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On the theory of Laplace hyperfunctions in several variables

By

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Abstract

We survey the theory of Laplace hyperfunctions in several variables in [1, 2, 9]. A Laplace hyperfunction in one variable was first introduced by H. Komatsu ([3]-[8]) to consider the Laplace transform for a hyperfunction. We here construct Laplace hyperfunctions in several variables and their Laplace transform.

§ 1. A vanishing theorem of cohomology groups for the sheaf of holomorphic functions of exponential type

We briefly recall the vanishing theorem of cohomology groups on a Stein open subset with coefficients in holomorphic functions of exponential type and the edge of the wedge theorem for them.

Let n be a natural number, and let M be an n-dimensional $\mathbb{R}$-vector space. Let E be the complexification of M. We denote by $\mathbb{D}_E$ the radial compactification of E which is defined by

$$\mathbb{D}_E := E \sqcup ((E \setminus \{0\})/\mathbb{R}_+)\infty.$$

Let U be an open subset in $\mathbb{D}_E$. A holomorphic function $f(z)$ in $U \cap E$ is said to be of exponential type if, for any compact subset K in U, there exist positive constants C_K and H_K such that

$$|f(z)| \leq C_K e^{H_K|z|} \qquad (z \in K \cap E). \tag{1.1}$$

We denote by $\mathcal{O}^{\exp}_{\mathbb{D}_E}$ the sheaf of holomorphic functions of exponential type on $\mathbb{D}_E$.

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To recall the vanishing theorem of cohomology groups on a Stein open subset for $\mathscr{O}^{\exp}_{\mathbb{D}_E}$, we give the definition of the regularity condition at ∞ for an open subset in $\mathbb{D}_E$. We denote by E_∞ the set $\mathbb{D}_E \setminus E$. For a subset V in $\mathbb{D}_E$, we define the set $\mathrm{clos}^1_\infty(V) \subset E_\infty$ as follows. A point $z\infty \in E_\infty$ belongs to $\mathrm{clos}^1_\infty(V)$ if and only if there exist points $\{z_k\}_{k\in\mathbb{N}}$ in $V \cap E$ which satisfy $z_k \to z\infty$ in $\mathbb{D}_E$ and $|z_{k+1}|/|z_k| \to 1$ $(k \to \infty)$. Set

$$N^1_\infty(V) := E_\infty \setminus \mathrm{clos}^1_\infty(E \setminus V). \tag{1.2}$$

Definition 1.1. An open subset U in $\mathbb{D}_E$ is said to be regular at ∞ if $N^1_\infty(U) = U \cap E_\infty$ is satisfied.

Note that this condition is equivalent to saying $E_\infty \setminus U = \mathrm{clos}^1_\infty(E \setminus U)$. Now we state our vanishing theorem of cohomology groups for $\mathscr{O}^{\exp}_{\mathbb{D}_E}$.

Theorem 1.2 ([2], Theorem 3.7). *Let U be an open subset in $\mathbb{D}_E$. Assume that $U \cap E$ is pseudo-convex in E and U is regular at ∞, then we have*

$$\mathrm{H}^k(U, \mathscr{O}^{\exp}_{\mathbb{D}_E}) = 0 \quad (k \neq 0). \tag{1.3}$$

The regularity condition of U at ∞ plays an essential role in our vanishing theorem of cohomology groups for $\mathscr{O}^{\exp}_{\mathbb{D}_E}$ as the following shows.

Example 1.3 ([2], Example 3.17). We consider the radial compactification $\mathbb{D}_{\mathbb{C}^2}$ of $\mathbb{C}^2$. Let $(1,0)\infty \in \mathbb{D}_{\mathbb{C}^2} \setminus \mathbb{C}^2$. Set

$$V := \left\{(z_1, z_2) \in \mathbb{C}^2; |\arg(z_1)| < \frac{\pi}{4}, |z_2| < |z_1|\right\},$$
$$U := \left(\overline{V}\right)^\circ \setminus \{(1,0)\infty\} \subset \mathbb{D}_{\mathbb{C}^2}.$$

It is easy to check that $U \cap E = V$ is pseudo-convex in $\mathbb{C}^2$ and U is not regular at ∞. In this case, we have $\mathrm{H}^1(U, \mathscr{O}^{\exp}_{\mathbb{D}_E}) \neq 0$.

