigma_{r-s}(l)e(lz), \tag{2.4}$$

which was first introduced and studied by Ramanujan [18] for the purpose of giving various evaluations of the holomorphic Eisenstein series $E_k(z)$ with $k = 2,4,6$.

We now state our first main result.

Theorem 1. *Let η be given by (1.7). Then the double holomorphic Eisenstein series $\widetilde{\zeta_{\mathbb{Z}^2}}(s;z)$ in (1.10) with (1.9) can be continued to an entire function to the s-space $\mathbb{C}^2$, and the formula*

$$\widetilde{\zeta_{\mathbb{Z}^2}}(s;z) = 2\cos^2\Big(\frac{\pi}{2}\langle s\rangle\Big)\zeta(\langle s\rangle) + \widetilde{\zeta_{\mathbb{Z}^2}}^*(s;z) \tag{2.5}$$

holds, where $\widetilde{\zeta_{\mathbb{Z}^2}}^*(s;z)$ is represented for any integer $N \geq 0$ as:

i) *if $z \in \mathscr{D}_1^+ \cup \mathscr{D}_1^-$,*

$$\widetilde{\zeta_{\mathbb{Z}^2}}^*(s;z) = \frac{2(2\pi)^{\langle s\rangle}}{\Gamma(\langle s\rangle)}\cos\Big(\frac{\pi}{2}\langle s\rangle\Big)\{S_N(s;z) + R_N(s;z)\} \tag{2.6}$$

in the region $\sigma_2 > -N$;

ii) *if $z \in \mathscr{D}_2^+ \cup \mathscr{D}_2^-$,*

$$\widetilde{\zeta_{\mathbb{Z}^2}}^*(s;z) = \frac{2(2\pi)^{\langle s\rangle}}{\Gamma(\langle s\rangle)}\cos\Big(\frac{\pi}{2}\langle s\rangle\Big)\{S_N(\widehat{s};\widehat{z}) + R_N(\widehat{s};\widehat{z})\} \tag{2.7}$$

in the region $\sigma_1 > -N$.

Here in both the cases i) and ii), $S_N(s;z)$ is further expanded as

$$S_N(s;z) = \sum_{n=0}^{N-1} \frac{(-1)^n (s_2)_n}{(\langle s\rangle)_n n!}\Phi_{\langle s\rangle-1+n,n}(e(\varepsilon(z_1)z_1))(4\pi\eta)^n, \tag{2.8}$$

giving the asymptotic series in the ascending order of η as $\eta \to 0$ through the sector $|\arg\eta| < \pi/2$; the remainder $R_N(s;z)$ is expressed by a certain Mellin-Barnes type integral, and satisfies the estimate

$$R_N(s;z) = O(e^{-2\pi|\operatorname{Im} z_1|}|\eta|^N) \tag{2.9}$$

as $\eta \to 0$ through $|\arg\eta| \leq \pi/2 - \delta$ with any small $\delta > 0$, while $z \in (\mathfrak{H}^\pm)^2$ moves within $|\operatorname{Im} z_j| \geq y_0 > 0$ $(j = 1,2)$. Here the constant implied in the O-symbol depends at most on s, y_0, N and δ. Furthermore, the explicit expression

$$R_N(s;z) = \frac{(-1)^N (s_2)_N}{(N-1)!(\langle s\rangle)_N}(4\pi\eta)^N \sum_{h,k=1}^{\infty} e(hk\varepsilon(z_1)z_1)h^{\langle s\rangle-1+N}k^N \times \int_0^1 (1-\xi)^{N-1}{}_1F_1\Big({s_2+N \atop \langle s\rangle+N}; -4\pi hk\eta\xi\Big)d\xi \tag{2.10}$$

holds in the same region of s above, where the case $N = 0$ should read without the factor $(-1)!$ and the ξ-integration.

Remark. The n-th indexed term on the right side of (2.8) is of order $\asymp e^{-2\pi|\operatorname{Im} z_1|}|\eta|^n$, since

$$\Phi_{r,s}(e(\varepsilon(z)z)) = e(\varepsilon(z)z) + O(e^{-4\pi|\operatorname{Im} z|}) \asymp e^{-2\pi|\operatorname{Im} z|} \tag{2.11}$$

as $\operatorname{Im} z \to \pm\infty$, by (2.4); this shows that the presence of the bound in (2.9) is reasonable.

The case $N = 0$ of Theorem 1 yields the following result.

Corollary 1.1. *The function $\widetilde{\zeta_{\mathbb{Z}^2}}^*(s;z)$ defined by (2.5) can be continued to an entire function of s to the s-space $\mathbb{C}^2$, and the following transformation formulae hold for all $s \in \mathbb{C}^2$:*

i) *if* $z \in \mathscr{D}_1^+ \cup \mathscr{D}_1^-$,

$$\widetilde{\zeta_{\mathbb{Z}^2}}^*(s;z) = \frac{2(2\pi)^{\langle s\rangle}}{\Gamma(\langle s\rangle)} \cos\left(\frac{\pi}{2}\langle s\rangle\right) \sum_{h,k=1}^{\infty} e(hk\varepsilon(z_1)z_1)h^{\langle s\rangle-1}{}_1F_1\left({s_2 \atop \langle s\rangle}; -4\pi hk\eta\right); \tag{2.12}$$

ii) *if* $z \in \mathscr{D}_2^+ \cup \mathscr{D}_2^-$,

$$\widetilde{\zeta_{\mathbb{Z}^2}}^*(s;z) = \frac{2(2\pi)^{\langle s\rangle}}{\Gamma(\langle s\rangle)} \cos\left(\frac{\pi}{2}\langle s\rangle\right) \sum_{h,k=1}^{\infty} e(hk\varepsilon(z_2)z_2)h^{\langle s\rangle-1}{}_1F_1\left({s_1 \atop \langle s\rangle}; -4\pi hk\eta\right). \tag{2.13}$$

We now state our second main result.

