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# Transformations of a series in function fields

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## 1. Introduction

Let

$$\eta(z)=e^{\pi iz/12}\prod_{n=1}^{\infty}\left(1-e^{2\pi inz}\right)\qquad(\mathrm{Im}(z)>0)$$

be the Dedekind $\eta$-function. Dedekind [2] described the transformation of $\log\eta(z)$ under the substitution $z'=(az+b)/(cz+d)$, $\begin{pmatrix}a&b\\c&d\end{pmatrix}\in SL_2(\mathbb{Z})$. More precisely, he proved that for $\begin{pmatrix}a&b\\c&d\end{pmatrix}\in SL_2(\mathbb{Z})$ with $a\neq 0, c>0$,

$$\log\eta\left(\frac{az+b}{cz+d}\right)=\log\eta(z)+\frac{1}{2}\log\left(\frac{cz+d}{i}\right)+\frac{\pi i(a+d)}{12c}-\pi iD(a,c), \tag{1.1}$$

where $D(a,c)$ is the Dedekind sum defined by

$$D(a,c)=\frac{1}{4c}\sum_{k=1}^{c-1}\cot\left(\frac{\pi ak}{c}\right)\cot\left(\frac{\pi k}{c}\right) \tag{1.2}$$

for coprime integers $a$ and $c>0$. We can use (1.1) to prove the so-called reciprocity law given by

$$D(a,c)+D(c,a)=-\frac{1}{4}+\frac{1}{12}\left(\frac{a}{c}+\frac{c}{a}+\frac{1}{ac}\right) \tag{1.3}$$

for coprime positive integers $a,c$. Details of the proofs of (1.1) and (1.3) can be found in the book [6].

An analogy exists between number fields and function fields. For example, $A:=\mathbb{F}_q[T], K:=\mathbb{F}_q(T)$, and $K_\infty:=\mathbb{F}_q((1/T))$ are analogous to $\mathbb{Z},\mathbb{Q}$, and $\mathbb{R}$, respectively. In [1] and [5], we introduced a function field analog $s(a,c)$ (see Section 2) of $D(a,c)$, and established its reciprocity law. In this report, we use the Dedekind sum $s(a,c)$ in function fields to describe the transformation of a certain series under the substitution $z'=(az+b)/(cz+d)$, $\begin{pmatrix}a&b\\c&d\end{pmatrix}\in GL_2(A)$. As an application, we prove the reciprocity law for $s(a,c)$.

## 2. Review of the Dedekind sum

Let $A=\mathbb{F}_q[T]$ and $K=\mathbb{F}_q(T)$. Let $K_\infty=\mathbb{F}_q((1/T))$ be the completion of $K$ at $\infty=(1/T)$, and let $C_\infty$ be the completion of an algebraic closure of $K_\infty$.

2.1. **The Carlitz exponential function.** Let $D_0 = 1$, $D_n = [n][n-1]^q \cdots [1]^{q^{n-1}}$ for $n > 0$ and $[n] = T^{q^n} - T$. Let $e(z)$ be the Carlitz exponential function defined by

$$e(z) = \sum_{n=0}^{\infty} \frac{z^{q^n}}{D_n},$$

which is entire over $C_\infty$. By definition, it holds that $de(z)/dz = e'(z) = 1$. The map $e : C_\infty \to C_\infty$ is $\mathbb{F}_q$-linear and surjective. The kernel $L := \mathrm{Ker}(e)$ is a free $A$-module of rank one. It is easy to see that $e(z)$ is $L$-periodic: $e(z+l) = e(z)$ for $l \in L$. Let $\overline{\pi}$ denote a generator of $L$. The function $e(z)$ can be written as

$$e(z) = z \prod_{0 \neq l \in L} \left(1 - \frac{z}{l}\right).$$

From this, we have

$$\frac{1}{e(z)} = \frac{e'(z)}{e(z)} = \sum_{l \in L} \frac{1}{l+z}.$$

The reader is referred to Goss [4] for additional details of $e(z)$.

2.2. **The Dedekind sum.** Let $a, c$ be the coprime elements of $A \setminus \{0\}$. The (inhomogeneous) Dedekind sum $s(a,c)$ is defined as

$$s(a,c) = \frac{1}{c} \sum_{0 \neq \mu \in A/cA} e\left(\frac{\overline{\pi} a \mu}{c}\right)^{-1} e\left(\frac{\overline{\pi}\mu}{c}\right)^{-1}.$$

When $c$ is a unit of $A$, $s(a,c)$ is defined to be zero. This is an analog of the classical Dedekind sum $D(a,c)$ defined in (1.2). For any $\epsilon \in \mathbb{F}_q \setminus \{0\}$,

$$s(\epsilon a, c) = \epsilon^{-1} s(a,c). \tag{2.1}$$

By replacing $\mu$ with $\epsilon\mu$ ($\epsilon \in \mathbb{F}_q \setminus \{0\}$) in the definition of $s(a,c)$, we see that $s(a,c) = 0$ if $q > 3$.

