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k-summability of formal solutions for certain partial differential equations and the method of successive approximation

By

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Abstract

We study the k-summability of divergent formal solutions to the Cauchy problem of certain linear partial differential operators of the first order with respect to t whose coefficients are polynomial in t. In order to prove the k-summability of divergent solutions, we employ the method of successive approximation for a construction of divergent solutions and the analysis of convolution equations associated with divergent solutions. We give a proof of the existence and uniqueness of local holomorphic solutions for the convolution equations.

§ 1. Result

Let linear partial differential operators L with polynomial coefficients in t be given by

(1.1) $$L = \partial_t - P(t, \partial_x), \quad P(t, \partial_x) = \sum_{\alpha:\text{finite}} a_\alpha(t)\partial_x^\alpha,$$

where $(t, x) \in \mathbb{C}^2$, $(\partial_t, \partial_x) = (\partial/\partial t, \partial/\partial x)$ and $a_\alpha(t) \in \mathbb{C}[t]$ for all α.

We consider the following Cauchy problem for L

(1.2) $$\begin{cases} LU(t,x) = (\partial_t - P(t, \partial_x))\, U(t,x) = 0 \\ U(0,x) = \varphi(x) \in \mathcal{O}_x, \end{cases}$$

where $\mathcal{O}_x$ denotes the set of holomorphic functions in a neighborhood of the origin $x = 0$. The Cauchy problem (1.2) has a unique formal solution of the form

(1.3) $$\hat{U}(t,x) = \sum_{n=0}^{\infty} U_n(x)\frac{t^n}{n!}, \quad U_0(x) = \varphi(x).$$

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We assume that for the operator $P = P(t, \partial_x)$

$$\text{(A-1)} \qquad \alpha_* := \max\{\alpha; a_\alpha(t) \not\equiv 0\} \geq 2,$$

which is called non-Kowalevskian condition. In this case, the formal solution is divergent in general.

Our purpose in this paper is to study the k-summability of this divergent solution under some conditions for L. In order to explain the conditions we define the Newton polygon of L.

Let $i(\alpha)$ be the order of zero of $a_\alpha(t)$ at $t = 0$. We define a domain $\mathrm{N}(\alpha)$ by

$$\mathrm{N}(\alpha) := \{(x, y) \in \mathbb{R}^2;\ x \leq \alpha,\ y \geq i(\alpha)\} \quad \text{for } a_\alpha(t) \not\equiv 0,$$

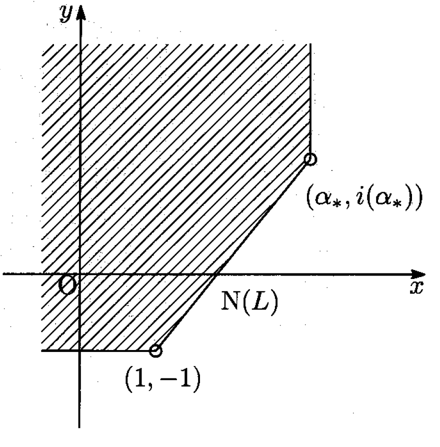
and $\mathrm{N}(\alpha) := \phi$ for $a_\alpha(t) \equiv 0$. Then the Newton polygon $\mathrm{N}(L)$ is defined by

$$\text{(1.4)} \qquad \mathrm{N}(L) := \mathrm{Ch}\left\{\mathrm{N}(1, -1) \cup \bigcup_{0 \leq \alpha \leq \alpha_*} \mathrm{N}(\alpha)\right\},$$

where $\mathrm{Ch}\{\cdots\}$ denotes the convex hull of the set $\mathrm{N}(1, -1) \cup \cup_\alpha \mathrm{N}(\alpha)$, and $\mathrm{N}(1, -1)$ $:= \{(x, y)\,;\, x \leq 1,\ y \geq -1\}$.

We assume that

(A-2) the Newton polygon $\mathrm{N}(L)$ has only one side of a positive slope with two end points $(1, -1)$ and $(\alpha_*, i(\alpha_*))$.