Theorem 2. *Let $\widetilde{\zeta_{\mathbb{Z}^2}}^*(s;z)$ be defined by (2.5). Then the following formulae hold for any $s \in \mathbb{C}^2$:*

i) *if* $z \in \mathscr{D}_1^+ \cup \mathscr{D}_1^-$,

$$\widetilde{\zeta_{\mathbb{Z}^2}}^*(s;z) = 2(2\pi)^{\langle s\rangle} \cos\left(\frac{\pi}{2}\langle s\rangle\right) \left\{ \frac{e^{\varepsilon(z_{12})\pi i s_2}}{\Gamma(s_1)} T_1(s;z) + \frac{e^{\varepsilon(z_{21})\pi i s_1}}{\Gamma(s_2)} T_2(s;z) \right\}; \tag{2.14}$$

ii) *if* $z \in \mathscr{D}_2^+ \cup \mathscr{D}_2^-$,

$$\widetilde{\zeta_{\mathbb{Z}^2}}^*(s;z) = 2(2\pi)^{\langle s\rangle} \cos\left(\frac{\pi}{2}\langle s\rangle\right) \left\{ \frac{e^{\varepsilon(z_{21})\pi i s_1}}{\Gamma(s_2)} T_1(\widehat{s};\widehat{z}) + \frac{e^{\varepsilon(z_{12})\pi i s_2}}{\Gamma(s_1)} T_2(\widehat{s};\widehat{z}) \right\}. \tag{2.15}$$

Here in both the cases i) and ii), $T_j(s;z)$ $(j=1,2)$ are represented for any integer $N \geq 0$ as

$$T_j(s;z) = S_{j,N}(s;z) + R_{j,N}(s;z) \tag{2.16}$$

in the region $-N < \sigma_j < N+1$ $(j=1,2)$, where

$$S_{1,N}(s;z) = \sum_{n=0}^{N-1} \frac{(-1)^n (s_2)_n (1-s_1)_n}{n!} \Phi_{s_1-n-1,-s_2-n}(e(\varepsilon(z_1)z_1))\{4\pi e^{\varepsilon(z_{12})\pi i}\eta\}^{-s_2-n}, \tag{2.17}$$

$$S_{2,N}(s;z) = \sum_{n=0}^{N-1} \frac{(-1)^n (s_1)_n (1-s_2)_n}{n!} \Phi_{s_2-n-1,-s_1-n}(e(\varepsilon(z_2)z_2))(4\pi\eta)^{-s_1-n}, \tag{2.18}$$

both giving the asymptotic series in the descending order of η as $\eta \to \infty$ through the sector $|\arg\eta| < \pi/2$; the remainders $R_{j,N}(s;z)$ $(j=1,2)$ are expressed by certain Mellin-Barnes type integrals, and satisfy the estimates

$$R_{1,N}(s;z) = O(e^{-2\pi|\operatorname{Im} z_1|}|\eta|^{-\sigma_2-N}), \tag{2.19}$$

$$R_{2,N}(s;z) = O(e^{-2\pi|\operatorname{Im} z_2|}|\eta|^{-\sigma_1-N}) \tag{2.20}$$

respectively as $\eta \to \infty$ through $|\arg\eta| \leq \pi/2-\delta$ with any small $\delta > 0$, while $z \in (\mathfrak{H}^{\pm})^2$ moves within $|\operatorname{Im} z_j| \geq y_0 > 0$ $(j=1,2)$. Here the constants implied in the O-symbols depend at most on s, y_0, N and δ. Furthermore, the explicit expressions

$$R_{1,N}(s;z) = \frac{(-1)^n (s_2)_N (1-s_1)_N}{(N-1)!} \sum_{h,k=1}^{\infty} e(hk\varepsilon(z_1)z_1)h^{\langle s\rangle-1} \times \int_0^1 \xi^{-s_2-N}(1-\xi)^{N-1} U(s_2+N;\langle s\rangle;4\pi hk e^{\varepsilon(z_{12})\pi i}\eta/\xi)d\xi, \tag{2.21}$$

$$R_{2,N}(s;z) = \frac{(-1)^N (s_1)_N (1-s_2)_N}{(N-1)!} \sum_{h,k=1}^{\infty} e(hk\varepsilon(z_2)z_2)h^{\langle s\rangle-1} \times \int_0^1 \xi^{-s_1-N}(1-\xi)^{N-1} U(s_1+N;\langle s\rangle;4\pi hk\eta/\xi)d\xi \tag{2.22}$$

hold in the same region of s above, where the case $N=0$ should read without the factor $(-1)!$ and the ξ-integration.

Remark. By virtue of (1.2), it is ensured that $|e(\varepsilon(z_j)z_j)| < 1$ $(j=1,2)$ in (2.5)–(2.22).

3. Specific values at $s \in \mathbb{N}^2$ and $s \in (-\mathbb{N}_0)^2$

The following Sections 3–6 are devoted to showing that our second main formula (2.5) with (2.14) (resp. (2.15)), (2.17), (2.18), (2.21) and (2.22) reduces to various expressions (in closed form) of the particular values of $\widetilde{\zeta_{\mathbb{Z}^2}}(\boldsymbol{s};\boldsymbol{z})$ and its partial derivatives when $\boldsymbol{s}$ is at any integer lattice points, as well as its central values at $\boldsymbol{s}=(s,s)$ $(s\in\mathbb{C})$. For this, let B_n $(n\in\mathbb{N}_0)$ denote the n-th Bernoulli number (cf. [3]), and write $\boldsymbol{0}=(0,0)$, $\boldsymbol{e}_1=(1,0)$, $\boldsymbol{e}_2=(0,1)$, and $\boldsymbol{1}=(1,1)$ for brevity. We hereafter use the customary notation $(q;q)_\infty=\prod_{k=1}^\infty(1-q^k)$ for any complex q with $|q|<1$, and set $q_j=e(\varepsilon(z_j)z_j)$ $(j=1,2)$. The assertions only when $\boldsymbol{z}\in\mathscr{D}_1^+\cup\mathscr{D}_1^-$ are to be stated in the sequel, since those when $\boldsymbol{z}\in\mathscr{D}_2^+\cup\mathscr{D}_2^-$ are readily obtained from the former by (2.2).

Corollary 2.1. *For specific values of $\widetilde{\zeta_{\mathbb{Z}^2}}(\boldsymbol{s};\boldsymbol{z})$ and its first derivatives at any lattice points $\boldsymbol{s}=\boldsymbol{m}\in\mathbb{N}^2$ with $\boldsymbol{m}=(m_1,m_2)$, the following formulae hold when $\boldsymbol{z}\in\mathscr{D}_1^+\cup\mathscr{D}_1^-$:*

i) *if $\langle\boldsymbol{m}\rangle=m_1+m_2$ is even,*

$$\widetilde{\zeta_{\mathbb{Z}^2}}(\boldsymbol{m};\boldsymbol{z})=-\frac{(2\pi i)^{\langle\boldsymbol{m}\rangle}B_{\langle\boldsymbol{m}\rangle}}{\langle\boldsymbol{m}\rangle!}+2(2\pi i)^{\langle\boldsymbol{m}\rangle}\times\left\{\frac{(-1)^{m_2}}{(m_1-1)!}S_{1,m_1}(\boldsymbol{m};\boldsymbol{z})+\frac{(-1)^{m_1}}{(m_2-1)!}S_{2,m_2}(\boldsymbol{m};\boldsymbol{z})\right\},\tag{3.1}$$

and in particular for $\boldsymbol{m}=\boldsymbol{1}$,

$$\widetilde{\zeta_{\mathbb{Z}^2}}(\boldsymbol{1};\boldsymbol{z})=\frac{\pi^2}{3}+\frac{2\pi}{\eta}\log\frac{(q_1;q_1)_\infty}{(q_2;q_2)_\infty};\tag{3.2}$$

ii) *if $\langle\boldsymbol{m}\rangle=m_1+m_2$ is odd,*

$$\widetilde{\zeta_{\mathbb{Z}^2}}(\boldsymbol{m};\boldsymbol{z})=0,\tag{3.3}$$

and further for $j=1,2$,

$$\frac{\partial\widetilde{\zeta_{\mathbb{Z}^2}}}{\partial s_j}(\boldsymbol{m};\boldsymbol{z})=\frac{1}{2}(2\pi i)^{\langle\boldsymbol{m}\rangle+1}\left\{\frac{(-1)^{m_1}}{(m_2-1)!}S_{1,m_1}(\boldsymbol{m};\boldsymbol{z})+\frac{(-1)^{m_2}}{(m_1-1)!}S_{2,m_2}(\boldsymbol{m};\boldsymbol{z})\right\}.\tag{3.4}$$