## 3. A series related to $s(a,c)$

Let $\Omega = C_\infty \setminus K_\infty$ be the Drinfeld upper half-plane, which is an analog of the classical upper half-plane $H := \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$. The group $GL_2(A)$ acts on $\Omega$ by fractional linear transformations $\begin{pmatrix} a & b \\ c & d \end{pmatrix} z = (az+b)/(cz+d)$. Let

$$\xi(z) = \sum_{0 \neq a \in A} \frac{1}{a e(\overline{\pi} a z)},$$

which is convergent for $z \in \Omega$. This can be written as

$$\xi(z) = \begin{cases} 0 & \text{if } q > 3, \\ -\sum_{a \in A_+} 1/a e(\overline{\pi} a z) & \text{if } q = 3, \\ \sum_{a \in A_+} 1/a e(\overline{\pi} a z) & \text{if } q = 2, \end{cases}$$

where $A_+$ is the set of monic elements in $A$.

In the classical case, it is known that for $\gamma \in SL_2(\mathbb{Z})$ and $z \in H$,

$$\log \eta(\gamma z) - \log \eta(z) = -2\pi i \int_z^{\gamma z} G_2(\tau) d\tau,$$

where $G_2(\tau) = -\frac{1}{24}+\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} ne^{2\pi imn\tau}$ is the Eisenstein series of weight 2. We have a similar result for $\xi(z)$. Since $de(z)/dz = 1$, $d\xi(z)/dz = \overline{\pi}\sum_{a\in A+} e(\overline{\pi}az)^{-2}$. As is well known, $g(z) := 1-(T^q-T)\sum_{a\in A_+} e(\overline{\pi}az)^{1-q}$ is a weight $q-1$ modular form for $GL_2(A)$ (see [3], (9.2)). Therefore, we see that for $\gamma \in GL_2(A)$ and $z \in \Omega$,

$$\xi(\gamma z) - \xi(z) = \begin{cases} -\overline{\pi}\int_z^{\gamma z} \frac{g(\tau)-1}{T^3-T} d\tau & if\ q = 3, \\ -\overline{\pi}\int_z^{\gamma z} \left(\frac{g(\tau)-1}{T^2-T}\right)^2 d\tau & if\ q = 2. \end{cases}$$

## 4. Transformation formula for $\xi(z)$

We provide the results for the transformations for $\xi(z)$.

**Proposition 1.** (1) *For* $\epsilon \in \mathbb{F}_q \setminus \{0\}$, $\xi(\epsilon z) = \epsilon^{-1}\xi(z)$.

(2) *For* $\gamma = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in GL_2(A)$, $\xi(\gamma z) = \det\gamma\xi(z)$.

**Theorem 2.** *We have*

$$\xi(-1/z) = \xi(z) - \frac{\alpha(2)}{\overline{\pi}}\left(z + \frac{1}{z}\right) - \frac{\alpha(1)^2}{\overline{\pi}},$$

*where* $\alpha(n) = \sum_{0\neq a\in A} a^{-n}$.

We require the following lemma to prove Theorem 4.

**Lemma 3.** *Let* $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ *be a matrix in* $GL_2(A)$ *with* $c \neq 0$. *Then,*

$$\gamma z = \frac{a}{c} - \frac{\det\gamma}{c(cz+d)}.$$

**Theorem 4.** *For* $\gamma = \begin{pmatrix} 0 & b \\ c & d \end{pmatrix} \in GL_2(A)$,

$$\xi(\gamma z) = \det\gamma\left[\xi(z) - \frac{\alpha(2)}{\overline{\pi}c}\left(cz + d + \frac{1}{cz+d}\right)\right] + \frac{\alpha(1)^2}{\overline{\pi}}. \tag{4.1}$$

The following is the main result.