We put $i_* := i(\alpha_*)$. Then we assume that the indices α of the operator P satisfy the following inequality.

$$\text{(A-3)} \qquad \frac{\alpha}{i(\alpha) + 1} \leq \frac{\alpha_*}{i_* + 1}.$$

We call this number $\alpha_*/(i_* + 1)$ the modified order of L and we put

$$\text{(1.5)} \qquad \frac{\alpha_*}{i_* + 1} =: \frac{p}{q}, \quad (p, q) = 1.$$

Moreover, we assume that for any α,

$$a_\alpha(t) = \sum_{i(\alpha)\le i\le i_*,\ i+1\in q\mathbb{N}_0} a_i^{(\alpha)} t^i, \tag{A-4}$$

where $\mathbb{N}_0$ denotes the set of the nonnegative integers. Especially, we have

$$a_{\alpha_*}(t) = a_{i_*}^{(\alpha_*)} t^{i_*}.$$

In order to state our result, we give the definition of a characteristic equation for L

$$a_{i_*}^{(\alpha_*)} z^{\alpha_*} - 1 = 0. \tag{1.6}$$

Let z_n $(n = 1, 2, \ldots, \alpha_*)$ be the roots of the characteristic equation.

Finally, we prepare the notation $S(d, \beta, \rho)$. For $d \in \mathbb{R}$, $\beta > 0$ and $\rho\,(0 < \rho \le \infty)$, we define a sector $S = S(d, \beta, \rho)$ by

$$S(d, \beta, \rho) := \{t \in \mathbb{C}; |d - \arg t| < \beta/2, 0 < |t| < \rho\},$$

where d, β and ρ are called the direction, the opening angle and the radius of S, respectively. We write $S(d, \beta, \infty) = S(d, \beta)$ for short.

Under the above preparations, our result is stated as follows.

Theorem 1.1. *We suppose the assumptions* (A-1)-(A-4). *Let $d \in \mathbb{R}$ be fixed and we put $d_n = qd/p - \arg z_n$ for $n = 1, 2, \ldots, \alpha_*$. Let*

$$k = \frac{i_* + 1}{\alpha_* - 1}. \tag{1.7}$$

We assume that the Cauchy data $\varphi(x) \in \mathcal{O}_x$ can be analytically continued in the region $\cup_{n=1}^{\alpha_*} S(d_n, \varepsilon)$ for some positive ε, and has the following exponential growth estimate

$$|\varphi(x)| \le C_\varphi \exp\left(\delta_\varphi |x|^{\frac{\alpha_*}{\alpha_*-1}}\right), \quad x \in \bigcup_{n=1}^{\alpha_*} S(d_n, \varepsilon), \tag{1.8}$$

for some positive constants C_φ and δ_φ. Then the divergent solution $\hat{U}(t, x)$ of the Cauchy problem (1.2) *is k-summable in d direction.*

We may assume that $k \ge 1$, which is only needed for the analysis of the convolution equations, without loss of generality by a change of variable, e.g. $t^{1/(\alpha_*-1)} = \tau$.

We remark that the roots of the characteristic equation are given by

$$z_n = (a_{i_*}^{(\alpha_*)})^{-1/\alpha_*} \omega_{\alpha_*}^{n-1} \quad (n = 1, \ldots, \alpha_*),$$

where $\omega_\alpha = e^{2\pi i/\alpha}$.

The k-summability of divergent solutions of non-Kowalevskian equations with constant coefficients has been developed by many mathematicians (e.g. [8] for the heat equation, [11] for the operator $\partial_t^p - \partial_x^q, (p < q)$, [2] for general equations, [9] for moment partial differential equations). But, there are not many study of k-summability of divergent solutions for equations with variable coefficients yet. In the papers [4] and [5], we treated the equations whose coefficients are monomial in t. In the paper [7], we treated the equations of the first order with respect to t whose coefficients are polynomial in t and modified order is equal to one. In this paper, we consider the equations of the first order with respect to t whose coefficients are polynomial in t and modified order is general, and in the paper [6], we treated the higher order case. In the following, we will give an outline of a proof of Theorem 1.1. Especially, we will give the proof of the local existence and uniqueness of holomorphic solutions for the convolution equations associated with L, which were admitted without proof in the papers [6] and [7].

§ 2. Review of k-summability

In this section, we give some notation and definitions in the way of Ramis or Balser (cf. W. Balser [1] for detail).