Here in both the cases above, $S_{j,m}(\boldsymbol{m};\boldsymbol{z})$ $(j=1,2)$ are given (in finite closed forms) as

$$S_{1,m_1}(\boldsymbol{m};\boldsymbol{z})=\sum_{n=0}^{m_1-1}\binom{m_1-1}{n}(m_2)_n\Phi_{m_1-n-1,-m_2-n}(q_1)(-4\pi\eta)^{-m_2-n},\tag{3.5}$$

$$S_{2,m_2}(\boldsymbol{m};\boldsymbol{z})=\sum_{n=0}^{m_2-1}\binom{m_2-1}{n}(m_1)_n\Phi_{m_2-n-1,-m_1-n}(q_2)(4\pi\eta)^{-m_1-n}.\tag{3.6}$$

Remark. Formula (3.2) gives a two variable analogue of the classical Kronecker limit formula for the (one variable) Epstein zeta-function as $s\to1$ (cf. [19]), while (3.1), (3.3) and (3.4) may be regarded as its variants at $\boldsymbol{s}=\boldsymbol{m}\in\mathbb{N}^2$; similar reductions from asymptotic expansions are also observed in [5][6].

Corollary 2.2. *For specific values of $\widetilde{\zeta_{\mathbb{Z}^2}}(\boldsymbol{s};\boldsymbol{z})$ at any lattice points $\boldsymbol{s}=-\boldsymbol{m}\in(-\mathbb{N}_0)^2$ with $\boldsymbol{m}=(m_1,m_2)$, the following formulae hold when $\boldsymbol{z}\in\mathscr{D}_1^+\cup\mathscr{D}_1^-$:*

$$\widetilde{\zeta_{\mathbb{Z}^2}}(-\boldsymbol{m};\boldsymbol{z})=\begin{cases}-1 & \text{if } \boldsymbol{m}=\boldsymbol{0},\\ 0 & \text{otherwise}.\end{cases}\tag{3.7}$$

Corollary 2.3. *For specific values of $(\partial\widetilde{\zeta_{\mathbb{Z}^2}}/\partial s_j)(\boldsymbol{s};\boldsymbol{z})$ $(j=1,2)$ at any lattice points $\boldsymbol{s}=-\boldsymbol{m}\in(-\mathbb{N}_0)^2$ with $\boldsymbol{m}=(m_1,m_2)$, the following formulae hold when $\boldsymbol{z}\in\mathscr{D}_1^+\cup\mathscr{D}_1^-$:*

i) *if $\boldsymbol{m}=\boldsymbol{0}$, for $j=1,2$,*

$$\frac{\partial\widetilde{\zeta_{\mathbb{Z}^2}}}{\partial s_j}(\boldsymbol{0};\boldsymbol{z})=-2\log\sqrt{2\pi}+2S_{j,1}(\boldsymbol{0};\boldsymbol{z})=-2\log\left\{\sqrt{2\pi}(q_j;q_j)_\infty\right\},\tag{3.8}$$

ii) *if* $\langle \boldsymbol{m} \rangle = m_1 + m_2$ *is even with* $\langle \boldsymbol{m} \rangle \geq 2$,

$$\frac{\partial \widetilde{\zeta_{\mathbb{Z}^2}}}{\partial s_1}(-\boldsymbol{m};\boldsymbol{z}) = \frac{\langle \boldsymbol{m} \rangle !}{(2\pi i)^{\langle \boldsymbol{m} \rangle}}\zeta(\langle \boldsymbol{m} \rangle + 1) + \frac{2m_1!}{(2\pi i)^{\langle \boldsymbol{m} \rangle}} S_{1,m_2+1}(-\boldsymbol{m};\boldsymbol{z}), \tag{3.9}$$

$$\frac{\partial \widetilde{\zeta_{\mathbb{Z}^2}}}{\partial s_2}(-\boldsymbol{m};\boldsymbol{z}) = \frac{\langle \boldsymbol{m} \rangle !}{(2\pi i)^{\langle \boldsymbol{m} \rangle}}\zeta(\langle \boldsymbol{m} \rangle + 1) + \frac{2m_2!}{(2\pi i)^{\langle \boldsymbol{m} \rangle}} S_{2,m_1+1}(-\boldsymbol{m};\boldsymbol{z}); \tag{3.10}$$

iii) *if* $\langle \boldsymbol{m} \rangle = m_1 + m_2$ *is odd with* $\langle \boldsymbol{m} \rangle \geq 1$,

$$\frac{\partial \widetilde{\zeta_{\mathbb{Z}^2}}}{\partial s_k}(-\boldsymbol{m};\boldsymbol{z}) = 0 \qquad (k = 1,2), \tag{3.11}$$

and further

$$\begin{aligned}\frac{\partial^2 \widetilde{\zeta_{\mathbb{Z}^2}}}{\partial s_1 \partial s_2}(-\boldsymbol{m};\boldsymbol{z}) = -\frac{B_{\langle \boldsymbol{m} \rangle+1}}{\langle \boldsymbol{m} \rangle + 1}\pi^2 - \frac{1}{2(2\pi i)^{\langle \boldsymbol{m} \rangle - 1}} \\ \times \left\{ m_1! S_{1,m_2+1}(-\boldsymbol{m};\boldsymbol{z}) + m_2! S_{2,m_1+1}(-\boldsymbol{m};\boldsymbol{z}) \right\}.\end{aligned} \tag{3.12}$$

Here in the cases i)–iii) above, $S_{j,m+1}(-\boldsymbol{m};\boldsymbol{z})$ $(j = 1,2)$ *are given (in finitely closed forms) as*

$$S_{1,m_2+1}(-\boldsymbol{m};\boldsymbol{z}) = \sum_{n=0}^{m_2} \binom{m_2}{n} (1+m_1)_n \Phi_{-m_1-n-1,m_2-n}(q_1)(-4\pi\eta)^{m_2-n}, \tag{3.13}$$

$$S_{2,m_1+1}(-\boldsymbol{m};\boldsymbol{z}) = \sum_{n=0}^{m_1} \binom{m_1}{n} (1+m_2)_n \Phi_{-m_2-n-1,m_1-n}(q_2)(4\pi\eta)^{m_1-n}. \tag{3.14}$$

Remark. Formula (3.8) gives a two variable analogue of the classical Kronecker limit formula for (the derivative of) Epstein zeta-function at $s = 0$ (cf. [19]), while (3.9)–(3.12) may be regarded as its variants at $\boldsymbol{s} = -\boldsymbol{m} \in (-\mathbb{N}_0)^2$; similar reductions from asymptotic expansions are also observed in [5][6].

