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SOME RESULTS ON THE ISOVARIANT BORSUK-ULAM CONSTANTS

Ikumitsu NAGASAKI

Department of Mathematics
Kyoto Prefectural University of Medicine

ABSTRACT. In the previous article [4], we introduced the isovariant Borsuk-Ulam constant of a compact Lie group and provided an estimate of this constant for the unitary group $\mathrm{U}(n)$. In this article, we shall continue the study of the isovariant Borsuk-Ulam constants for simple compact Lie groups and announce some results of [5].

1. REVIEW OF THE ISOVARIANT BORSUK-ULAM CONSTANT

Let G be a compact Lie group. A (continuos) G-map $f : X \to Y$ between G-spaces is called *G-isovariant* if f preserves the isotropy groups; i.e., $G_{f(x)} = G_x$ for every $x \in X$. The isovariant Borsuk-Ulam theorem was first studied by A. G. Wasserman [9]. In particular, the following result is deduced from Wasserman's results.

Theorem 1.1 (Isovariant Borsuk-Ulam theorem). *Let G be a* solvable *compact Lie group. If there exists a G-isovariant map $f : S(V) \to S(W)$ between linear G-spheres, then*

$$\dim V - \dim V^G \leq \dim W - \dim W^G$$

holds.

We call G a *Borsuk-Ulam group* (BUG for short) if the isovariant Borsuk-Ulam theorem holds for G. Therefore solvable G is a Borsuk-Ulam group. A fundamental problem is: Which groups are Borsuk-Ulam groups? This is not completely solved; however, several examples are known, see [6, 7, 9]. Wasserman also conjectures that all finite groups are Borsuk-Ulam groups. On the other hand, a connected compact Lie group being a Borsuk-Ulam group other than a torus is not known.

In [4], we introduced the isovariant Borsuk-Ulam constant c_G as follows.

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Definition. The *isovariant Borsuk-Ulam constant* c_G of G is defined to be the supremum of $c \in \mathbb{R}$ such that:

If there is a G-isovariant map $f : S(V) \to S(W)$, then

$$c(\dim V - \dim V^G) \leq \dim W - \dim W^G$$

holds. (If $G = 1$, then set $c_G = 1$ as convention.)

Clearly $c_G = 1$ if and only if G is a Borsuk-Ulam group.

In equivariant case, the (equivariant) Borsuk-Ulam constant a_G is introduced and studied by Bartsch [2]. In particular, if G is not a p-toral group, then $a_G = 0$. Contrary to this, in section 3, we present the positivity of c_G for any compact Lie group G.

We here recall some properties of c_G that are generalization of Wasserman's results. The detail is described in [5].

Proposition 1.2. (1) *If* $1 \to K \to G \to Q \to 1$ *is an exact sequence of compact Lie groups, then*

$$\min\{c_K, c_Q\} \leq c_G \leq c_Q.$$

In particular, if $c_K = 1$*, then* $c_G = c_Q$*.*

(2) *If* $1 = H_0 \triangleleft H_1 \triangleleft H_2 \triangleleft \cdots \triangleleft H_r = G$*, then*

$$\min_{1 \leq i \leq r} \{c_{H_i/H_{i-1}}\} \leq c_G.$$

Using this proposition, we have

Corollary 1.3. $c_{G_1 \times \cdots \times G_r} = \min_i\{c_{G_i}\}$.

Corollary 1.4. *Let* G *be a connected compact Lie group. Then* $c_G = \min_i\{G_i\}$*, where* G_i *are simple factors of* G*.*

2. Main results — Estimates of c_G

Let G be a *simple* compact Lie group. Let T denote the maximal torus T of G. We set

$$d_G = \sup\left\{\frac{\dim U^T}{\dim U} \,\middle|\, U : \text{nontrivial irreducible } G\text{-representation}\right\},$$

called the *zero weight ratio* of G. The following is a key result for estimation of c_G.

Proposition 2.1 ([5]). $c_G \geq K_G := 1 - d_G$.

By representation theory, d_G can be determined, see [5] for the proof.

