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On The Borsuk-Ulam Theorem and Bordism

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1 Introduction

In [1], Crabb, Goncalves, Libardi, and Perche studied the Borsuk-Ulam property by using bordism relation. Let $\tau: \widetilde{M} \to \widetilde{M}$ be a free involution between m-dimensional manifold $\widetilde{M}$, and Y a topological space. Then the pair $((\widetilde{M}, \tau), Y)$ satisfies the Borsuk-Ulam property (the BUP) iff for any continuous map $f : \widetilde{M} \to Y$ there exists $x \in \widetilde{M}$ such that $f(x) = f(\tau(x))$ ([[1]]). Set $M = \widetilde{M}/\langle T\rangle$. Then the quotient map $q : \widetilde{M} \to M$ is a double covering map. Let λ denote the real line bundle associated with the covering map. Then by usinf the obstruction theory and cobordism theory, Crabb, Goncalves, Libardi, and Perche obtained the following ([[1]]), where R_i denotes the unoriented cobordism group of dimension i, $R_i(X)$the unoriented bordism group of a topological space X, and $R(\mathbf{Z}_2)$ the unoriented bordism group of free $\mathbf{Z}_2$ action on closed smooth manifolds. Note that $R(\mathbf{Z}_2) \cong R(B\mathbf{Z}_2)$ holds, and that $R(\mathbf{Z}_2)$ is generated by the classes $[A_i] \in R(\mathbf{Z}_2)$ represented by the untipodal maps $A_i : \mathbf{S}^n \to \mathbf{S}^n$. For general references about bordism theory, see [2].

Theorem 1.1. *$((\widetilde{M}, \tau), \mathbf{R}^m)$ satisfies the BUP if and only if $w_1(\lambda)^m \neq 0$*

Theorem 1.2. *Suppose that $m > 1$. Let $\alpha \in R_m(\mathbf{Z}_2)$. Then α is wrriten as $\alpha = a_0p_m + a_1p_{m-1} + \cdots + a_{m-1}p_1 + a_mp_0$ for some $a_i \in R_i$.*

(i) If $\alpha = [(\widetilde{M}, \tau)]$, then $a_0 = \langle w_1(\lambda)^m, [\widetilde{M}]\rangle$.

(ii) $\alpha = [(\widetilde{M}, \tau)]$ holds for some connected $\widetilde{M}$.

(iii) If $a_0 = 1$ and $\alpha = [(\widetilde{M}, \tau)]$, then $((\widetilde{M}, \tau), \mathbf{R}^m)$ satisfies the BUP.

(iv) If $a_0 = 0$ and $\alpha = [(\widetilde{M}, \tau)]$ with $\widetilde{M}$ connected, then $((\widetilde{M}, \tau), \mathbf{R}^m)$ does not satisfy the BUP.

Theorem 1.3. *Supose that $1 < n < m$. Then,*

(i) There is a pair $(\widetilde{M}, \tau)$ such that $\alpha = [(\widetilde{M}, \tau)]$ with $\widetilde{M}$ connected, and that $((\widetilde{M}, \tau), \mathbf{R}^n)$ satisfies the BUP.

(ii) If $(a_0, a_1, \ldots a_{m-n}) \neq (0, 0, \ldots 0)$ and $\alpha = [(\widetilde{M}, \tau)]$, then $(\widetilde{M}, \tau)$ satisfies the BUP.

(iii) If $(a_0, a_1, \ldots a_{m-n}) = (0, 0, \ldots 0)$, then there exists a pair $(\widetilde{M}, \tau)$ such that $\alpha = [(\widetilde{M}, \tau)]$ with $\widetilde{M}$ connected, and that $(\widetilde{M}, \tau)$ does not satisfy the BUP.

It seems interesting to restrict τ to spin stucture preserving case. Following corollary is obtained by Theorem 1.1.

Corollary 1.1. *Suppese that $\widetilde{M}$ is a spin manifold and that τ is spin structure preserving. Then $((\widetilde{X},\tau),\mathbf{R}^2)$satisfies the BUP if and only if the types of spin structures that are preserved by τ are unique.*

Our aim of this paper is to generalize the BUP property to compact Lie groups and to consider similar result for the case $G = \mathbf{U}(1)$. We work in smooth category.

2 Definition of the BUP for compact Lie gruop

Let $\widetilde{M}$ be a closed smooth m-dimensional manifolds such that a compact Lie group G acts freely. Suppose that G acts on $\mathbf{R}^n$ via a representation $\rho : G \to GL(n,\mathbf{R})$ Let $f : \widetilde{M} \to \mathbf{R}^n$ a continuous map.

If G is finite (resp. not finite), set $f_G(x) = \frac{1}{\sharp(G)}\sum_{g\in G} g^{-1}f(g(x))$ (resp. $f_G(x) = \int_G g^{-1}f(g(x))\,dx$). Note that $f_G(gx) = gf_G(x)$ holds for all $g \in G$.

