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Cutting Pascal's triangle modulo a prime by a straight line

Tomohiro Ikkai
Graduate School of Mathematics,
Nagoya University

We use the following notations in this article.

- $\mathbb{N}$: the set of positive integers,
  $\mathbb{N}_0$: the set of non-negative integers.
- $\mathcal{T}=\{(x,y)\in\mathbb{R}^2\,;\,x\geq y\geq 0\}$,
  $\mathcal{T}_{\mathbb{N}_0}=\mathcal{T}\cap(\mathbb{N}_0\times\mathbb{N}_0)=\{(m,n)\in\mathbb{N}_0\times\mathbb{N}_0\,;\,m\geq n\}$.
- p: a given prime number.
- $I_p=\{0,1,\ldots,p-1\}$.
- $\theta_p=\dfrac{\log\frac{p(p-1)}{2}}{\log p}$.

The set $\mathcal{T}_{\mathbb{N}_0}$ can be regarded as Pascal's triangle in the sense that each point $(m,n)\in\mathcal{T}_{\mathbb{N}_0}$ corresponds to a binomial coefficient $\binom{m}{n}$. We consider the subset

$$\operatorname{Pas}(p)=\left\{(m,n)\in\mathcal{T}_{\mathbb{N}_0}\,;\,\binom{m}{n}\not\equiv 0\pmod p\right\}$$

of $\mathcal{T}_{\mathbb{N}_0}$. This set $\operatorname{Pas}(p)$ has "self-similarity" as seen in Figure 1 and Figure 2.

Essouabri [1] introduced the zeta functions associated to $\operatorname{Pas}(p)$ and proved some of their analytic properties. The zeta function associated to $\operatorname{Pas}(p)$ is defined, for two-variable polynomials $P,Q\in\mathbb{R}[X,Y]$ and $s\in\mathbb{C}$, by the following general Dirichlet series

$$Z_p(P,Q;s)=\sum_{\substack{(m,n)\in\operatorname{Pas}(p)\\ P(m,n)\neq 0}}\frac{Q(m,n)}{P(m,n)^{s/\deg P}},$$

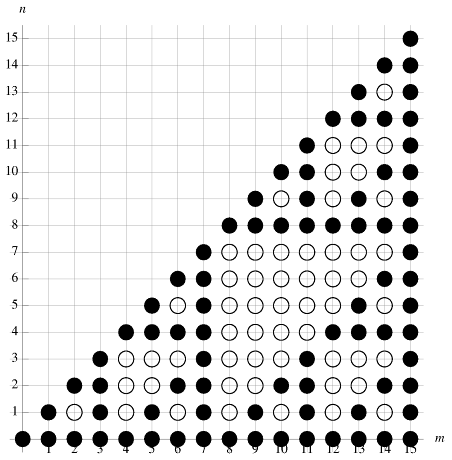

Figure 1: $0 \leq m < 16$ in Pas (2)

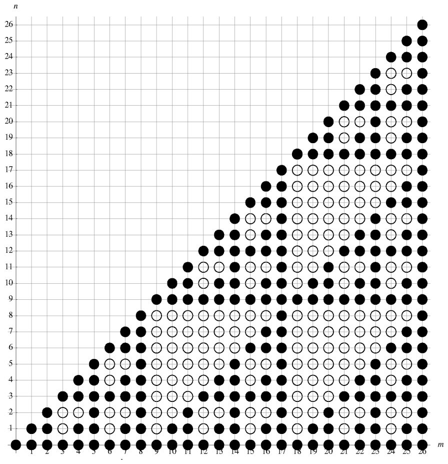

Figure 2: $0 \leq m < 27$ in Pas (3)

where $P(X, Y)$ satisfies the following two conditions:

- $\mathcal{T}$*-ellipticity*: Let $P_*(X, Y)$ denote the highest degree part of $P(X, Y)$. (That is, $P_*(X, Y) \in \mathbb{R}[X, Y]$ is the unique homogeneous polynomial such that $\deg(P - P_*) < \deg P$, where $\deg(0) := -\infty$.)

  Then $P_*(x, y) > 0$ for any $(x, y) \in \mathcal{T} \setminus \{(0, 0)\}$.

- $\mathcal{T}$*-positivity*: For any $(x, y) \in \mathcal{T}$, we have $P(x, y) \geq 0$.

We do not decide the order of summation in the definition of $Z_p(P, Q; s)$ in this article, so that we only consider the absolute convergence. In other words, we disregard the matter of conditional convergence of $Z_p(P, Q; s)$.

The series defining our zeta function $Z_p(P, Q; s)$ absolutely and uniformly converges at least on any compact set contained in the half-plane $\{\mathrm{Re}(s) > 2 + \deg Q\}$. The convergent region of $Z_p(P, Q; s)$ can be actually larger than this half-plane. For example, when $Q(X, Y) = 1$, we have the abscissa of absolute convergence $\sigma_a = \theta_p$ of $Z_p(P, 1; s)$.