Let $k > 0$, $S = S(d, \beta)$ and $B(\sigma) := \{x \in \mathbb{C}; |x| \le \sigma\}$. Let $v(t,x) \in \mathcal{O}(S \times B(\sigma))$ which means that $v(t,x)$ is holomorphic in $S \times B(\sigma)$. Then we define that $v(t,x) \in \mathrm{Exp}_t^k(S \times B(\sigma))$ if, for any closed subsector S' of S, there exist some positive constants C and δ such that

$$\max_{|x|\le\sigma} |v(t,x)| \le Ce^{\delta|t|^k}, \quad t \in S'. \tag{2.1}$$

For $k > 0$, we define that $\hat{v}(t,x) = \sum_{n=0}^{\infty} v_n(x)t^n \in \mathcal{O}_x[[t]]_{1/k}$ (we say $\hat{v}(t,x)$ is a formal power series of Gevrey order $1/k$) if $v_n(x)$ are holomorphic on a common closed disk $B(\sigma)$ for some $\sigma > 0$ and there exist some positive constants C and K such that for any n,

$$\max_{|x|\le\sigma} |v_n(x)| \le CK^n\Gamma\left(1 + \frac{n}{k}\right). \tag{2.2}$$

Here when $v_n(x) \equiv v_n$ (constants) for all n, we use the notation $\mathbb{C}[[t]]_{1/k}$ instead of $\mathcal{O}_x[[t]]_{1/k}$.

Let $k > 0$, $\hat{v}(t,x) = \sum_{n=0}^{\infty} v_n(x)t^n \in \mathcal{O}_x[[t]]_{1/k}$ and $v(t,x)$ be an analytic function on $S(d,\beta,\rho) \times B(\sigma)$. Then we define that

$$v(t,x) \cong_k \hat{v}(t,x) \quad \text{in } S = S(d,\beta,\rho), \tag{2.3}$$

if for any closed subsector S' of S, there exist some positive constants C and K such that for any $N \geq 1$, we have

$$\max_{|x|\leq\sigma}\left|v(t,x)-\sum_{n=0}^{N-1}v_n(x)t^n\right|\leq CK^N|t|^N\Gamma\left(1+\frac{N}{k}\right),\quad t\in S'. \tag{2.4}$$

For $k>0$, $d\in\mathbb{R}$ and $\hat{v}(t,x)\in\mathcal{O}_x[[t]]_{1/k}$, we say that $\hat{v}(t,x)$ is *k-summable* in d direction, and denote it by $\hat{v}(t,x)\in\mathcal{O}_x\{t\}_{k,d}$, if there exist a sector $S=S(d,\beta,\rho)$ with $\beta>\pi/k$ and an analytic function $v(t,x)$ on $S\times B(\sigma)$ such that $v(t,x)\cong_k\hat{v}(t,x)$ in S.

We remark that the function $v(t,x)$ above for a *k-summable* $\hat{v}(t,x)$ is unique if it exists. Therefore such a function $v(t,x)$ is called the *k-sum* of $\hat{v}(t,x)$ in d direction.

§ 3. Construction of a formal solution

Decomposition of operator $P(t,\partial_x)$ We give a decomposition of the operator P.

For $\ell\in\mathbb{N}_0$, we define

$$K_\ell:=\left\{(\alpha,i);\ \ell=p(i+1)/q-\alpha,\ a_i^{(\alpha)}\neq 0\right\}$$

and we put $P_\ell(t,\partial_x):=\sum_{(\alpha,i)\in K_\ell}a_i^{(\alpha)}t^i\partial_x^\alpha$. Then we obtain

$$P(t,\partial_x)=\sum_{\ell=0}^{\alpha_*}P_\ell(t,\partial_x).$$

In fact, we have $\ell=p(i+1)/q-\alpha\leq p(i_*+1)/q=\alpha_*$.

The sequence of Cauchy problems By employing the decomposition of the operator P, we consider the following sequence of Cauchy problems for $\nu\geq 0$.

$$(E_\nu)\qquad \begin{cases}\partial_t u_\nu(t,x)=\displaystyle\sum_{\ell=0}^{\min\{\alpha_*,\nu\}}P_\ell(t,\partial_x)u_{\nu-\ell}(t,x),\\ u_\nu(0,x)=\begin{cases}\varphi(x) & (\nu=0),\\ 0 & (\nu\geq 1).\end{cases}\end{cases}$$

For each ν, the Cauchy problem (E_ν) has a unique formal solution of the form

$$(Sol_\nu)\qquad \hat{u}_\nu(t,x)=\sum_{n\geq 0}u_{\nu,n}(x)t^n/n!.$$

Then $\hat{U}(t,x)=\sum_{\nu\geq 0}\hat{u}_\nu(t,x)$ is the formal power series solution of the original Cauchy problem (1.2).