Furthermore, by showing a Martineau type theorem for $\mathscr{O}^{\exp}_{\mathbb{D}_E}$, we have the following theorem, which is a kind of the edge of the wedge type theorem for $\mathscr{O}^{\exp}_{\mathbb{D}_E}$. Let $\overline{M}$ be the closure of M in $\mathbb{D}_E$.

Theorem 1.4 ([1], Corollary 3.16). *The closed subset $\overline{M} \subset \mathbb{D}_E$ is purely n-codimentional relative to the sheaf $\mathscr{O}^{\exp}_{\mathbb{D}_E}$, i.e.,*

$$\mathscr{H}^k_{\overline{M}}(\mathscr{O}^{\exp}_{\mathbb{D}_E}) = 0 \qquad (k \neq n). \tag{1.4}$$

§ 2. Laplace hyperfunctions and their Laplace transform

In this section we construct Laplace transform for Laplace hyperfunctions with support in an $\mathbb{R}_+$-conic closed convex cone in $\overline{M}$ and their inverse Laplace transforms. We first recall the definition of Laplace hyperfunctions:

Definition 2.1. The sheaf of Laplace hyperfunctions on $\overline{M}$ is defined by

$$\mathcal{B}^{\exp}_{\overline{M}} := \mathscr{H}^n_{\overline{M}}(\mathcal{O}^{\exp}_{\mathbb{D}_E}) \underset{\mathbb{Z}_{\overline{M}}}{\otimes} \omega_{\overline{M}}. \tag{2.1}$$

Here $\omega_{\overline{M}}$ is the orientation sheaf $\mathscr{H}^n_{\overline{M}}(\mathbb{Z}_{\mathbb{D}_E})$ and $\mathbb{Z}_{\mathbb{D}_E}$ is the constant sheaf on $\mathbb{D}_E$ having stalk $\mathbb{Z}$.

Let $a \in M$ and K be an $\mathbb{R}_+$-conic closed convex cone in M. Let us denote by K_a the set $\{z + a; z \in K\}$ and denote by $\overline{K_a}$ the closure of K_a in $\overline{M}$. We first get the representation of $\Gamma_{\overline{K_a}}(\overline{M}, \mathcal{B}^{\exp}_{\overline{M}})$ by the relative Čech cohomology groups with coefficients in $\mathcal{O}^{\exp}_{\mathbb{D}_E}$.

Let us prepare some notation and the proposition below. For a subset $Z \subset \mathbb{D}_E$, set

$$N_\infty(Z) := E_\infty \setminus \overline{(E \setminus Z)}. \tag{2.2}$$

For an open subset $U \subset E$, define

$$\widehat{U} := U \cup N_\infty(U). \tag{2.3}$$

Definition 2.2. Let Ω be an open subset in $\overline{M}$ and Γ an $\mathbb{R}^+$-conic open cone in M. Let U be an open subset in $\mathbb{D}_E$. We call U a wedge of the type $\Omega \times \sqrt{-1}\Gamma$ if U satisfies the following conditions.

1. $U \subset (\Omega \times \widehat{\sqrt{-1}\Gamma})$,
2. For any open proper subcone Γ' of Γ, there exists an open neighborhood V of Ω in $\mathbb{D}_E$ such that

$$(M \times \widehat{\sqrt{-1}\Gamma'}) \cap V \subset U. \tag{2.4}$$

We have the following proposition.

Proposition 2.3. *Let K be an $\mathbb{R}_+$-conic closed cone in M and Γ a proper open cone in M. Assume that Γ is given by the intersection of finite number of half-spaces in M. Then there exist an open neighborhood Ω of $\overline{K}$ in $\overline{M}$ and an open subset U in $\mathbb{D}_E$ such that the following conditions are satisfied.*

1. *U is a wedge of the type $\Omega \times \sqrt{-1}\Gamma$.*
2. *U is Stein and regular at ∞.*
3. *U is an open neighborhood of $\Omega \setminus \overline{K}$ in $\mathbb{D}_E$.*