4. Specific values in connection with generalized Lambert series

We evaluate in this section several specific values of $\widetilde{\zeta_{\mathbb{Z}^2}}(\boldsymbol{s};\boldsymbol{z})$ in terms of the generalized Lambert series defined for any $s, w \in \mathbb{C}$ with $w \neq q^{-k}$ $(k \in \mathbb{N})$ by

$$\mathscr{L}_s(w;q) = \sum_{k=1}^{\infty} \frac{k^s w q^k}{1 - wq^k}. \tag{4.1}$$

Let $\delta(m)$ $(m \in \mathbb{Z})$ be the symbol which equals 1 or 0 according to $m = 0$ or otherwise, and $\mathfrak{S}_k^n$ $(n, k \in \mathbb{N}_0)$ denote the Stirling numbers of the first kind defined by

$$\frac{(e^w - 1)^k}{k!} = \sum_{n=0}^{\infty} \frac{\mathfrak{S}_k^n}{n!} w^n. \tag{4.2}$$

Proposition 1. *For any* $r \in \mathbb{N}_0$ *and* $s \in \mathbb{C}$ *we have the relation*

$$\Phi_{r,s}(q) = \sum_{j=0}^{r} \mathfrak{S}_j^r \mathscr{L}_s^{(j)}(q), \tag{4.3}$$

where

$$\mathscr{L}_s^{(j)}(q) = \left(\frac{\partial}{\partial w}\right)^j \mathscr{L}_s(w;q)\bigg|_{w=1} = j! \sum_{k=1}^{\infty} \frac{k^s q^{k\{j+\delta(j)\}}}{(1-q^k)^{j+1}} \qquad (j \in \mathbb{N}_0). \tag{4.4}$$

We remark that the Lambert series of the type in (4.4) play underlying rôles in Ramanujan's theories of Eisenstein series, theta functions and elliptic functions (cf. [1][2][21]), where, for e.g.,

$$P(q) = 1 - 24\mathscr{L}_1^{(0)}(q), \quad Q(q) = 1 + 240\mathscr{L}_3^{(0)}(q), \quad R(q) = 1 - 504\mathscr{L}_5^{(0)}(q). \tag{4.5}$$

are the classical Eisenstein series reformulated by Ramanujan [18]. It is possible to transfer, through (4.3), from the expressions in (3.5), (3.6), (3.13) and (3.14) to those in terms of $\mathscr{L}_s^{(j)}(q)$ ($j \in \mathbb{N}_0$). We can for instance show the following Corollaries 2.4–2.6.

Corollary 2.4. *For specific values of $\widetilde{\zeta_{\mathbb{Z}^2}}(\boldsymbol{s};z)$ and its first derivatives at $\boldsymbol{s} = \mathbf{1}$, $(1,2)$, $(2,1)$ and $2\cdot\mathbf{1}$, the following formulae hold when $z \in \mathscr{D}_1^+ \cup \mathscr{D}_1^-$:*

$$\widetilde{\zeta_{\mathbb{Z}^2}}(\mathbf{1};z) = \frac{\pi^2}{3} + \frac{2\pi}{\eta}\mathscr{L}_{-1}^{(0)}(q_1) - \frac{2\pi}{\eta}\mathscr{L}_{-1}^{(0)}(q_2), \tag{4.6}$$

$$\frac{\partial\widetilde{\zeta_{\mathbb{Z}^2}}}{\partial s_j}(1,2;z) = -\frac{\pi^2}{2\eta^2}\{\mathscr{L}_{-2}^{(0)}(q_1) - \mathscr{L}_{-2}^{(0)}(q_2)\} + \frac{2\pi^3}{\eta}\mathscr{L}_{-1}^{(1)}(q_2), \tag{4.7}$$

$$\frac{\partial\widetilde{\zeta_{\mathbb{Z}^2}}}{\partial s_j}(2,1;z) = \frac{\pi^2}{2\eta^2}\{\mathscr{L}_{-2}^{(0)}(q_1) - \mathscr{L}_{-2}^{(0)}(q_2)\} - \frac{2\pi^3}{\eta}\mathscr{L}_{-1}^{(1)}(q_1), \tag{4.8}$$

both for $j = 1,2$, and

$$\widetilde{\zeta_{\mathbb{Z}^2}}(2\cdot\mathbf{1};z) = \frac{\pi^4}{45} + \frac{2\pi^2}{\eta^2}\{\mathscr{L}_{-2}^{(1)}(q_1) + \mathscr{L}_{-2}^{(1)}(q_2)\} - \frac{\pi}{\eta^3}\{\mathscr{L}_{-3}^{(0)}(q_1) - \mathscr{L}_{-3}^{(0)}(q_2)\}. \tag{4.9}$$

Corollary 2.5. *For specific values of $(\partial\widetilde{\zeta_{\mathbb{Z}^2}}/\partial s_j)(\boldsymbol{s};z)$ at $\boldsymbol{s} = \mathbf{0}$, $-\mathbf{1}$, $-2\cdot\mathbf{1}$, $-m_j\boldsymbol{e}_j$ with even m_j $(j = 1,2)$, the following formulae hold when $z \in \mathscr{D}_1^+ \cup \mathscr{D}_1^-$:*

$$\frac{\partial\widetilde{\zeta_{\mathbb{Z}^2}}}{\partial s_j}(\mathbf{0};z) = -\log\sqrt{2\pi} + 2\mathscr{L}_{-1}^{(0)}(q_j) \qquad (j = 1,2), \tag{4.10}$$

$$\frac{\partial\widetilde{\zeta_{\mathbb{Z}^2}}}{\partial s_1}(-\mathbf{1};z) = -\frac{1}{2\pi^2}\zeta(3) + \frac{2\eta}{\pi}\mathscr{L}_{-2}^{(1)}(q_1) - \frac{1}{\pi^2}\mathscr{L}_{-3}^{(0)}(q_1), \tag{4.11}$$

$$\frac{\partial\widetilde{\zeta_{\mathbb{Z}^2}}}{\partial s_2}(-\mathbf{1};z) = -\frac{1}{2\pi^2}\zeta(3) - \frac{2\eta}{\pi}\mathscr{L}_{-2}^{(1)}(q_2) - \frac{1}{\pi^2}\mathscr{L}_{-3}^{(0)}(q_2), \tag{4.12}$$