Construction of formal solutions $\hat{u}_\nu(t,x)$ We give a construction of the formal solutions $\hat{u}_\nu(t,x)$ of the Cauchy problems (E_ν).

Lemma 3.1. *Let $\nu \geq 0$. For each ν, we have*

$$u_{\nu,n}(x) = A_\nu(n)\varphi^{(\frac{p}{q}n-\nu)}(x) \quad (pn/q - \nu \in \mathbb{N}_0), \tag{3.1}$$

and $u_{\nu,n}(x) \equiv 0$ $(pn/q-\nu \notin \mathbb{N}_0)$. Here $\{A_\nu(n)\}$ satisfy the following recurrence formula:

$$(R_\nu) \qquad \begin{cases} A_\nu(n+1) = \displaystyle\sum_{\ell=0}^{\min\{\alpha_*,\nu\}} \sum_{K_\ell} a_i^{(\alpha)}[n]_i A_{\nu-\ell}(n-i) \quad (n \geq 0), \\ A_\nu(0) = \begin{cases} 1 & (\nu = 0), \\ 0 & (\nu \geq 1). \end{cases} \end{cases}$$

where we interpret as $A_\nu(n) = 0$ for all ν if $n < 0$. Here the notation $[n]_i$ is defined by

$$[n]_i := \begin{cases} n(n-1)(n-2)\cdots(n-i+1), & i \geq 1, \\ 1, & i = 0. \end{cases}$$

By substituting (Sol_ν) into the equation (E_ν), we can see that $u_{\nu,n}(x) = A_\nu(n) \times\varphi^{(pn/q-\nu)}(x)$, where $\{A_\nu(n)\}$ satisfy the recurrence formula (R_ν). Especially, if $pn/q-\nu \notin \mathbb{N}_0$, $A_\nu(n) = 0$. We omit the details.

§ 4. Gevrey order of the formal solution $\hat{U}(t,x)$

We give the Gevrey order of the formal solution $\hat{U}(t,x)$. For the purpose, we give a result of Gevrey order of formal solutions $\hat{u}_\nu(t,x)$ of (E_ν) without proof (see [7]).

Proposition 4.1. *We assume $\varphi(x) \in \mathcal{O}_x$. For each ν, we have $\hat{u}_\nu(t,x) \in \mathcal{O}_x[[t]]_{1/k}$, $k = (i_*+1)/(\alpha_*-1)$. More exactly, we have*

$$\max_{|x|\leq\sigma} \left| \frac{u_{\nu,n}(x)}{n!} \right| \leq \frac{AB^{\nu+n}}{\nu!} \Gamma\left(1 + \frac{n}{k}\right) \tag{4.1}$$

by some positive constants A, B and σ for any n with $pn/q - \nu \in \mathbb{N}_0$.

We can see that $\hat{U}(t,x) \in \mathcal{O}_x[[t]]_{1/k}$ by Proposition 4.1 immediately, because of

$$\hat{U}(t,x) = \sum_{\nu=0}^{\infty} \hat{u}_\nu(t,x) = \sum_\nu \sum_n \frac{u_{\nu,n}(x)}{n!} t^n = \sum_n \frac{\sum_\nu u_{\nu,n}(x)}{n!} t^n =: \sum_n \frac{U_n(x)}{n!} t^n.$$

§ 5. Preliminaries for proof of Theorem 1.1

In this section, we prepare some results which are employed to prove Theorem 1.1. First, we give an important lemma for the summability theory (cf. [1], [8]).

Lemma 5.1. *Let $k > 0$, $d \in \mathbb{R}$ and $\hat{v}(t,x) = \sum_{n=0}^{\infty} v_n(x)t^n \in \mathcal{O}_x[[t]]_{1/k}$. Then the following statements are equivalent:*

i) $\hat{v}(t,x) \in \mathcal{O}_x\{t\}_{k,d}$.

ii) We put

$$v_B(s,x) = (\hat{\mathcal{B}}_k \hat{v})(s,x) := \sum_{n=0}^{\infty} \frac{v_n(x)}{\Gamma(1+n/k)} s^n, \tag{5.1}$$

which is a formal k-Borel transformation of $\hat{v}(t,x)$, that is convergent in a neighborhood of $(s,x) = (0,0)$. Then $v_B(s,x) \in \mathrm{Exp}_s^k(S(d,\varepsilon) \times B(\sigma))$ for some $\varepsilon > 0$ and $\sigma > 0$.

