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ZEROS OF THE L-FUNCTION ATTACHED TO A CUSP FORM AND SOME APPLICATIONS OF SELBERG'S ORTHOGONALITY

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1. Introduction

In this article, we survey and announce some new results which are obtained by using *Selberg's orthogonality*. The following contents are based on the author's talk at the conference "Analytic Number Theory and Related Areas" in 2015, which was held at RIMS in Kyoto.

First we shall describe the definition of Selberg's orthogonality and related things. Selberg [28] introduced a class of Dirichlet series, which is called the Selberg class $\mathcal{S}$. This class is defined to be the set of Dirichlet series $L(s)=\sum_{n=1}^{\infty}a_L(n)n^{-s}$ which satisfy, roughly, the following axioms:

(i) Ramanujan bound: $a_L(n)\ll_{\varepsilon}n^{\varepsilon}$ for any $\varepsilon>0$,
(ii) Analytic continuation (except for a possible pole at $s=1$),
(iii) Functional equation,
(iv) Euler product expression.

For the precise definition of the class $\mathcal{S}$ and various results on $\mathcal{S}$, see e.g. [28] and [22].

We denote by $\mathcal{S}\setminus\{1\}$ the set of all functions in the Selberg class $\mathcal{S}$ except for the constant function 1. A function $L(s)\in\mathcal{S}\setminus\{1\}$ is called *primitive* if it cannot be factored as a product of two functions in $\mathcal{S}$ non-trivially, which means that $L(s)=L_1(s)L_2(s)$ with $L_1(s),L_2(s)\in\mathcal{S}$ implies $L_1(s)=1$ or $L_2(s)=1$. Selberg [28] gave the following conjecture for the set of primitive functions.

Conjecture 1 (Selberg orthonormality conjecture). *For any two primitive functions $L_1(s)=\sum_{n=1}^{\infty}a_{L_1}(n)n^{-s}, L_2(s)=\sum_{n=1}^{\infty}a_{L_2}(n)n^{-s}\in\mathcal{S}$, we have*

$$\sum_{p\le x}\frac{a_{L_1}(p)\overline{a_{L_2}(p)}}{p}=\begin{cases}\log\log x+O(1) & \text{if } L_1(s)=L_2(s),\\ O(1) & \text{if } L_1(s)\neq L_2(s),\end{cases}\tag{1.1}$$

as $x\to\infty$. Here and below the letter p denotes a prime number.

It is known that this conjecture implies several interesting consequences. See e.g. [22, Section 3].

In this article we define *Selberg's orthogonality* as follows.

Definition 1. *Let $\mathcal{D}$ be a set of Dirichlet series. We say that $\mathcal{D}$ satisfies Selberg's orthogonality if for any two Dirichlet series $D_1(s)=\sum_{n=1}^{\infty}a_1(n)n^{-s}, D_2(s)=\sum_{n=1}^{\infty}a_2(n)n^{-s}$ in $\mathcal{D}$ we have*

$$\sum_{p\le x}\frac{a_1(p)\overline{a_2(p)}}{p}=\begin{cases}c_1\log\log x+O(1) & \text{if } D_1(s)=D_2(s),\\ O(1) & \text{if } D_1(s)\neq D_2(s),\end{cases}\tag{1.2}$$

as $x\to\infty$, where c_1 is a positive constant depending on $D_1(s)$.

This article will give new applications of Selberg's orthogonality to the following two topics:

(1) Zeros of the L-function attached to a holomorphic cusp form for $SL(2,\mathbb{Z})$,

(2) Independence of general L-functions (without assuming the Generalized Ramanujan Conjecture).

See Theorems A and B and Corollaries C and D below. We remark that actually, weaker versions of (1.2) are sufficient for the proofs of those theorems.

We have also a new application of Selberg's orthogonality to the topic:

(3) Sign changes of Fourier coefficients of a holomorphic cusp form (not necessarily a Hecke eigen cusp form) for $SL(2,\mathbb{Z})$.

This result was however omitted in the author's talk and is omitted in this article also.

These three kinds of results (1) (2) and (3) (and their proofs) may be considered to indicate that the L-functions attached to primitive cusp forms are *independent* in the theory of complex analysis, in the theory of transcendental numbers and functions, and in the theory of Fourier coefficients of cusp forms (i.e., Dirichlet coefficients of the associated L-functions), respectively.