Essouabri proved that $Z_p(P, Q; s)$ is meromorphically continued to the whole plane $\mathbb{C}$. He also gave the following theorem on the poles of $Z_p(P, Q; s)$.

Theorem 1 (Essouabri, 2005 [1]). *The meromorphic function on* $\mathbb{C}$ *defined by*

$$Z_p(X, 1; s) = \sum_{\substack{(m,n) \in \mathrm{Pas}(p) \\ m \neq 0}} \frac{1}{m^s}$$

has at least two non-real poles on its axis of absolute convergence $\{\mathrm{Re}(s) = \theta_p\}$.

The key in Essouabri's proof of Theorem 1 is an estimate of the number of points $\mathrm{Pas}(p)$. Let

$$\phi_p^*(q) = \#\{(m,n) \in \mathrm{Pas}(p)\ ;\ m = q\}$$

and

$$N_p^*(u) = \#\{(m,n) \in \mathrm{Pas}(p)\ ;\ m < u\} = \sum_{q=0}^{u-1} \phi_p^*(q)$$

for $q \in \mathbb{N}_0$ and $u \in \mathbb{N}$, then $Z_p(X,1;s)$ can be written as an ordinary Dirichlet series

$$Z_p(X,1;s) = \sum_{q=1}^{\infty} \frac{\phi_p^*(q)}{q^s}$$

whenever $Z_p(X,1;s)$ absolutely converges. The summatory function $N_p^*(u)$ of coefficients of $Z_p(X,1;s)$ is estimated that $N_p^*(u) \asymp u^{\theta_p}$ as $u \to \infty$. On the other hand, it is known that the limit value $\lim_{u\to\infty} N_p^*(u)/u^{\theta_p}$ does not exist from former studies; Harborth [2] and Stolarsky [6] showed the case when $p = 2$, and Stein [5] and Wilson [7] showed the general cases. We use the fact that $\lim_{u\to\infty} N_p^*(u)/u^{\theta_p}$ does not exist and a Tauberian argument to prove Theorem 1.

Theorem 1 can be extended to the cases of some other polynomials. The following is the main theorem in this article.

Theorem 2 (I. [3]). (1) *For any prime p, the meromorphic function $Z_p(X+Y,1;s)$ has at least two non-real poles on its axis of absolute convergence* $\{\mathrm{Re}(s) = \theta_p\}$.

(2) *The meromorphic function $Z_2(X+2Y,1;s)$ has at least two non-real poles on its axis of absolute convergence* $\{\mathrm{Re}(s) = \theta_2\}$.

Theorem 2 can be proved by imitating Essouabri's proof of Theorem 1. Here is given just an outline of a proof of Theorem 2 in this article. For the details of the proof, see [3]. To carry out the same argument as his, we need to evaluate certain limit values like $\lim_{u\to\infty} N_p^*(u)/u^{\theta_p}$. Let

$$N_p(P;u) = \#\{(m,n) \in \mathrm{Pas}(p)\ ;\ P(m,n) < u\}$$

for $u \in \mathbb{N}$ and $P \in \mathbb{R}[X,Y]$ satisfying $\mathcal{T}$-ellipticity and $\mathcal{T}$-positivity. Then the following holds.

Theorem 3 (I. [3]). (1) *For each prime p, it holds that $N_p(X+Y;u) \asymp u^{\theta_p}$ as $u \to \infty$, however the limit value $\lim_{u\to\infty} N_p(X+Y;u)/u^{\theta_p}$ does not exist.*

(2) *It holds that $N_2(X+2Y;u) \asymp u^{\theta_2}$ as $u \to \infty$, however the limit value $\lim_{u\to\infty} N_2(X+2Y;u)/u^{\theta_2}$ does not exist.*

To prove the absence of the limit values stated in Theorem 3, we observe the coefficients of the ordinary Dirichlet series representation of $Z_p(P,1;s)$, where $P(X,Y) = X+Y$ or $X+pY$. Specifically, we can write

$$Z_p(P,1;s) = \sum_{q=1}^{\infty} \frac{\phi_p(P;q)}{q^s}$$

with

$$\phi_p(P;q) = \#\{(m,n) \in \operatorname{Pas}(p)\ ;\ P(m,n) = q\}$$

for $P(X,Y) = X+Y$, $X+pY$ when $\operatorname{Re}(s) > \theta_p$.