Next, we introduce three formal series. For each $\nu \geq 0$, we define

$$\hat{f}_\nu(t) := \sum_{n \geq 0, \frac{pn}{q} - \nu \in \mathbb{N}_0} A_\nu(n) t^n = \sum\nolimits'_{n \geq 0} A_\nu(n) t^n, \tag{5.2}$$

which is the generating function of $\{A_\nu(n)\}$, and

$$\hat{g}_\nu(t) := \sum\nolimits'_{n \geq 0} A_\nu(n) \frac{\left(\frac{p}{q}n - \nu\right)!}{n!} t^n \in \mathbb{C}[[t]]_{1/k}, \tag{5.3}$$

$$\hat{h}_\nu(t) := \sum\nolimits'_{n \geq 0} A_\nu(n) \frac{\left(\frac{p}{q}n\right)!}{n!} t^n \in \mathbb{C}[[t]]_{1/k}, \tag{5.4}$$

which are called the moment series of $\hat{f}_\nu$. We note that we can find $\hat{g}_\nu(t)$ in $\hat{u}_\nu(t,x)$ by the formal use of the Cauchy integral formula.

$$\hat{u}_\nu(t,x) = \sum\nolimits'_{n \geq 0} A_\nu(n) \varphi^{\left(\frac{p}{q}n - \nu\right)}(x) \frac{t^n}{n!} = \frac{1}{2\pi i} \oint \varphi(x+\zeta) \zeta^{\nu-1} \hat{g}_\nu\left(\frac{t}{\zeta^{p/q}}\right) d\zeta.$$

Moreover, we have a formal relationship between $\hat{g}_\nu$ and $\hat{h}_\nu$. For $\nu \geq 1$,

$$\hat{g}_\nu(t) = \frac{1}{\Gamma(\nu)} \int_0^1 \tau^{-\nu}(1-\tau)^{\nu-1} \hat{h}_\nu(\tau^{p/q} t) d\tau \tag{5.5}$$

and $\hat{g}_0(t) = \hat{h}_0(t)$ when $\nu = 0$.

Finally, we prepare a lemma for the summability of $\hat{h}_\nu(t)$ which is given by (5.4), where an outline of its proof is given in section 7.

Lemma 5.2. *Let $\hat{h}_\nu(t)$ be given by (5.4) and $k = (i_* + 1)/(\alpha_* - 1)$. Then, for θ satisfying*

$$\theta \not\equiv (-\arg a_{i_*}^{(\alpha_*)} + 2\pi m)/(i_* + 1) \pmod{2\pi} \quad (m = 0, 1, \ldots, i_*), \tag{5.6}$$

we obtain the following estimates

$$|h_{\nu B}(s)| \leq C_h K_h^\nu \exp(\delta_h |s|^k), \quad s \in S(\theta, \varepsilon_0), \tag{5.7}$$

where positive constants C_h, K_h and δ_h are independent of ν, and $\varepsilon_0 > 0$.

We remark that Lemma 5.2 means that $\hat{h}_\nu(t) \in \mathbb{C}\{t\}_{k,\theta}$ with θ satisfying (5.6).

§ 6. Proof of Theorem 1.1

By employing Lemmas 5.1 and 5.2, we obtain the following result without proof (cf. [7], [8], [9], [10], [11]).

Proposition 6.1. *Let d be fixed and put $u_{\nu B}(s,x) = (\hat{\mathcal{B}}_k \hat{u}_\nu)(s,x)$. We assume that the Cauchy data $\varphi(x)$ satisfies the same assumptions as in Theorem 1.1. Then for each ν, we have*

$$\max_{|x|\le\sigma} |u_{\nu B}(s,x)| \le C\frac{K^\nu}{\nu!} \exp\left(\delta|s|^k\right), \quad s \in S(d,\varepsilon) \tag{6.1}$$

by some positive constants C, K, δ and σ, which are independent of ν.

We remark that Proposition 6.1 means that $\hat{u}_\nu(t,x) \in \mathcal{O}_x\{t\}_{k,d}$.

We can immediatly prove Theorem 1.1 by using Proposition 6.1.