2. Zeros of L-functions I

2.1. Dirichlet L-functions. Let χ be a primitive Dirichlet character and let $L(s,\chi)$ denote the associated Dirichlet L-function, which is defined by

$$L(s,\chi) := \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} \qquad \text{for } \operatorname{Re} s > 1.$$

If χ is the principal character mod 1 then $L(s,\chi)$ is the Riemann zeta-function $\zeta(s)$. It is well known that $L(s,\chi)$ has analytic continuation to the whole plane $\mathbb{C}$ and a certain functional equation whose critical line is $\operatorname{Re} s = 1/2$. Also, $L(s,\chi)$ has the Euler product expression

$$L(s,\chi) = \prod_p \left(1 - \frac{\chi(p)}{p^s}\right)^{-1} \qquad \text{for } \operatorname{Re} s > 1.$$

This implies that $L(s,\chi)$ has no zeros in the half-plane $\operatorname{Re} s > 1$.

We then have the following famous conjecture:

Conjecture 2 (Generalized Riemann Hypothesis for $L(s,\chi)$). *Let χ be a primitive Dirichlet character. Then the Dirichlet L-function $L(s,\chi)$ has no zeros in the strip $1/2 < \operatorname{Re} s < 1$.*

2.2. The Davenport-Heilbronn function. We now consider a function which has a functional equation similar to that of the Riemann zeta-function $\zeta(s)$ but has zeros in the strip $\operatorname{Re} s > 1/2$, i.e., does not satisfy an analogue of the Riemann hypothesis.

Inspired by the papers [6] [7] of Davenport-Heilbronn, Titchmarsh [30, Chap. X, 10.25] introduced the following function $L(s)$:

$$\begin{aligned} L(s) &= \frac{1}{2}\sec\theta\left(e^{-i\theta}L(s,\chi) + e^{i\theta}L(s,\overline{\chi})\right) \\ &= \frac{1}{1^s} + \frac{\tan\theta}{2^s} + \frac{-\tan\theta}{3^s} + \frac{-1}{4^s} + \frac{1}{6^s} + \cdots, \end{aligned}$$

where $0 < \theta < \frac{1}{4}\pi$ is a real number satisfying $\tan\theta = \frac{\sqrt{10-2\sqrt{5}}-2}{\sqrt{5}-1}$ and $L(s,\chi)$ is the Dirichlet L-function mod 5 given by

$$L(s,\chi) = \frac{1}{1^s} + \frac{i}{2^s} + \frac{-i}{3^s} + \frac{-1}{4^s} + \frac{1}{6^s} + \cdots.$$

This function $L(s)$ is usually called the Davenport-Heilbronn function (see [2, p. 239]). We note that $L(s)$ is a linear combination of Dirichlet L-functions. As in [30, Chap. X, 10.25],

$L(s)$ satisfies the functional equation

$$\left(\frac{5}{\pi}\right)^{s/2}\Gamma\left(\frac{1}{2}+\frac{1}{2}s\right)L(s)=\left(\frac{5}{\pi}\right)^{1/2-s/2}\Gamma\left(1-\frac{1}{2}s\right)L(1-s).$$

This functional equation is similar to that of the Riemann zeta-function $\zeta(s)$. Titchmarsh [30, Chap. X, 10.25], however, showed the following result.

Theorem 1 (Titchmarsh). *The function $L(s)$ has infinitely many zeros in the half-plane $\mathrm{Re}\, s > 1$ (as well as on the line $\mathrm{Re}\, s = 1/2$).*

Further, for the case of the strip $1/2 < \mathrm{Re}\, s < 1$, Voronin (see [12, p. 214, Theorem 1]) showed the following result, which is in contrast to Conjecture 2. Related results to Theorems 1 and 2 are given in, for example, [10] and [27].

Theorem 2 (Voronin). *Let σ_1 and σ_2 be any real numbers with $1/2 < \sigma_1 < \sigma_2 < 1$. Then the function $L(s)$ has infinitely many zeros in the strip $\sigma_1 < \mathrm{Re}\, s < \sigma_2$.*

3. Zeros of L-functions II

3.1. **L-functions attached to cusp forms for $SL(2, Z)$.** Let $k \geq 12$ be an even positive integer. Let S_k denote the set of holomorphic cusp forms of weight k for $SL(2,\mathbb{Z})$. Let $f(z) \in S_k$. We write its Fourier expansion as

$$f(z)=\sum_{n=1}^{\infty}a_f(n)n^{(k-1)/2}e^{2\pi i n z}.$$

Then the (normalized) L-function $L(s, f)$ attached to $f(z)$ is defined by

$$L(s,f):=\sum_{n=1}^{\infty}\frac{a_f(n)}{n^s}.$$

This function has analytic continuation to the whole plane $\mathbb{C}$ and a certain functional equation whose critical line is $\mathrm{Re}\, s = 1/2$.