Now let us consider the representation of $\Gamma_{\overline{K_a}}(\overline{M}, \mathcal{B}^{\exp}_{\overline{M}})$ by the relative Čech cohomology with coefficients in $\mathcal{O}^{\exp}_{\mathbb{D}_E}$. Choose vectors $\gamma_0, \dots, \gamma_n \in S^{n-1}$. By Proposition 2.3, we can take an open neighborhood Ω of $\overline{K_a}$ in $\overline{M}$ and an open subset $U_j \subset \mathbb{D}_E$ which is the wedge of the type $\Omega \times \sqrt{-1}\gamma_j^\circ$, Stein and regular at ∞, and furthermore, an open neighborhood of $\Omega \setminus \overline{K_a}$. Here γ_j° denotes the polar set $\{y \in M; y\gamma_j > 0\}$ of γ_j. We also take a neighborhood U of $\overline{K_a}$ in $\mathbb{D}_E$ which is Stein and regular at ∞. Then $\mathfrak{U} = \{U, U_0, \ldots, U_n\}$ and $\mathfrak{U}' = \{U_0, \ldots, U_n\}$ give a relative open covering of the pair $(U, U \setminus \overline{K_a})$. Hence we have

$$\Gamma_{\overline{K_a}}(\overline{M}, \mathcal{B}^{\exp}_{\overline{M}}) = \frac{\mathrm{Ker}\{\bigoplus_{j=0}^n \mathcal{O}^{\exp}_{\mathbb{D}_E}(\bigcap_{l\neq j} U_l) \to \mathcal{O}^{\exp}_{\mathbb{D}_E}(\bigcap_{l=0}^n U_l)\}}{\mathrm{Im}\{\bigoplus_{j\neq k} \mathcal{O}^{\exp}_{\mathbb{D}_E}(\bigcap_{l\neq j,\,k} U_l) \to \bigoplus_{j=0}^n \mathcal{O}^{\exp}_{\mathbb{D}_E}(\bigcap_{l\neq j} U_l)\}}. \tag{2.5}$$

Let us define the Laplace transform for an element $f = \bigoplus_{j=0}^n F_j$ of the above representation of $\Gamma_{\overline{K_a}}(\overline{M}, \mathcal{B}^{\exp}_{\overline{M}})$. Set, for $j = 0, 1, \ldots, n$,

$$D_j := \{x + \sqrt{-1}y \in E \ ; \ x \in \Gamma,\ y = \varphi(x)\gamma\},$$

where we take an appropriate closed cone $\Gamma \subset \Omega$ which contains K and a point $\gamma \in \bigcap_{l\neq j}\gamma_l^\circ$. Further, the continuous function $\varphi : \Gamma \to \mathbb{R}_+ \cup \{0\}$ is chosen to satisfy the following conditions: (1) $\varphi(x) = 0$ in $\partial\Gamma$, (2) $\overline{D_j} \cap \overline{K_a} = \emptyset$, (3) $\overline{D}_j \subset U_j$. Note that such Γ, γ and φ always exist for each j.

Definition 2.4. Under the above situation, the Laplace transform of $f = \bigoplus_{j=0}^n F_j \in \Gamma_{\overline{K_a}}(\overline{M}, \mathcal{B}^{\exp}_{\overline{M}})$ is defined by the integral

$$\mathscr{L}(f)(\lambda) := \sum_{j=0}^{n} \sigma_j \int_{D_j} F_j(z) e^{-\lambda z} dz, \tag{2.6}$$

where $\sigma_j := \mathrm{sgn}\ (\ \det(\omega_0, \cdots, \omega_{j-1}, \omega_{j+1}, \cdots, \omega_n)\ )$.

Note that the Laplace transform does not depend on the choice of Γ, γ and φ.

Definition 2.5. Let Ω be an open subset in $\mathbb{D}_E$. The set $\mathcal{O}^{a,\inf}_{\mathbb{D}_E}(\Omega)$ consists of a holomorphic function $f(z)$ on $\Omega \cap E$ such that, for any compact subset $K \subset \Omega$ and $\epsilon > 0$, $f(z)$ satisfies

$$|e^{az} f(z)| \leq C_{K,\epsilon} e^{\epsilon|z|}, \qquad z \in K \cap E. \tag{2.7}$$

with a positive constant $C_{K,\epsilon}$.