$$\begin{aligned}\frac{\partial\widetilde{\zeta_{\mathbb{Z}^2}}}{\partial s_1}(-2\cdot\mathbf{1};z) = {} & \frac{3}{2\pi^2}\zeta(5) + \frac{4\eta^2}{\pi^2}\{\mathscr{L}_{-3}^{(1)}(q_1) + \mathscr{L}_{-3}^{(2)}(q_1)\}, \\ & - \frac{6\eta}{\pi^3}\mathscr{L}_{-4}^{(1)}(q_1) + \frac{3}{\pi^4}\mathscr{L}_{-5}^{(0)}(q_1),\end{aligned} \tag{4.13}$$

$$\begin{aligned}\frac{\partial\widetilde{\zeta_{\mathbb{Z}^2}}}{\partial s_2}(-2\cdot\mathbf{1};z) = {} & \frac{3}{2\pi^2}\zeta(5) + \frac{4\eta^2}{\pi^2}\{\mathscr{L}_{-3}^{(1)}(q_2) + \mathscr{L}_{-3}^{(2)}(q_2)\} \\ & + \frac{6\eta}{\pi^3}\mathscr{L}_{-4}^{(1)}(q_2) + \frac{3}{\pi^4}\mathscr{L}_{-5}^{(0)}(q_2),\end{aligned} \tag{4.14}$$

and for any even $m_j \in \mathbb{N}_0$ $(j = 1,2)$,

$$\frac{\partial\widetilde{\zeta_{\mathbb{Z}^2}}}{\partial s_j}(-m_j\boldsymbol{e}_j;z) = \frac{m_j!}{(2\pi i)^{m_j}}\zeta(m_j+1) + \frac{2m_j!}{(2\pi i)^{m_j}}\mathscr{L}_{-m_j-1}^{(0)}(q_j). \tag{4.15}$$

Corollary 2.6. *For specific values $(\partial^2\widetilde{\zeta_{\mathbb{Z}^2}}/\partial s_1\partial s_2)(s;z)$ at $s=-e_j$ for $j=1,2$, $(-2,-1)$, and $(-1,-2)$, the following formulae hold when $z\in\mathscr{D}_1^+\cup\mathscr{D}_1^-$:*

$$\frac{\partial^2\widetilde{\zeta_{\mathbb{Z}^2}}}{\partial s_1\partial s_2}(-e_1;z)=-\frac{\pi^2}{12}-2\pi\eta\mathscr{L}_{-1}^{(1)}(q_2)-\frac{1}{2}\{\mathscr{L}_{-2}^{(0)}(q_1)+\mathscr{L}_{-2}^{(0)}(q_2)\},\tag{4.16}$$

$$\frac{\partial^2\widetilde{\zeta_{\mathbb{Z}^2}}}{\partial s_1\partial s_2}(-e_2;z)=-\frac{\pi^2}{12}+2\pi\eta\mathscr{L}_{-1}^{(1)}(q_1)-\frac{1}{2}\{\mathscr{L}_{-2}^{(0)}(q_1)+\mathscr{L}_{-2}^{(0)}(q_2)\},\tag{4.17}$$

$$\begin{aligned}\frac{\partial^2\widetilde{\zeta_{\mathbb{Z}^2}}}{\partial s_1\partial s_2}(-2,-1;z)=\frac{\pi^2}{120}&+2\eta^2\{\mathscr{L}_{-2}^{(1)}(q_2)+\mathscr{L}_{-2}^{(2)}(q_2)\}-\frac{\eta}{\pi}\{\mathscr{L}_{-3}^{(1)}(q_1)\\&-2\mathscr{L}_{-3}^{(1)}(q_2)\}+\frac{3}{4\pi^2}\{\mathscr{L}_{-4}^{(0)}(q_1)+\mathscr{L}_{-4}^{(0)}(q_2)\},\end{aligned}\tag{4.18}$$

$$\begin{aligned}\frac{\partial^2\widetilde{\zeta_{\mathbb{Z}^2}}}{\partial s_1\partial s_2}(-1,-2;z)=\frac{\pi^2}{120}&+2\eta^2\{\mathscr{L}_{-2}^{(1)}(q_1)+\mathscr{L}_{-2}^{(2)}(q_1)\}-\frac{\eta}{\pi}\{2\mathscr{L}_{-3}^{(1)}(q_1)\\&-\mathscr{L}_{-3}^{(1)}(q_2)\}+\frac{3}{4\pi^2}\{\mathscr{L}_{-4}^{(0)}(q_1)+\mathscr{L}_{-4}^{(0)}(q_2)\}.\end{aligned}\tag{4.19}$$

Using the relation (4.3), we can further show that $S_{j,m}(\boldsymbol{m};z)$ in (3.5) and (3.6), as well as $S_{j,m+1}(-\boldsymbol{m};z)$ in (3.13) and (3.14) are (as functions of z) the elements from the $\mathbb{Z}[(4\pi\eta)^{\pm}]$-modules spanned by certain sets of the generalized Lambert series $\mathscr{L}_s^{(j)}(q)$.

Corollary 2.7. *For the functions in (3.5), (3.6), (3.13) and (3.14), the following algebraic properties hold when $z\in\mathscr{D}_1^+\cup\mathscr{D}_1^-$:*

i) *at any lattice points $s=\boldsymbol{m}\in\mathbb{N}^2$ with $\boldsymbol{m}=(m_1,m_2)$,*

$$S_{1,m_1}(\boldsymbol{m};z)\in\Big\langle\{\mathscr{L}_{l-\langle\boldsymbol{m}\rangle+1}^{(j)}(q_1)\mid 0\le j\le l\le m_1-1\}\Big\rangle_{\mathbb{Z}[1/4\pi\eta]},\tag{4.20}$$

$$S_{2,m_2}(\boldsymbol{m};z)\in\Big\langle\{\mathscr{L}_{l-\langle\boldsymbol{m}\rangle+1}^{(j)}(q_2)\mid 0\le j\le l\le m_2-1\}\Big\rangle_{\mathbb{Z}[1/4\pi\eta]};\tag{4.21}$$

ii) *at any lattice points $s=-\boldsymbol{m}\in(-\mathbb{N}_0)^2$ with $\boldsymbol{m}=(m_1,m_2)$,*

$$S_{1,m_2+1}(-\boldsymbol{m};z)\in\Big\langle\{\mathscr{L}_{l-\langle\boldsymbol{m}\rangle-1}^{(j)}(q_1)\mid 0\le j\le l\le m_2\}\Big\rangle_{\mathbb{Z}[4\pi\eta]},\tag{4.22}$$

$$S_{2,m_1+1}(-\boldsymbol{m};z)\in\Big\langle\{\mathscr{L}_{l-\langle\boldsymbol{m}\rangle-1}^{(j)}(q_2)\mid 0\le j\le l\le m_1\}\Big\rangle_{\mathbb{Z}[4\pi\eta]}.\tag{4.23}$$

5. Specific values at $s\in\mathbb{N}\times(-\mathbb{N}_0)$ or $s\in(-\mathbb{N}_0)\times\mathbb{N}$

Our second main formula in Theorem 2 further yields various evaluations (in finitely closed forms) for specific values of $\widetilde{\zeta_{\mathbb{Z}^2}}(s;z)$ and its partial derivatives at any lattice points $s=\boldsymbol{m}$ in either $\mathbb{N}\times(-\mathbb{N}_0)$ or $(-\mathbb{N}_0)\times\mathbb{N}$.