Assume that $f(z)$ is a Hecke eigen cusp form in S_k. Then $L(s, f)$ has the Euler product expression

$$L(s,f)=a_f(1)\prod_p\left(1-\frac{\lambda_f(p)}{p^s}+\frac{1}{p^{2s}}\right)^{-1},$$

where we write $a_f(n) = a_f(1)\lambda_f(n)$. We have the estimate

$$|\lambda_f(p)| \leq 2 \qquad \text{for every prime } p, \tag{3.1}$$

due to Deligne. This Euler product expression implies that

(3.2) $L(s, f)$ has no zeros in the half-plane $\mathrm{Re}\, s > 1$.

Further, it is believed that the following conjecture is true.

Conjecture 3 (Generalized Riemann Hypothesis for $L(s, f)$). *If $f(z) \in S_k$ is a Hecke eigen cusp form, then the L-function $L(s, f)$ has no zeros in the strip $1/2 < \mathrm{Re}\, s < 1$.*

Next we shall consider the case that $f(z)$ is not a Hecke eigen cusp form. In contrast to (3.2), Conrey and Ghosh [5, Theorem 2] proved that $L(s, \Delta^2)$ has infinitely many zeros in the half-plane $\mathrm{Re}\, s > 1$. Here, as usual, $\Delta(z)$ denotes the function given by

$$\Delta(z)=e^{2\pi i z}\prod_{n=1}^{\infty}\left(1-e^{2\pi i n z}\right)^{24},$$

which belongs to S_{12}, so that $\Delta(z)^2$ belongs to S_{24}. In general, Booker and Thorne [3] showed the next theorem. These results are analogues of Theorem 1.

Theorem 3 (Booker & Thorne). *Assume that $f(z) \in S_k$ is not a Hecke eigen cusp form. Then $L(s,f)$ has infinitely many zeros in the half-plane* $\operatorname{Re} s > 1$.

Later, Righetti [25] gave a related result on the existence of such zeros in the half-plane of absolute convergence for an axiomatically-defined class of L-functions. One of the axioms of this class is (a weak version of) Selberg's orthogonality (1.2) given in Definition 1.

The first main theorem of the present article is the following, which concerns the existence of zeros in the strip $1/2 < \operatorname{Re} s < 1$. This is in contrast to Conjecture 3.

Theorem A. *Assume that $f(z) \in S_k$ is not a Hecke eigen cusp form. Let σ_1 and σ_2 be any real numbers with $1/2 < \sigma_1 < \sigma_2 < 1$. Then $L(s,f)$ has infinitely many zeros in the strip $\sigma_1 < \operatorname{Re} s < \sigma_2$. More precisely, we have*

$$N_f(\sigma_1,\sigma_2,T) \gg_{f,\sigma_1,\sigma_2} T \qquad \text{as } T \to \infty, \tag{3.3}$$

where $N_f(\sigma_1,\sigma_2,T)$ denotes the number of zeros ρ of $L(s,f)$ satisfying $\sigma_1 < \operatorname{Re}\rho < \sigma_2$ and $0 < \operatorname{Im}\rho < T$.

As an improvement of (3.3), it would be conjectured that

$$N_f(\sigma_1,\sigma_2,T) = C_{f,\sigma_1,\sigma_2} T + o(T) \qquad \text{as } T \to \infty,$$

where C_{f,σ_1,σ_2} is some positive constant depending on f, σ_1 and σ_2. For related results, see e.g. [8] and [14].

Let us now describe an outline of the proof of Theorem A. It is well known that the set S_k is a linear space with $\dim S_k < \infty$. Let $\mathcal{H}_k$ denote the set of normalized (i.e. $a_f(1) = 1$) Hecke eigen cusp forms in S_k. Then $\mathcal{H}_k$ is a basis of S_k. Therefore, if $f(z) \not\equiv 0$ is not a Hecke eigen cusp form, then we can write

$$f(z) = \alpha_1 f_1(z) + \cdots + \alpha_r f_r(z)$$

with $r \geq 2$, nonzero complex numbers α_j and distinct forms $f_j(z)$ in $\mathcal{H}_k$, and hence

$$L(s,f) = \alpha_1 L(s,f_1) + \cdots + \alpha_r L(s,f_r). \tag{3.4}$$

Thus, value-distribution of the L-function $L(s,f)$ will be known from *joint* value-distribution of $L(s,f_r)$'s.