Then we find that the Laplace transform gives the following morphism.

$$\mathscr{L}\ :\ \Gamma_{\overline{K}_a}(\overline{M}, \mathcal{B}^{\exp}_{\overline{M}}) \longrightarrow \mathcal{O}^{a,\inf}_{\mathbb{D}_E}(N_\infty(K^\circ)). \tag{2.8}$$

Here K° denotes the dual open cone of K in E. Since the above morphism does not depend on the representation of $\Gamma_{\overline{K}_a}(\overline{M}, \mathcal{B}^{\exp}_{\overline{M}})$, $\mathscr{L}$ is well-defined.

Definition 2.6. Let T be an open subset in E_∞, and U an open subset in $\mathbb{D}_E$. We say that U has the opening wider than or equal to T at ∞ if $T \subset N_\infty(U)$ is satisfied.

We have the following lemma which plays an important role in establishing the inverse Laplace transform.

Lemma 2.7. *The following conditions are equivalent:*

1. $f \in \mathcal{O}_{\mathbb{D}_E}^{a,\,\mathrm{inf}}(N_\infty(K^\circ))$.
2. *There exists an open subset U in E whose opening is wider than or equal to $N_\infty(K^\circ)$ such that f is holomorphic on U and, for any compact subset K in $\widehat{U}$, there exists an infra-linear function $\phi_K(s)$ satisfying*

$$|e^{az}f(z)| \le e^{\phi_K(|z|)}, \qquad z \in K \cap E.$$

3. *There exists an infra-linear function $\phi(s)$ and an open subset U in E whose opening is wider than or equal to $N_\infty(K^\circ)$ such that f is holomorphic on U with*

$$|e^{az}f(z)| \le e^{\phi(|z|)}, \qquad z \in U.$$

Let us define the inverse Laplace transform.

Definition 2.8. We define the morphism

$$\mathscr{S} \;:\; \mathcal{O}_{\mathbb{D}_E}^{a,\,\mathrm{inf}}(N_\infty(K^\circ)) \longrightarrow \mathcal{B}_{\overline{M}}^{\exp}(\overline{M}) \tag{2.9}$$

by

$$\mathscr{S}(f) = \bigoplus_{0\le k\le n} \sigma_k f_k, \qquad f \in \mathcal{O}_{\mathbb{D}_E}^{a,\,\mathrm{inf}}(N_\infty(K^\circ)).$$

Here f_k is given by the integral

$$f_k(z) := \frac{1}{(2\pi\sqrt{-1})^n}\int_{T_k} f(\lambda)e^{\lambda z}d\lambda. \tag{2.10}$$

The path of the integration T_k is given as follows. Set

$$\Sigma_k := \{\eta \in M; \eta = \sum_{j\ne k} t_j\gamma_j,\, t_j \ge 0\}.$$

Let ψ be an infra-linear function, and let $\hat{\xi}$ be a point in the dual open cone of K in M. Then we put

$$T_k := \left\{\ \lambda = \xi + \sqrt{-1}\eta \in E\ ;\ \eta \in \Sigma_k, \quad \xi = \psi(|\eta|)\hat{\xi}\ \right\}. \tag{2.11}$$

Note that the integral f_k does not depend on the choice of ψ and $\hat{\xi}$ if ψ is rapidly increasing. We can see that f_k is a holomorphic function of exponential type on $\widehat{(M \times \sqrt{-1}\bigcap_{j\ne k}\gamma_j^\circ)}$ by Lemma 2.7.

Furthermore, we have:

Lemma 2.9. $\mathrm{supp}(\mathscr{S}(f)) \subset \overline{K_a}$ for $f \in \mathcal{O}_{\mathbb{D}_E}^{a,\,\mathrm{inf}}(N_\infty(K^\circ))$.

Hence we have the inverse Laplace transform, and we can show that it satisfies the following theorem.

Theorem 2.10. $\mathscr{S} \circ \mathscr{L} = id_{\Gamma_{\overline{K_a}}(\overline{M}, \mathcal{B}_{\overline{M}}^{\exp})}, \qquad \mathscr{L} \circ \mathscr{S} = id_{\mathcal{O}_{\mathbb{D}_E}^{a,\,\mathrm{inf}}(N_\infty(K^\circ))}.$

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