**Theorem 5.** *Let* $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(A)$. *If* $a \neq 0$ *and* $c \neq 0$, *then*

$$\xi(\gamma z) = \det\gamma\left[\xi(z) - \frac{\alpha(2)}{\overline{\pi}c}\left(cz + d + \frac{1}{cz+d}\right)\right] + \frac{\alpha(1)^2}{\overline{\pi}c} + \overline{\pi}s(a,c). \tag{4.2}$$

## 5. Outline of the proof of Theorem 5

**5.1. Case $\gamma \in SL_2(A)$.** Let

$$R_1 = \sum_{\substack{0\neq f\in A \\ f\equiv 0 \pmod c}} \frac{1}{fe(\overline{\pi}f\gamma z)} - \sum_{\substack{0\neq f\in A \\ f\equiv 0 \pmod c}} \frac{1}{fe(\overline{\pi}fz)},$$

$$R_2 = \sum_{\substack{0\neq f\in A \\ f\not\equiv 0 \pmod c}} \frac{1}{fe(\overline{\pi}f\gamma z)} - \sum_{\substack{0\neq f\in A \\ f\not\equiv 0 \pmod c}} \frac{1}{fe(\overline{\pi}fz)}.$$

Then, we have $\xi(\gamma z) - \xi(z) = R_1 + R_2$, for which we compute $R_1$ and $R_2$, separately.

We first compute $R_1$. By Lemma 3, $1/e(\bar{\pi} f \gamma z) = 1/e(-\bar{\pi} f/c(cz+d))$. Hence,

$$R_1 = \frac{1}{\bar{\pi}} \sum_{\substack{0 \neq f \in A \\ f \equiv 0 \pmod{c}}} \sum_{g \in A} \frac{c(cz+d)}{f(gc(cz+d)-f)} + \frac{1}{\bar{\pi}} \sum_{\substack{0 \neq f \in A \\ f \equiv 0 \pmod{c}}} \sum_{g \in A} \frac{1}{f(g - fz + h)}.$$

Setting $f' = f/c$, $R_1$ becomes

$$\frac{1}{\bar{\pi} c} \sum_{0 \neq f' \in A} \sum_{g \in A} \frac{cz+d}{f'(g(cz+d)-f')} + \frac{1}{\bar{\pi} c} \sum_{0 \neq f' \in A} \sum_{g \in A} \frac{1}{f'(g - f'cz + h)}.$$

We set $h = -f'd$ and then divide $R_1$ into the part $g = 0$ and the part $g \neq 0$. Then,

$$\begin{aligned} R_1 = &-\frac{\alpha(2)}{\bar{\pi} c}\left(cz+d+\frac{1}{cz+d}\right) \\ &+ \frac{1}{\bar{\pi} c} \sum_{0 \neq f' \in A} \sum_{0 \neq g \in A} \frac{cz+d}{f'(g(cz+d)-f')} + \frac{1}{\bar{\pi} c} \sum_{0 \neq f' \in A} \sum_{0 \neq g \in A} \frac{1}{f'(g - f'(cz+d))}. \end{aligned}$$

In the last of the two double summation terms above, we interchange $f'$ and $g$. Then, $R_1$ becomes

$$-\frac{\alpha(2)}{\bar{\pi} c}\left(cz+d+\frac{1}{cz+d}\right) + \frac{\alpha(1)^2}{\bar{\pi} c}.$$

We next compute the two sums in $R_2$. As for the first sum, we have

$$R_3 := \sum_{\substack{f \in A \\ f \not\equiv 0 \pmod{c}}} \frac{1}{f e(\bar{\pi} f \gamma z)} = \sum_{0 \neq \mu \in A/cA} \sum_{\substack{f \in A \\ f \equiv \mu \pmod{c}}} \frac{1}{f e(\bar{\pi} f \gamma z)}.$$

When $f \equiv \mu \pmod{c}$, $f$ can be written as $f = \mu + ch$ for a certain $h \in A$. Hence, by Lemma 3,

$$\frac{\bar{\pi}}{e(\bar{\pi} f \gamma z)} = c \sum_{g \in A} \frac{1}{cg + a\mu - \frac{f}{cz+d}}.$$

Setting $r = cg + a\mu$, $\bar{\pi}/e(\bar{\pi} f \gamma z)$ can be written as

$$c(cz+d) \sum_{\substack{r \in A \\ r \equiv a\mu \pmod{c}}} \frac{1}{r(cz+d) - f}.$$

Thus, we have

$$R_3 = \frac{c}{\bar{\pi}} \sum_{0 \neq \mu \in A} \sum_{\substack{f \in A \\ f \equiv \mu \pmod{c}}} \sum_{\substack{g \in A \\ g \equiv a\mu \pmod{c}}} \frac{cz+d}{f(g(cz+d)-f)}.$$