As a related result to the proof of Theorem A, the author [19] [18] proved the following joint denseness theorem for the L-functions $L(s,f_1),\dots,L(s,f_r)$ on a vertical line in the strip $1/2 < \operatorname{Re} s < 1$. (The paper [18, Theorem 1.1] considers a more complicated case, precisely, the case of the Rankin-Selberg L-functions $L(s,f_1 \otimes g),\dots,L(s,f_r \otimes g)$.)

Theorem 4 (Nagoshi). *Let $1/2 < \sigma_0 < 1$. Let $f_1(z),\dots,f_r(z)$ be distinct forms in $\mathcal{H}_k$. Then the set*

$$\{(L(\sigma_0+it,f_1),\dots,L(\sigma_0+it,f_r)) \in \mathbb{C}^r : t \in \mathbb{R}\}$$

is dense in $\mathbb{C}^r$.

We note that (a weak version of) Selberg's orthogonality for $L(s,f_1),\dots,L(s,f_r)$ is used in the proof of Theorem 4.

Very roughly speaking, Theorem A is proved by combining arguments of the papers [3] [25] (which deal with the case $\operatorname{Re} s > 1$) and arguments of a proof of Theorem 4 (which handle the case $1/2 < \operatorname{Re} s < 1$; see also [12, Chap. VII, 3]).

We make a remark on another proof of Theorem A. Recently, Lee, Nakamura and Pańkowski [15, Theorem 1.2] obtained the so-called joint universality for L-functions in the Selberg class under a stronger version of Selberg's orthonormality. (Note that the above L-functions

$L(s, f_1)$, ..., $L(s, f_r)$ satisfy this strong version of Selberg's orthonormality.) This theorem is a joint denseness result for L-functions in a certain set of holomorphic functions, whereas Theorem 4 is a joint denseness result for L-functions in $\mathbb{C}^r$. The proof of Theorem 4 (see also [18, Theorem 1.1]) and that of their theorem [15, Theorem 1.2] are analogous to each other. Using their theorem and (3.4), we can obtain the so-called *strong universality* (see [29, Section 11.3]) for the L-function $L(s, f)$ with $f(z)(\not\equiv 0)$ being as in Theorem A. This strong universality yields Theorem A.

4. Hypertranscendence of an L-function without assuming GRC

In this and the next section, we discuss differential independence properties for general L-functions. In this section, we shall concentrate on the case of a single general L-function.

Let F be a field of meromorphic functions. A meromorphic function $f(s)$ on $\mathbb{C}$ is called *hypertranscendental* over F, if $y = f(s)$ does not satisfy any nontrivial algebraic differential equation over F (that is, any equation of the form $P(y, y', \dots, y^{(n)}) = 0$, where n is a non-negative integer, and where P is a non-zero polynomial in $y, y', \dots, y^{(n)}$ whose coefficients belong to F). The field F is usually required to be a differential field (that is, F is closed under differentiation).

The following result was stated by Hilbert [9, p. 428] in 1900 in his famous lecture at the ICM in Paris. His proof is based on a result of Hölder (which asserts that the Gamma function $\Gamma(s)$ is hypertranscendental over $\mathbb{C}(s)$) and an usual functional equation for $\zeta(s)$.

Theorem 5 (Hilbert). *The Riemann zeta-function $\zeta(s)$ is hypertranscendental over the field $\mathbb{C}(s)$ of rational functions.*

Another proof of Theorem 5 and a general result for a wide class of Dirichlet series were obtained by Ostrowski [21].

Much later, from the viewpoint of value-distribution of $\zeta(s)$, Voronin [32] [12, p. 254] obtained yet another proof of Theorem 5 and the following stronger theorem, which is called *functional independence (in the sense of Voronin)* of $\zeta(s)$ and its derivatives (see [29, p. 196]). Voronin's proof is based on his result [31] which asserts that if σ is a real number with $1/2 < \sigma \leq 1$ then the set

$$\left\{ \left(\zeta(\sigma + it), \zeta'(\sigma + it), \dots, \zeta^{(K)}(\sigma + it) \right) \in \mathbb{C}^{K+1} : t \in \mathbb{R} \right\} \tag{4.1}$$

is dense in $\mathbb{C}^{K+1}$.