Corollary 2.8. *For specific values of $\widetilde{\zeta_{\mathbb{Z}^2}}(s;z)$ at any lattice points $s=\boldsymbol{m}\in\mathbb{N}\times(-\mathbb{N}_0)$ with $\boldsymbol{m}=(m_1,-m_2)$, the following formulae hold when $z\in\mathscr{D}_1^+\cup\mathscr{D}_1^-$ upon $N_1=\min(m_1,m_2+1)$:*

i) *if $\langle\boldsymbol{m}\rangle=m_1-m_2$ is even,*

$$\widetilde{\zeta_{\mathbb{Z}^2}}(\boldsymbol{m};z)=2\zeta(m_1-m_2)+\frac{2(2\pi i)^{m_1-m_2}(-1)^{m_2}}{(m_1-1)!}S_{1,N_1}(\boldsymbol{m};z);\tag{5.1}$$

ii) *if $\langle\boldsymbol{m}\rangle=m_1-m_2$ is odd,*

$$\widetilde{\zeta_{\mathbb{Z}^2}}(\boldsymbol{m};z)=0,\tag{5.2}$$

and further for $k=1,2$,

$$\frac{\partial\widetilde{\zeta_{\mathbb{Z}^2}}}{\partial s_k}(\boldsymbol{m};z)=\frac{\pi^2}{2}\delta(\langle\boldsymbol{m}\rangle-1)+\frac{(2\pi i)^{m_1-m_2+1}}{2(m_1-1)!}S_{1,N_1}(\boldsymbol{m};z).\tag{5.3}$$

Here in both the cases i) and ii), $S_{1,N_1}(\boldsymbol{m};z)$ *is given by*

$$S_{1,N_1}(\boldsymbol{m};z)=\sum_{n=0}^{N_1-1}\binom{m_2}{n}(1-m_1)_n\Phi_{m_1-n-1,m_2-n}(q_1)(-4\pi\eta)^{m_2-n}. \tag{5.4}$$

Corollary 2.9. *For specific values of* $\widetilde{\zeta_{\mathbb{Z}^2}}(s;z)$ *at any lattice points* $\boldsymbol{s}=\boldsymbol{m}\in(-\mathbb{N}_0)\times\mathbb{N}$ *with* $\boldsymbol{m}=(-m_1,m_2)$*, the following formulae hold when* $z\in\mathscr{D}_1^+\cup\mathscr{D}_1^-$ *upon* $N_2=\min(m_1+1,m_2)$:

i) *if* $\langle\boldsymbol{m}\rangle=m_2-m_1$ *is even,*

$$\widetilde{\zeta_{\mathbb{Z}^2}}(\boldsymbol{m};z)=2\zeta(m_2-m_1)+\frac{2(2\pi i)^{m_2-m_1}(-1)^{m_1}}{(m_2-1)!}S_{2,N_2}(\boldsymbol{m};z); \tag{5.5}$$

ii) *if* $\langle\boldsymbol{m}\rangle=m_2-m_1$ *is odd,*

$$\widetilde{\zeta_{\mathbb{Z}^2}}(\boldsymbol{m};z)=0, \tag{5.6}$$

and further for $k=1,2$,

$$\frac{\partial\widetilde{\zeta_{\mathbb{Z}^2}}}{\partial s_k}(\boldsymbol{m};z)=\frac{\pi^2}{2}\delta(\langle\boldsymbol{m}\rangle-1)+\frac{(2\pi i)^{m_2-m_1+1}}{2(m_2-1)!}S_{2,N_2}(\boldsymbol{m};z). \tag{5.7}$$

Here in both the cases i) and ii), $S_{2,N_2}(\boldsymbol{m};z)$ *is given by*

$$S_{2,N_2}(\boldsymbol{m};z)=\sum_{n=0}^{N_2-1}\binom{m_1}{n}(1-m_2)_n\Phi_{m_2-n-1,m_1-n}(q_2)(4\pi\eta)^{m_1-n}. \tag{5.8}$$

We next define for any $r\in\mathbb{N}_0$ the functions $\mathscr{S}_r(q)$ by

$$\mathscr{S}_r(q)=\frac{1}{2}\zeta(-r)+\Phi_{0,r}(q)=\begin{cases}\dfrac{B_1}{2}+\Phi_{0,0}(q) & \text{if } r=0,\\ -\dfrac{B_{r+1}}{2(r+1)}+\Phi_{0,r}(q) & \text{if } r\geq 1,\end{cases} \tag{5.9}$$

which was first introduced and studied by Ramanujan [18] in the course of developing his theory of Eisenstein series and elliptic functions (see, for e.g., [1][2][21]); the Eisenstein series in (4.5), due to Ramanujan [18], are for instance connected with $\mathscr{S}_r(q)$ as

$$P(q)=-24\mathscr{S}_1(q),\qquad Q(q)=240\mathscr{S}_3(q),\qquad R(q)=-504\mathscr{S}_5(q),$$

while Weierstraß' elliptic function $\wp(w\mid 2\pi(1,z))$, associated with the basis $2\pi(1,z)$ for $z\in\mathfrak{H}^+$, is expanded into the Laurent series involving $\mathscr{S}_r(q)$ with $q=e(z)$ in its coefficients as

$$\wp(w\mid 2\pi(1,z))=\frac{1}{w^2}+2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(2n)!}\mathscr{S}_{2n+1}(q)w^{2n} \tag{5.10}$$

for $0<|w|<2\pi\min(1,|z|)$. The case $\boldsymbol{m}=m_j\boldsymbol{e}_j$ $(j=1,2)$ of Corollaries 2.8 and 2.9 in particular implies the following relations:

Corollary 2.10. *The following formulae hold for any* $m_j\in\mathbb{N}$ $(j=1,2)$ *when* $z\in\mathscr{D}_1^+\cup\mathscr{D}_1^-$:

i) *if* $m_j\in\mathbb{N}$ $(j=1,2)$ *is even,*

$$\widetilde{\zeta_{\mathbb{Z}^2}}(m_j\boldsymbol{e}_j;z)=2\zeta(m_j)+\frac{2(2\pi i)^{m_j}}{(m_j-1)!}\Phi_{m_j-1,0}(q_j)=\frac{2(2\pi i)^{m_j}}{(m_j-1)!}\mathscr{S}_{m_j-1}(q_j), \tag{5.11}$$

and in particular when $m_j\geq 4$,

$$\widetilde{\zeta_{\mathbb{Z}^2}}(m_j\boldsymbol{e}_j;z)=\frac{(2\pi)^{m_j}}{(m_j-1)!}\left(\frac{\partial}{\partial w}\right)^{m_j-2}\left\{\wp(w\mid 2\pi(1,\varepsilon(z_j)z_j))-\frac{1}{w^2}\right\}\bigg|_{w=0}; \tag{5.12}$$

ii) *if* $m_j\in\mathbb{N}$ $(j=1,2)$ *is odd,*

$$\widetilde{\zeta_{\mathbb{Z}^2}}(m_j\boldsymbol{e}_j;z)=0, \tag{5.13}$$

and further for $k=1,2$,

$$\frac{\partial\widetilde{\zeta_{\mathbb{Z}^2}}}{\partial s_k}(m_j\boldsymbol{e}_j;z)=\frac{\pi^2}{2}\delta(m_j-1)+\frac{(2\pi i)^{m_j+1}}{2(m_j-1)!}\Phi_{m_j-1,0}(q_j)=\frac{(2\pi i)^{m_j+1}}{2(m_j-1)!}\mathscr{S}_{m_j-1}(q_j). \tag{5.14}$$

Remark. It is interesting to note that the even index cases of $\mathscr{S}_r(q)$ appear in the formulae for $\partial\widetilde{\zeta_{\mathbb{Z}^2}}/\partial s_k$ above, whereas the cases cannot be observed in Ramanujan's theories of Eisenstein series, elliptic functions and theta functions.

Corollary 2.11. *For specific values of $\widetilde{\zeta_{\mathbb{Z}^2}}(s;z)$ and its derivatives at any lattice points $s = (m_1,-1) \in \mathbb{N}\times(-\mathbb{N}_0)$, the following formulae hold when $z \in \mathscr{D}_1^+ \cup \mathscr{D}_1^-$:*

i) *if m_1 is odd,*

$$\widetilde{\zeta_{\mathbb{Z}^2}}(m_1,-1;z) = 2\zeta(m_1-1) - \frac{2(2\pi i)^{m_1-1}}{(m_1-1)!}\{-4\pi\eta\,\Phi_{m_1-1,1}(q_1) + (1-m_1)\Phi_{m_1-2,0}(q_1)\}, \tag{5.15}$$

and in particular if $m_1 \geq 3$,

$$\widetilde{\zeta_{\mathbb{Z}^2}}(m_1,-1;z) = \frac{2(2\pi i)^{m_1}}{(m_1-2)!}\left\{\mathscr{S}_{m_1-2}(q_1) + \frac{4\pi\eta}{m_1-1}\Phi_{m_1-1,1}(q_1)\right\}; \tag{5.16}$$

ii) *if m_1 is even,*

$$\widetilde{\zeta_{\mathbb{Z}^2}}(m_1,-1;z) = 0, \tag{5.17}$$

and further for $k = 1,2$,

$$\begin{aligned}\frac{\partial\widetilde{\zeta_{\mathbb{Z}^2}}}{\partial s_k}(m_1,-1;z) &= \frac{\pi^2}{2}\delta(m_1-2) - \frac{(2\pi i)^{m_1}}{2(m_1-2)!}\{-4\pi\eta\,\Phi_{m_1-1,1}(q_1) \\ &\quad + (1-m_1)\Phi_{m_1-2,0}(q_1)\} \\ &= \frac{(2\pi i)^{m_1}}{2(m_1-2)!}\left\{\mathscr{S}_{m_1-2}(q_1) + \frac{4\pi\eta}{m_1-1}\Phi_{m_1-1,1}(q_1)\right\}.\end{aligned} \tag{5.18}$$

Corollary 2.12. *For specific values of $\widetilde{\zeta_{\mathbb{Z}^2}}(s;z)$ and its derivatives at any lattice points $s = (-1,m_2) \in (-\mathbb{N}_0)\times\mathbb{N}$, the following formulae hold when $z \in \mathscr{D}_1^+ \cup \mathscr{D}_1^-$:*

i) *if m_2 is odd,*

$$\widetilde{\zeta_{\mathbb{Z}^2}}(-1,m_2;z) = 2\zeta(m_2-1) - \frac{2(2\pi i)^{m_1-1}}{(m_2-1)!}\{4\pi\eta\,\Phi_{m_2-1,1}(q_2) + (1-m_2)\Phi_{m_2-2,0}(q_2)\}, \tag{5.19}$$

and in particular if $m_2 \geq 3$,

$$\widetilde{\zeta_{\mathbb{Z}^2}}(-1,m_2;z) = \frac{2(2\pi i)^{m_2-1}}{(m_2-2)!}\left\{\mathscr{S}_{m_2-2}(q_2) - \frac{4\pi\eta}{m_2-1}\Phi_{m_2-1,1}(q_2)\right\}; \tag{5.20}$$

ii) *if m_2 is even,*

$$\widetilde{\zeta_{\mathbb{Z}^2}}(-1,m_2;z) = 0, \tag{5.21}$$

and further for $k = 1,2$,

$$\begin{aligned}\frac{\partial\widetilde{\zeta_{\mathbb{Z}^2}}}{\partial s_k}(-1,m_2;z) &= \frac{\pi^2}{2}\delta(m_2-2) - \frac{(2\pi i)^{m_2}}{2(m_2-1)!}\{4\pi\eta\,\Phi_{m_2-1,1}(q_2) \\ &\quad + (1-m_2)\Phi_{m_2-2,0}(q_2) \\ &= \frac{(2\pi i)^{m_2}}{2(m_2-2)!}\left\{\mathscr{S}_{m_2-2}(q_2) - \frac{4\pi\eta}{m_2-1}\Phi_{m_2-1,1}(q_2)\right\}.\end{aligned} \tag{5.22}$$

We next define the functions $\phi_j(w;q)$ $(j=1,2)$ by the Fourier series expansions

$$\begin{aligned}\phi_1(w;q) &= \frac{1}{4}\cot\frac{w}{2} + \sum_{n=1}^{\infty}\frac{q^n}{1-q^n}\sin nw, \\ \phi_2(w;q) &= -\frac{1}{24} + \sum_{n=1}^{\infty}\frac{q^n}{(1-q^n)^2}\cos nw\end{aligned} \tag{5.23}$$

for $|\operatorname{Im} w| < 2\pi |\operatorname{Im} z|$ with $q = e(z)$, and their analytic continuations. These were first introduced by Ramanujan, who made extensive use of these functions for developing his theories of elliptic functions, theta functions and Eisenstein series (see, for e.g., [1][2][21]); Weierstraß' elliptic function $\wp(w \mid 2\pi(1,z))$ for $z \in \mathfrak{H}^+$ is in fact connected with these functions as

$$\wp(w \mid 2\pi(1,z)) = 2\left\{\phi_2(0;q) - \frac{\partial}{\partial w}\phi_1(w;q)\right\}.$$

In view of the Laurent series expansions

$$\begin{aligned} \phi_1(w;q) &= \frac{1}{2w} + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n-1)!}\mathscr{S}_{2n-1}(q) w^{2n-1}, \\ \phi_2(w;q) &= -\frac{1}{24} + \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}\Phi_{1,2n}(q) w^{2n} \end{aligned} \tag{5.24}$$

for $0 < |w| < 2\pi \min(1,|z|)$, we can show the following relations.