Definition 2.1. *Let $\rho : G \to GL(n,\mathbf{R})$ be a representation. Then $((\widetilde{M},G),\rho)$ satisfies the BUP if and only if for any continuous function $f : \widetilde{M} \to \mathbf{R}^n$ there exists $x \in \widetilde{M}$ such that $f_G(x) = 0$.*

3 Borsuk Ulam property for (G,ρ)

Let $\rho_2 : \mathbf{U}(1) \to GL(2,n)$ denote the representation defined by $\rho_2(e^{\theta i}) = \begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}$. Then define the representation $\rho : \mathbf{U}(n) \to GL(n,\mathbf{R})$ as follows;

$$\rho = \begin{cases}\rho_2 \oplus \cdots \oplus \rho_2 & (n : even)\\ \rho_2 \oplus \cdots \oplus \rho_2 \oplus 1 & (n : odd)\end{cases}$$

From now on, we consider the case for $G = \mathbf{U}(1)$ together with the above representation. In this case, the quotient map $\widetilde{M} \to M = \widetilde{M}/\mathbf{U}(1)$ is the projection map of a principal $\mathbf{U}(1)$-bundle λ equipped with the classifying map $f : M \to B\mathbf{U}(1)$.

In case for odd n, we have the nowhere-zero section $s : M \to E(n\lambda) = \widetilde{M} \times_\rho \mathbf{R}^n$ given by $s(x) = [\tilde{x},(0,0,\ldots,0,1)]$. Therefore we obtain;

Proposition 3.1. *If n is odd, then $((\widetilde{M},\rho),\mathbf{R}^n)$ does not satisfy the BUP.*

We also have;

Proposition 3.2. *$((\widetilde{M}^{2n+1},\rho);\mathbf{R}^{2n})$ satisfies the BUP if and only if $c_1(\lambda)^n \neq 0 \in H^{2n}(M;\mathbf{Z})$ holds.*

$((\widetilde{M}^{4n+1},\rho);\mathbf{R}^{4n})$ satisfies the BUP if and only if $p_1(\lambda)^n \neq 0 \in H^{4n}(M;\mathbf{Z})$ holds.

3.1 Unoriented bordism category

W first consider in the unoriented bordism category R_*.

Theorem 3.1. *Let* $m = 2n + 1$.
Suppose that $\alpha = a_0 p_n + a_2 p_{n-1} + \cdots + a_{2n} p_0 \in R_{2n}(B\mathbf{U}(1))$, *where* $a_i \in R_i$
(i) If $\alpha = [(M^{2n}, f)]$, *then* $a_0 \equiv \langle (c_1(\lambda)^n)_2, [M]_2 \rangle$ *holds modulo 2.*
(ii) There exists (M^{2n}, f) *such that* $\alpha = [(M, f)]$ *and that* M *is connected.*
(iii) If $a_0 \neq 0$ *and* $\alpha = [(M^{2n}, f)]$, *then* $((\widetilde{X}, \rho); \mathbf{R}^{2n})$ *satisfies the BUP.*

It seems natural to consider in the oriented bordism category rather than in the unoriented bordism category,

3.2 Oriented bordism category

Next we consider in the oriented bordism category Ω_*. Our consequences are following;

Theorem 3.2. *Suppose that* $\alpha \in \Omega_{2n+1}(\mathbf{U}(1)) \cong \Omega_{2n}(B\mathbf{U}(1))$
(i) If $[(M^{2n}, f)] = \alpha$ *holds, then* $a_0 = \langle c_1(\lambda)^n, [M] \rangle \in \mathbf{Z} \cong \Omega_0$
(ii) There exists (M^{2n}, f) *such that* $\alpha = [M, f]$ *and that* M *is connected.*
(iii) If $a_0 \neq 0$ *and* $\alpha = [(M^{2n}, f)]$ *hold, then* $((\widetilde{M}, \rho); \mathbf{R}^{2n})$ *satisfies the BUP.*
(iv) If $a_0 = 0$, *then there exists* (M^{2n}, f) *such that* $[(M, f)] = \alpha$, *that* $\widetilde{M}$ *is connected , and that* $((\widetilde{M}^{2n+1}, \rho); \mathbf{R}^{2n})$ *does not satisfies the BUP.*

Theorem 3.3. *Suppose that* $m \geq 2n + 2$ *and that* $\alpha = a_0 p_m + a_1 p_{m-1} + \cdots + a_m p_0 \in \Omega_i(B\mathbf{Z})$.
(i) There exists $^{\exists}(\widetilde{M}, \tau)$ *such that* $[(\widetilde{X}, \tau)] = \alpha$, *and that* $\widetilde{X}$ *is connected, and that* $((\widetilde{X}, \tau); \mathbf{R}^n)$ *satisfies the BUP.*
(ii) If $(a_0, a_1, \ldots a_{m-n}) \neq (0, 0, \ldots, 0))$ *and* $\Big([(\widetilde{M}, \tau)] = \alpha$ *hold, then* $((\widetilde{M}, \tau); \mathbf{R}^n))$ *does not satisfies the BUP.*
(iii) If $(a_0, a_1, \ldots a_{m-n}) = (0, 0, \ldots, 0))$ *holds, then there exists* $(\widetilde{X}, \tau)$ *such that* $[(\widetilde{M}, \tau)] = \alpha$, *that* $\widetilde{M}$ *is connected and that* $((\widetilde{M}, \tau); \mathbf{R}^n)$ *satisfies the BUP.*

References

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