Theorem 6 (Voronin). *Let K and N be non-negative integers. Let $H_0, \dots, H_N : \mathbb{C}^{K+1} \to \mathbb{C}$ be continuous functions, not all identically zero. Then*

$$\sum_{n=0}^{N} s^n H_n\left(\zeta(s), \zeta'(s), \dots, \zeta^{(K)}(s)\right) = 0$$

does not hold identically for $s \in \mathbb{C} \setminus \{1\}$.

We introduced the L-function $L(s, \chi)$ attached to a primitive Dirichlet character χ in Section 2 and the L-function $L(s, f)$ attached to a normalized Hecke eigen cusp form $f(z) \in S_k$ in Section 3. We now introduce a generalization of those functions. Let $\pi = \otimes'_{p \leq \infty} \pi_p$ be an irreducible cuspidal automorphic representation of $GL_m(\mathbb{A}_\mathbb{Q})$ with unitary central character, where $\mathbb{Q}$ denotes the field of rational numbers and $\mathbb{A}_\mathbb{Q}$ its ring of adeles. The L-function $L(s, \pi)$ attached to π is defined as an Euler product of the form

$$L(s, \pi) = \prod_{p < \infty} \prod_{j=1}^{m} \left(1 - \frac{\alpha_\pi(p, j)}{p^s} \right)^{-1}. \tag{4.2}$$

Here $\alpha_\pi(p,j)$ $(1 \le j \le m)$ are complex numbers defined in terms of certain parameters of π_p (Satake parameters if π_p is unramified, and Langlands parameters in general). See e.g. [26].

The Generalized Ramanujan Conjecture (GRC) at non-archimedean places for π is the assertion that if π_p $(p < \infty)$ is unramified then

$$|\alpha_\pi(p,j)| = 1 \qquad \text{for all } 1 \le j \le m.$$

This is verified for certain representations π (for example, π of $GL_2(\mathbb{A}_\mathbb{Q})$ corresponding to a cusp form in $\mathcal{H}_k$ for $SL(2,\mathbb{Z})$; see (3.1)) but not in general.

It is proved (see e.g. [29, p. 283]) that if π satisfies the GRC (including the case of the archimedean place), then $L(s,\pi)$ has a denseness property as in (4.1) and hence we have an analogue of Theorem 6 for $L(s,\pi)$.

The author [20] recently obtained a certain type of functional difference-differential independence for $L(s,\pi)$, without any assumptions (such as assuming the GRC) on π, as in the next theorem. Let μ be any non-negative integer, $h_0, h_1, \dots, h_\mu$ be any real numbers with $h_0 < h_1 < \cdots < h_\mu$, and $\nu_0, \nu_1, \dots, \nu_\mu$ be any non-negative integers. We set

$$M := \sum_{j=0}^{\mu} (\nu_j + 1).$$

Following Reich's paper [24, p. 29], we say that a function $\Phi : \mathbb{C}^n \to \mathbb{C}$ is "locally not trivial", if for every non-empty open set $U \subset \mathbb{C}^n$ the restriction of Φ to U is not identically zero. For example, every holomorphic function $\Phi : \mathbb{C}^n \to \mathbb{C}$ which is not identically zero is "locally not trivial", according to the identity theorem.

Theorem 7 (Nagoshi). *Let π be an irreducible cuspidal automorphic representation of $GL_m(\mathbb{A}_\mathbb{Q})$ with unitary central character, where m is any positive integer. Let N be a non-negative integer. Let $\Phi_N : \mathbb{C}^M \to \mathbb{C}$ be a continuous and "locally not trivial" function. When $N \ge 1$, for each integer $0 \le n \le N-1$ let $\Phi_n : \mathbb{C}^M \to \mathbb{C}$ be a continuous function. Then*

$$\sum_{n=0}^{N} s^n \Phi_n\big(L(s+h_0,\pi), L'(s+h_0,\pi), \cdots, L^{(\nu_0)}(s+h_0,\pi), L(s+h_1,\pi),$$
$$\cdots, L^{(\nu_1)}(s+h_1,\pi), \cdots, L(s+h_\mu,\pi), \cdots, L^{(\nu_\mu)}(s+h_\mu,\pi)\big) = 0$$

does not hold identically for $s \in \mathbb{C}$ with $\operatorname{Re} s + h_0 > 1$.