Corollary 2.13. *The following evaluations are valid if $z \in \mathscr{D}_1^+ \cup \mathscr{D}_1^-$:*

i) *for any $m_j \in \mathbb{N}$ $(j = 1,2)$,*

$$\widetilde{\zeta_{\mathbb{Z}^2}}(m_j e_j; z) = -\frac{(2\pi)^{m_j}}{(m_j-1)!}\left(\frac{\partial}{\partial w}\right)^{m_j-1}\left\{\phi_1(w;q_j) - \frac{1}{2w}\right\}\bigg|_{w=0}, \tag{5.25}$$

and in particular for any $m_j \geq 3$,

$$\widetilde{\zeta_{\mathbb{Z}^2}}(m_j e_j; z) = \frac{(2\pi)^{m_j}}{(m_j-1)!}\left(\frac{\partial}{\partial w}\right)^{m_j-2}\left\{\wp(w \mid 2\pi(1,\varepsilon(z_j)z_j)) - \frac{1}{w^2}\right\}\bigg|_{w=0}, \tag{5.26}$$

which unifies (5.11) and (5.12) for $m_j \geq 3$;

ii) *for any $m_1 \in \mathbb{N}$ with $m_1 \geq 2$,*

$$\begin{aligned} \widetilde{\zeta_{\mathbb{Z}^2}}(m_1,-1;z) = \frac{2(2\pi)^{m_1-1}}{(m_1-2)!}\Bigg[&-\left(\frac{\partial}{\partial w}\right)^{m_1-2}\left\{\phi_1(w;q_1) - \frac{1}{2w}\right\}\bigg|_{w=0} \\ &-\frac{4\pi\eta}{m_1-1}\left(\frac{\partial}{\partial w}\right)^{m_1-1}\phi_2(w;q_1)\bigg|_{w=0}\Bigg]; \end{aligned} \tag{5.27}$$

iii) *for any $m_2 \in \mathbb{N}$ with $m_2 \geq 2$,*

$$\begin{aligned} \widetilde{\zeta_{\mathbb{Z}^2}}(-1,m_2;z) = \frac{2(2\pi)^{m_2-1}}{(m_2-2)!}\Bigg[&-\left(\frac{\partial}{\partial w}\right)^{m_2-2}\left\{\phi_1(w;q_2) - \frac{1}{2w}\right\}\bigg|_{w=0} \\ &+\frac{4\pi\eta}{m_2-1}\left(\frac{\partial}{\partial w}\right)^{m_2-1}\phi_2(w;q_2)\bigg|_{w=0}\Bigg]; \end{aligned} \tag{5.28}$$

Using again the relation (4.3), we can show the following results.

Corollary 2.14. *For the functions in (5.4) and (5.8), the following algebraic properties hold when $z \in \mathscr{D}_1^+ \cup \mathscr{D}_1^-$:*

i) *at any $s = \boldsymbol{m} = (m_1, -m_2) \in \mathbb{N} \times (-\mathbb{N}_0)$,*

$$S_{1,N_1}(\boldsymbol{m};z) \in \left\langle \left\{ \mathscr{L}^{(j)}_{\langle \boldsymbol{m}\rangle -1+l}(q_1) \,\middle|\, \begin{array}{c} 0 \leq j \leq l; \\ \max(0, 1-\langle \boldsymbol{m}\rangle) \leq l \leq m_2 \end{array} \right\} \right\rangle_{\mathbb{Z}[4\pi\eta]}; \tag{5.29}$$

ii) *at any $s = \boldsymbol{m} = (-m_1, m_2) \in (-\mathbb{N}_0) \times \mathbb{N}$,*

$$S_{2,N_2}(\boldsymbol{m};z) \in \left\langle \left\{ \mathscr{L}^{(j)}_{\langle \boldsymbol{m}\rangle -1+l}(q_2) \,\middle|\, \begin{array}{c} 0 \leq j \leq l; \\ \max(0, 1-\langle \boldsymbol{m}\rangle) \leq l \leq m_1 \end{array} \right\} \right\rangle_{\mathbb{Z}[4\pi\eta]}. \tag{5.30}$$

6. Central values

Next let $K_\nu(Z)$ denote the Bessel function of the third kind (cf. [4]), and write $d(l)=\sigma_0(l)$ for $l\in\mathbb{N}$. Then the case $N=0$ of our main formula (2.5) with (2.14), (2.21) and (2.22) yields the following results on the central values of $\widetilde{\zeta_{\mathbb{Z}^2}}(\boldsymbol{s};\boldsymbol{z})$ along the complex line $\boldsymbol{s}=s\mathbf{1}$ ($s\in\mathbb{C}$), and further its 'extremal' central value at $\boldsymbol{s}=(1/2)\mathbf{1}$.

Corollary 2.15. *The following formula holds for any $s\in\mathbb{C}$ when $\boldsymbol{z}\in\mathscr{D}_1^+\cup\mathscr{D}_1^-$:*

$$\frac{\widetilde{\zeta_{\mathbb{Z}^2}}(s\mathbf{1};\boldsymbol{z})}{\cos(\pi s)}=2\cos(\pi s)\zeta(2s)+\frac{4i\pi^s\eta^{1/2-s}}{\Gamma(s)}\Bigg[\sum_{l=1}^{\infty}l^{1/2-s}\sigma_{2s-1}(l)e(l\varepsilon(z_1)(z_1+z_2)/2)$$
$$\times\left\{\varepsilon(z_{12})K_{s-1/2}(2\pi le^{\varepsilon(z_{12})\pi i}\eta)+\varepsilon(z_{21})e^{\varepsilon(z_{21})\pi i(1/2-s)}K_{s-1/2}(2\pi l\eta)\right\}\Bigg],\tag{6.1}$$

whose limiting case as $s\to1/2$ asserts that

$$\frac{\partial\widetilde{\zeta_{\mathbb{Z}^2}}}{\partial s_j}(1/2;\boldsymbol{z})=\frac{\pi^2}{2}-2\pi i\sum_{l=1}^{\infty}d(l)e(l\varepsilon(z_1)(z_1+z_2)/2)$$
$$\times\left\{\varepsilon(z_{12})K_0(2\pi le^{\varepsilon(z_{12})\pi i}\eta)+\varepsilon(z_{21})K_0(2\pi l\eta)\right\}\tag{6.2}$$

for $j=1,2$.

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E-mail address: katsurad@z3.keio.jp; takumi@ge.ce.nihon-u.ac.jp