Theorem 2.2. *The zero weight ratios are given in the following table.*

Type of G	$A_n\,(n \geq 1)$	$B_n\,(n \geq 2)$	$C_n\,(n \geq 3)$	$D_n\,(n \geq 4)$
d_G	$\frac{1}{n+2}$	$\frac{1}{2n+1}$	$\frac{1}{2n+1}$	$\frac{1}{2n-1}$
K_G	$\frac{n+1}{n+2}$	$\frac{2n}{2n+1}$	$\frac{2n}{2n+1}$	$\frac{2n-2}{2n-1}$

TABLE 1. Classical case

Type of G	E_6	E_7	E_8	F_4	G_2
d_G	$\frac{1}{13}$	$\frac{1}{19}$	$\frac{1}{31}$	$\frac{1}{13}$	$\frac{1}{7}$
K_G	$\frac{12}{13}$	$\frac{18}{19}$	$\frac{30}{31}$	$\frac{12}{13}$	$\frac{6}{7}$

TABLE 2. Exceptional case

This implies the following isovariant Borsuk-Ulam type result. Set

$$d(V,W) = \frac{\dim W - \dim W^G}{\dim V - \dim V^G}.$$

Corollary 2.3. *If $d(V,W) < K_G$ for G simple, then there is no G-isovariant map $f : S(V) \to S(W)$.*

3. Remarks and applications

As a consequence of Theorem 2.2, $c_G > 0$ for connected G. In [3], we also see that $c_G > 0$ for finite G. Therefore we obtain a positivity result on c_G by Proposition 1.2.

Corollary 3.1. *$c_G > 0$ for any compact Lie group G.*

This implies that the weak isovariant Borsuk-Ulam theorem holds for any G which was first proved in [3]. We recall the weak isovariant Borsuk-Ulam theorem.

Definition (Isovariant Borsuk-Ulam function $\varphi_G : \mathbb{N} \to \mathbb{N}$). $\varphi_G(n)$ is defined as the minimum of $\dim W - \dim W^G$ such that there exists a G-isovariant maps $f : S(V) \to S(W)$ with $\dim V - \dim V^G \geq n$.

Proposition 3.2. (1) *If $n \leq m$, then $\varphi_G(n) \leq \varphi_G(m)$.*

(2) $\varphi_G(n+m) \leq \varphi_G(n) + \varphi_G(m)$ *(subadditivity).*
(3) $\varphi_G(n) \leq n$ *for* $n \in \mathcal{D}_G := \{\dim V \mid V^G = 0\}$.

From the definition of c_G, one can see

Proposition 3.3. (1)

$$c_G = \lim_{n\to\infty} \frac{\varphi_G(n)}{n} = \inf_n \frac{\varphi_G(n)}{n}.$$

(2)

$$\varphi(n) \geq c_G n \quad \textit{for} \quad n \in \mathcal{D}_G.$$

Definition. We say that the weak isovariant Borsuk-Ulam theorem holds for G if

$$\lim_{n\to\infty} \varphi_G(n) = \infty.$$

Clearly the positivity of c_G shows

Corollary 3.4 ([3]). *The weak isovariant Borsuk-Ulam theorem holds for any G.*

Bartsch [1] showed that when G is finite, the weak Borsuk-Ulam theorem holds for G if and only if G is a finite p-group. Our result is an isovariant version of Bartsch's result.

As an application of the positivity of c_G, one can see another isovariant Borsuk-Ulam type theorem using by a similar argument of [1].

Corollary 3.5. *Let G be a compact Lie group. Then there is no G-isovariant map f : $S(V) \to S(W)$ for $W \subsetneqq V$ $(V^G = 0)$.*

Remark. This is an isovariant version of Bartsch's result that there is no G-map f : $S(V) \to S(W)$ for $W \subsetneqq V$ $(V^G = 0)$ if and only if G is a p-toral, where G is called a p-toral if G has an exact sequence $1 \to T \to G \to P \to 1$, T: torus, P: finite p-group.

Also, an isovariant version of an infinite Borsuk-Ulam type theorem holds.

Corollary 3.6. *Let G be a compact Lie group. Suppose that* $\dim V = \infty$ *and* $\dim V^G < \infty$. *If there exists a G-isovariant map $f : S(V) \to S(W)$, then* $\dim W = \infty$.

Proof. Suppose $\dim W < \infty$. The Peter-Weyl theorem [8] shows that there exists a finite-dimensional subrepresentation V' of V with arbitrary higher dimension. Then there exists a G-isovariant map $f' := f_{|S(V')} : S(V') \to S(W)$; however, this contradicts $c_G > 0$. □

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Department of Mathematics, Kyoto Prefectural University of Medicine, 1-5 Shimogamo Hangi-cho, Sakyo-ku, Kyoto 606-0823, Japan
E-mail address: nagasaki@koto.kpu-m.ac.jp