This theorem implies the following algebraic difference-differential independence of $L(s,\pi)$ over $\mathbb{C}(s)$ and, in particular, the hypertranscendence (in the above sense) of $L(s,\pi)$ over $\mathbb{C}(s)$, which is a generalization of Theorem 5.

Corollary 8. *Let π be as in Theorem 7. Let*

$$P(s; z_1, \dots, z_M) = \sum_{a_1,\dots,a_M} C_{a_1,\dots,a_M}(s)\, z_1^{a_1} \cdots z_M^{a_M}$$

be a non-zero polynomial in M-variables $z_1, \dots, z_M$ whose coefficients $C_{a_1,\dots,a_M}(s)$ belong to $\mathbb{C}(s)$. Then

$$P\big(s; L(s+h_0,\pi), L'(s+h_0,\pi), \cdots, L^{(\nu_0)}(s+h_0,\pi), L(s+h_1,\pi),$$
$$\cdots, L^{(\nu_1)}(s+h_1,\pi), \cdots, L(s+h_\mu,\pi), \cdots, L^{(\nu_\mu)}(s+h_\mu,\pi)\big) = 0$$

does not hold identically for $s \in \mathbb{C}$ with $\operatorname{Re} s + h_0 > 1$. *In particular, $L(s,\pi)$ is hypertranscendental over $\mathbb{C}(s)$.*

We can actually obtain the hypertranscendence of $L(s,\pi)$ over a certain field $\mathcal{F}_s$ which contains $\mathbb{C}(s)$. See [20].

The proof of Theorem 7 in [20] makes use of the following:

- The prime number theorem
$$\sum_{n\le x} \Lambda(n)\,|a_\pi(n)|^2 \sim x \qquad (\text{as } x\to\infty)$$
for π, where $\Lambda(n)$ denotes the von Mangoldt function and $a_\pi(p^k) := \sum_{j=1}^{m} \alpha_\pi(p,j)^k$ (see [16]).
- Towards the GRC, it is proved (see [26, Proposition A.1]) that there exists a positive constant $\theta < 1/2$ (depending only on m) such that
$$|\alpha_\pi(p,j)| \le p^\theta$$
for all primes p and $1\le j\le m$.
- Reich's approach [24], which is from the viewpoint of the theory of value-distribution of Dirichlet series.

5. Algebraic difference-differential independence of L-functions without assuming GRC

After obtaining Theorem 6, Voronin [33] gave a similar functional independence property for any set of Dirichlet L-functions $L(s,\chi_j)$ with χ_j's being pairwise non-equivalent Dirichlet characters.

Further, the author [17] announced the following more general theorem. Let $\mathcal{L}$ be the set of Dirichlet series $L(s) = \sum_{n=1}^{\infty} a_L(n)n^{-s}$ satisfying the following two axioms:

(i) Ramanujan bound: $a_L(n) \ll_\varepsilon n^\varepsilon$ for any $\varepsilon > 0$,

(ii) Polynomial Euler product expression: there exists a positive integer m_L and complex numbers $\alpha_L(p,j)$, for all primes p and $1\le j\le m$, such that
$$L(s) = \prod_p \prod_{j=1}^{m_L} \left(1 - \frac{\alpha_L(p,j)}{p^s}\right)^{-1}.$$

We do not need to assume analytic continuation to the half-plane $\mathrm{Re}\, s \le 1$ and a functional equation for $L(s)$.

Theorem 9 (Nagoshi). *Let $L_1(s),\dots,L_N(s) \in \mathcal{L}$ be distinct Dirichlet series which satisfy Selberg's orthogonality. Let K be a positive integer. Then $L_1(s),\dots,L_N(s)$, $L_1^{(1)}(s)$, $\dots$, $L_N^{(1)}(s)$, $\dots$, $L_1^{(K)}(s),\dots,L_N^{(K)}(s)$ are functionally independent (in the sense of Voronin). In particular, $L_1(s),\dots,L_N(s)$ are algebraically differentially independent over $\mathbb{C}(s)$.*

We actually can prove a stronger result. The proof makes use of value-distribution of functions in $\mathcal{L}$ and their derivatives on the half-plane $\mathrm{Re}\, s > 1$, and is analogous to that of Theorem 4 above. See a forthcoming paper.