When $f \equiv \mu \pmod{c}$, using $k := a\mu - \mu$, $f + k \equiv a\mu \pmod{c}$. As for the second summation in $R_2$, noting that $a, c$ are coprime, we obtain

$$R_4 := \sum_{\substack{f \in A \\ f \not\equiv 0 \pmod{c}}} \frac{1}{f e(\bar{\pi} f z)} = \sum_{0 \neq \mu \in A/cA} \sum_{\substack{f \in A \\ f \equiv \mu \pmod{c}}} \frac{1}{(f+k) e(\bar{\pi}(f+k)z)}.$$

Now, we compute $\overline{\pi}/e(\overline{\pi}(f+k)z)$ as follows.

$$\frac{\overline{\pi}}{e(\overline{\pi}(f+k)z)} = -\sum_{g\in A}\frac{1}{g-(f+k)z+h},$$

where $h \in A$ is a fixed element. Because $f \equiv \mu \pmod{c}$,

$$\frac{a\mu - k - d(f+k)}{c} = \frac{-bc\mu - d(f-\mu)}{c} \in A.$$

Letting $h = (a\mu - k - d(f+k))/c$, $\overline{\pi}/e(\overline{\pi}(f+k)z)$ becomes

$$-c\sum_{\substack{r\in A\\ r\equiv a\mu \pmod{c}}}\frac{1}{r-k-(f+k)(cz+d)}.$$

Hence, we have

$$R_4 = -\frac{c}{\overline{\pi}}\sum_{0\neq\mu\in A/cA}\ \sum_{\substack{f\in A\\ f\equiv\mu \pmod{c}}}\ \sum_{\substack{g\in A\\ g\equiv a\mu \pmod{c}}}\frac{1}{(f+k)(g-k-(f+k)(cz+d))}.$$

Noting that $f \equiv \mu \pmod{c}$ if and only if $f+k \equiv a\mu \pmod{c}$, $R_4$ becomes

$$-\frac{c}{\overline{\pi}}\sum_{0\neq\mu\in A/cA}\ \sum_{\substack{f\in A\\ f\equiv\mu \pmod{c}}}\ \sum_{\substack{g\in A\\ g\equiv a\mu \pmod{c}}}\frac{1}{g(f-g(cz+d))}.$$

Using $R_3$ and $R_4$, we obtain

$$R_2 = R_3 - R_4 = \frac{c}{\overline{\pi}}\sum_{0\neq\mu\in A/cA}\ \sum_{\substack{f\in A\\ f\equiv\mu \pmod{c}}}\ \sum_{\substack{g\in A\\ g\equiv a\mu \pmod{c}}}\frac{1}{fg}.$$

As

$$\sum_{\substack{f\in A\\ f\equiv\mu \pmod{c}}}\frac{1}{f} = \sum_{s\in A}\frac{1}{\mu+cs} = \frac{\overline{\pi}}{ce(\overline{\pi}\mu/c)},$$

$$\sum_{\substack{g\in A\\ g\equiv a\mu \pmod{c}}}\frac{1}{g} = \sum_{t\in A}\frac{1}{a\mu+ct} = \frac{\overline{\pi}}{ce(\overline{\pi}a\mu/c)},$$

we see that $R_2 = \overline{\pi}s(a,c)$. Therefore, we conclude that $R_1 + R_2$ is the right-hand side of (4.2).

5.2. **Case $\gamma \in GL_2(A)$.** The cases $q > 3$ and $q = 2$ are trivial. It suffices to show the case of $q = 3$ and $\det\gamma = -1$. Noting that $\begin{pmatrix} a & -b\\ c & -d\end{pmatrix}$ belongs to $SL_2(A)$, using the result obtained in Subsection 5.1, we have

$$\begin{aligned}\xi(\gamma z) &= \xi(\begin{pmatrix} a & -b\\ c & -d\end{pmatrix}(-z))\\ &= \xi(-z) - \frac{\alpha(2)}{\overline{\pi}c}\left(c(-z) - d + \frac{1}{c(-z)-d}\right) + \frac{\alpha(1)^2}{\overline{\pi}c} + \overline{\pi}s(a,c),\end{aligned}$$

which is the right-hand side of (4.2).

## 6. Application

In this section, we prove the following theorem.

**Theorem 6** (Reciprocity law [1, 5]). *Let $a, c \in A \setminus \{0\}$ be coprime.*
(1) *If $q = 3$, then*

$$s(a,c) + s(c,a) = \frac{1}{T^3 - T}\left(\frac{a}{c} + \frac{c}{a} + \frac{1}{ac}\right).$$

(2) *If $q = 2$, then*

$$s(a,c) + s(c,a) = \frac{1}{T^4 + T^2}\left(\frac{a}{c} + \frac{c}{a} + \frac{1}{a} + \frac{1}{c} + \frac{1}{ac} + 1\right).$$

We have already proved this theorem ([1, 5]), by computing the residues of a rational function. Using Theorem 5, we now provide another proof.