Appealing to Lucas' formula

$$\binom{\sum_{i=0}^{h} m_i p^i}{\sum_{i=0}^{h} n_i p^i} \equiv \prod_{i=0}^{h} \binom{m_i}{n_i} \quad (\operatorname{mod} p)\ (h \in \mathbb{N}_0,\ m_i, n_i \in I_p),$$

we have that

$$\left(\sum_{i=0}^{h} m_i p^i, \sum_{i=0}^{h} n_i p^i\right) \in \operatorname{Pas}(p) \Longleftrightarrow m_i \geq n_i \text{ for } i = 0,1,\ldots,h. \tag{1}$$

(We shall approve the situation in which the top digits $n_h, n_{h-1}, \ldots, n_{h-k}$ of $\sum_{i=0}^{h} n_i p^i$ are all 0 for some $0 \leq k \leq h$.)

Using this fact (1), we find the following recurrences on $\{\phi_p(X+Y;q)\}_{q=0}^{\infty}$ and $\{\phi_p(X+pY;q)\}_{q=0}^{\infty}$;

$$\begin{cases} \phi_p(X+Y;r) & = \left\lfloor \frac{r}{2}+1 \right\rfloor, \\ \phi_p(X+Y;pq+r) & = \left\lfloor \frac{r}{2}+1 \right\rfloor \phi_p(X+Y;q) + \left\lfloor \frac{p-r}{2} \right\rfloor \phi_p(X+Y;q-1), \end{cases} \tag{2}$$

$$\begin{cases} \phi_p(X+pY;r) & = 1, \\ \phi_p(X+pY;pq+r) & = \sum_{a=0}^{r} \phi_p(X+pY;q-a), \end{cases} \tag{3}$$

hold for any $r \in I_p$ and $q \in \mathbb{N}_0$.

Using that $N_p(P;pu) = \sum_{q=0}^{pu-1} \phi_p(P;q)$ and these recurrences (2), (3), we have

$$\begin{aligned} &N_p(X+Y;pu) = p^{\theta_p} N_p(X+Y;u) - B_p \phi_p(X+Y;u-1), \\ &N_p(X+pY;pu) \\ &= p^{\theta_p} N_p(X+pY;u) - \sum_{b=1}^{p-1} \frac{(p-b)(p-b+1)}{2} \phi_p(X+pY;u-b), \end{aligned} \tag{4}$$

where

$$B_p = \begin{cases} 1 & (p = 2), \\ \dfrac{(p-1)(p+1)}{4} & (p \geq 3). \end{cases}$$

We can compute the limit value $\lim_{k\to\infty} N_p\left(P; p^k u\right)/(p^k u)^{\theta_p}$ for each $u \in \mathbb{N}$ by the above formulas. To conclude the proof, we find (at least) two distinct u's which give distinct limit values $\lim_{k\to\infty} N_p\left(P; p^k u\right)/(p^k u)^{\theta_p}$.

The greater the modulus p gets, the more complicated the recurrences (3) and (4) become. Therefore this method does not work in the general case. Moreover, it is usually hard to produce the recurrence on $\{\phi_p(P; q)\}_{q=0}^{\infty}$ even if $P(X + \lambda Y)$ with $\lambda \in \mathbb{N}$. The author guesses that we need a quite different method to treat more general cases.

Finally, let us see some connection between fractal geometry and the poles of $Z_p(P, 1; s)$ on its axis of absolute convergence.

A bounded open set of $\mathbb{R}$ is called a *fractal string*, which is important in fractal geometry. For a two-variable polynomial $P \in \mathbb{R}[X, Y]$ satisfying $\mathcal{T}$-ellipticity and $\mathcal{T}$-positivity, consider the family of pairwise disjoint open intervals

$$\mathcal{L}_P = \{J_{m,n} \subset \mathbb{R}\,;\ (m,n) \in \operatorname{Pas}(p), P(m,n) \neq 0\},$$

labeled by the elements of $\operatorname{Pas}(p)$ with $\operatorname{vol}_1(J_{m,n}) = P(m,n)^{-2}$, where vol_1 denotes the Lebesgue measure on $\mathbb{R}$. Then

$$\Omega_P = \bigcup_{J\in\mathcal{L}_P} = \bigcup_{\substack{(m,n)\in\operatorname{Pas}(p) \\ P(m,n)\neq 0}} J_{m,n}$$

forms a fractal string. The "geometric zeta function" of Ω_P is given by

$$\sum_{J\in\mathcal{L}_P} \operatorname{vol}_1(J)^{-s} = Z_p(P, 1; 2s \deg P) \quad (s \in \mathbb{C}).$$

The "*Minkowski measurability*," which is one of the geometric properties of (the boundary of) a fractal string, is connected with the analytic information on the existence of non-real poles of the geometric zeta function of the corresponding fractal string on its axis of absolute convergence. In addition, the Minkowski measurability of fractal string is relevant to the distribution of non-trivial zeros of the Riemann zeta function via spectrum theory. For details of the theory of fractal strings, see [4] for example.

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Tomohiro Ikkai
Graduate School of Mathematics
Nagoya University
Furocho, Chikusa-ku, Nagoya, 464-8602, Japan
E-mail address: m13006z@math.nagoya-u.ac.jp