We shall now discuss new related results for a much larger class than $\mathcal{L}$, under Selberg's orthogonality. Let $\mathcal{D}$ denote the set of Dirichlet series $D(s)$ which are absolutely convergent if $\mathrm{Re}\, s$ is sufficiently large. In particular, we do not assume the Ramanujan bound and hence we can obtain Corollary D below.

Conrey and Ghosh [4] showed essentially that if $L_1(s)$, $\dots$, $L_N(s) \in \mathcal{D}$ are distinct Dirichlet series which satisfy Selberg's orthogonality, then they are multiplicatively independent. This result implies that if Selberg's orthonormality conjecture (Conjecture 1) is true, then any function in the Selberg class $\mathcal{S}$ has unique factorization into primitive functions up to the order of factors. For an unconditional result on linear independence for a large class, which contains $\mathcal{S}$, see [11].

The second main theorem of the present article is the following, which is stronger than the above result of Conrey-Ghosh.

Theorem B. *Let $L_1(s), \ldots, L_N(s) \in \mathcal{D}$ be distinct Dirichlet series which satisfy Selberg's orthogonality. Then $L_1(s), \ldots, L_N(s)$ are algebraically difference-differentially independent over $\mathbb{C}(s)$ (in the sense of Corollary 8).*

Corollary C. *Assume that Selberg's orthonormality conjecture (Conjecture 1) is true. Then the primitive functions in the Selberg class $\mathcal{S}$ are algebraically difference-differentially independent over $\mathbb{C}(s)$ (in the sense of Corollary 8).*

Remark 1. We recall that h_j's (in Corollary 8) are real numbers. For example, $L(s, \chi)$ and $L(s + i\alpha, \chi)$ are distinct primitive functions in $\mathcal{S}$, where χ is a primitive Dirichlet character mod q with $q > 1$ and $\alpha \neq 0$ is a real number. Therefore in Corollary C it is not possible to let the condition of h_j's to be arbitrary distinct complex numbers.

It is known (see [1]) that the set of the L-functions $L(s, \pi)$ (given in (4.2)) for $GL_m(\mathbb{A}_\mathbb{Q})$ with $1 \leq m \leq 4$ satisfies Selberg's orthogonality unconditionally. Hence from Theorem B we have

Corollary D. *The L-functions $L(s, \pi)$ for $GL_m(\mathbb{A}_\mathbb{Q})$ with $1 \leq m \leq 4$ are algebraically difference-differentially independent over $\mathbb{C}(s)$ (in the sense of Corollary 8).*

Let us describe a sketch of the proof of Theorem B. For simplicity, we shall now show only the weaker assertion that $L_1(s), \ldots, L_N(s)$ are algebraically independent over $\mathbb{C}$. We use the following result due to Popken [23] (see also [13, Theorem 1]):

Lemma 1 (Popken). *Assume that Dirichlet series $D_1(s) = \sum_{n=1}^\infty a_1(n)n^{-s}, \ldots, D_r(s) = \sum_{n=1}^\infty a_r(n)n^{-s} \in \mathcal{D}$ are algebraically dependent over $\mathbb{C}$. Then there exist complex numbers A_j $(1 \leq j \leq r)$, not all zero, such that the relation*

$$\sum_{j=1}^{r} A_j a_j(p) = 0$$

holds for all primes p except for finitely many.

We write $L_j(s) = \sum_{n=1}^\infty a_j(n)n^{-s}$ for each $1 \leq j \leq N$. Let $(B_1, \ldots, B_N) \in \mathbb{C}^N$ be arbitrary with $(B_1, \ldots, B_N) \neq (0, \ldots, 0)$. Then, using Selberg's orthogonality, we have

$$\begin{aligned} &\sum_{p \leq x} \frac{|B_1 a_1(p) + \cdots + B_N a_N(p)|^2}{p} \\ &= \sum_{p \leq x} \frac{\sum_{j=1}^N |B_j|^2 |a_j(p)|^2 + \sum_{j \neq k} B_j \overline{B_k} a_j(p) \overline{a_k(p)}}{p} \\ &\gg \log\log x, \end{aligned}$$

which goes to ∞ as $x \to \infty$. Hence, in particular,

$$B_1 a_1(p) + \cdots + B_N a_N(p) \neq 0$$

for infinitely many primes p. This and Lemma 1 give the algebraic independence of $L_1(s)$, $\ldots$, $L_N(s)$ over $\mathbb{C}$. Theorem B is proved by extending this argument.

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