*Proof of Theorem 6.* There exist $b, d \in A$ such that $ad - bc = 1$. Then, $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ belongs to $SL_2(A)$. It holds that $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\gamma z = \begin{pmatrix} -c & -d \\ a & b \end{pmatrix} z$. We compute the values of the series $\xi$ on both sides.

Using Theorem 2,

$$\xi(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\gamma z) = \xi(\gamma z) - \frac{\alpha(2)}{\overline{\pi}}\left(\gamma z + \frac{1}{\gamma z}\right) - \frac{\alpha(1)^2}{\overline{\pi}}.$$

Combining Lemma 3 with Theorem 5, this can be written as

$$\begin{aligned}(6.1)\qquad & \xi(z) - \frac{\alpha(2)}{\overline{\pi}c}\left(cz + d + \frac{1}{cz+d}\right) + \frac{\alpha(1)^2}{\overline{\pi}c} + \overline{\pi}s(a,c) \\ & -\frac{\alpha(2)}{\overline{\pi}}\left(\frac{a}{c} - \frac{1}{c(cz+d)} + \frac{c}{a} + \frac{1}{a(az+b)}\right) - \frac{\alpha(1)^2}{\overline{\pi}}.\end{aligned}$$

Using (2.1) and Theorem 5, $\xi(\begin{pmatrix} -c & -d \\ a & b \end{pmatrix} z)$ becomes

$$(6.2)\qquad \xi(z) - \frac{\alpha(2)}{\overline{\pi}a}\left(az + b + \frac{1}{az+b}\right) + \frac{\alpha(1)^2}{\overline{\pi}a} - \overline{\pi}s(c,a).$$

Equating (6.1) with (6.2), we obtain

$$(6.3)\qquad s(a,c) + s(c,a) = \frac{\alpha(2)}{\overline{\pi}^2}\left(\frac{a}{c} + \frac{c}{a} + \frac{1}{ac}\right) + \frac{\alpha(1)^2}{\overline{\pi}^2}\left(\frac{1}{a} - \frac{1}{c} + 1\right).$$

Combining (6.3) with the following lemma enables us to complete the proof.

**Lemma 7.** (1) *If $q = 3$, then*

$$\alpha(2) = \frac{\overline{\pi}^2}{T^3 - T}, \quad \alpha(1) = 0.$$

(2) *If $q = 2$, then*

$$\alpha(2) = \alpha(1)^2 = \frac{\overline{\pi}^2}{T^4 + T^2}.$$

## 7. An analog of the sawtooth function

The sawtooth function $((x))$ is defined by

$$((x)) = \begin{cases} \{x\} - 1/2 & if\ x \in \mathbb{R} \setminus \mathbb{Z}, \\ 0 & if\ x \in \mathbb{Z}, \end{cases}$$

where $\{x\}$ is the fraction part of $x$. This function has the following Fourier expansion:

$$((x)) = -\frac{1}{2\pi i} \sum_{0 \neq n \in \mathbb{Z}} \frac{\exp(2\pi i n x)}{n}.$$

Inspired by the definition of $\xi(z)$, we define an analog of $((x))$ as follows. For a given $x \in K_\infty$, let $S_x$ be the set of all $a \in A$ such that $e(\overline{\pi} a x) \neq 0$. Then, we define

$$F(x) = \begin{cases} -\frac{1}{\overline{\pi}} \sum_{a \in S_x} \frac{1}{a e(\overline{\pi} a x)} & if\ S_x \neq \phi, \\ 0 & if\ S_x = \phi. \end{cases}$$

This function has the following properties.

- For $\epsilon \in \mathbb{F}_q \setminus \{0\}$, $F(\epsilon x) = \epsilon^{-1} F(x)$.
- For $b \in A$, $F(b) = 0$.
- For $b \in A$, $F(x + b) = F(x)$.

Moreover, the value $F(x)$ at $x \in K$ can be described in terms of the Dedekind sum in function fields:

**Theorem 8.** *For coprime* $a, c \in A \setminus \{0\}$, $F(a/c) = s(-a, c)$.

We note that for the function $((x))$, the result corresponding to Theorem 8 does not hold. It would be interesting to investigate $F(x)$ in the future.

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