Proof. Let $\hat{U}(t,x) = \sum_{\nu\ge0} \hat{u}_\nu(t,x)$ be the formal solution of original Cauchy problem (1.2). Then it is enough to show that $U_B(s,x) = (\hat{\mathcal{B}}_k \hat{U})(s,x) = \sum_{\nu\ge0} u_{\nu B}(s,x) \in \mathrm{Exp}_s^k(S(d,\varepsilon)\times B(\sigma))$. Therefore, we obtain the desired estimate of $U_B(s,x)$ for $s \in S(d,\varepsilon)$.

$$\begin{aligned}\max_{|x|\le\sigma} |U_B(s,x)| &\le \sum_{\nu\ge0} \max_{|x|\le\sigma} |u_{\nu B}(s,x)| \le C\exp(\delta|s|^k)\sum_{\nu\ge0}\frac{K^\nu}{\nu!}\\ &= Ce^K \exp(\delta|s|^k).\end{aligned}$$

□

§ 7. Proof of Lemma 5.2

We shall give the proof of Lemma 5.2. For the purpose, we will obtain the differential equations of $\hat{h}_\nu$ and the convolution equations of $h_{\nu B} = \hat{\mathcal{B}}_k \hat{h}_\nu$. After that, we will prove Lemma 5.2 by employing the method of successive approximation for the convolution equations.

§ 7.1. Differential equation of $\hat{h}_\nu$

We recall that

$$\hat{h}_\nu(t) = \sum\nolimits'_{n\ge0} A_\nu(n)\frac{\left(\frac{p}{q}n\right)!}{n!}t^n = \sum_{n\ge0,\frac{p}{q}n-\nu\in\mathbb{N}_0} A_\nu(n)\frac{\left(\frac{p}{q}n\right)!}{n!}t^n.$$

For $n \geq 0$ satisfying $pn/q - \nu \in \mathbb{N}_0$, we put

$$m(n) := \left(\tfrac{p}{q}n\right)!/n!. \tag{7.1}$$

For $n+1, i+1 \in q\mathbb{N}_0$ with $n \geq i \geq 0$, we have by using formula $n! = [n]_i(n-i)!$

$$\frac{m(n+1)}{m(n-i)} = \frac{\left[\frac{p}{q}(n+1)\right]_{\frac{p}{q}(i+1)}}{[n+1]_{i+1}}.$$

By multiplying both sides of (R_ν) by $(n+1)m(n+1)t^{n+1}$ and taking sum over $n \geq 0$ with $p(n+1)/q - \nu \in \mathbb{N}_0$, we get

$$\begin{aligned}\sum_{n\geq 0, p(n+1)/q-\nu\in\mathbb{N}_0} (n+1)A_\nu(n+1)m(n+1)t^{n+1} &= \sum_{\ell=0}^{\min\{\alpha_*,\nu\}} \sum_{K_\ell} a_i^{(\alpha)} t^{i+1} \\ \times \sum_{n\geq 0, p(n+1)/q-\nu\in\mathbb{N}_0} (n+1)[n]_i & \frac{\left[\frac{p}{q}(n+1)\right]_{\frac{p}{q}(i+1)}}{[n+1]_{i+1}} A_{\nu-\ell}(n-i)m(n-i)t^{n-i}.\end{aligned}$$

Here we notice that $(n+1)[n]_i/[n+1]_{i+1} = 1$. After multiplying both sides by t^k, we obtain a differential equation of $\hat{h}_\nu(t)$.

$$t^k\delta_t\hat{h}_\nu(t) = \sum_{\ell=0}^{\min\{\alpha_*,\nu\}} \sum_{K_\ell} a_i^{(\alpha)} t^{i+1+k} \left[\frac{p}{q}\delta_t + \frac{p}{q}(i+1)\right]_{\frac{p}{q}(i+1)} \hat{h}_{\nu-\ell}(t). \tag{7.2}$$

§ 7.2. A canonical form for differential equation of $\hat{h}_\nu$

We shall reduce the differential equation of $\hat{h}_\nu(t)$ to a certain canonical form (cf. [7]). We give the following lemma without proof.

Lemma 7.1. *Let $a \in \mathbb{R}$ and $k > 0$. Then for $n \geq 0$, we have*

$$[a\delta_t + n]_n = \sum_{m=0}^{n} d_{n,m}^{[a]} t^{-km}(t^k\delta_t)^m, \tag{7.3}$$

where $d_{00}^{[a]} = 1$ and

$$d_{n,m}^{[a]} = a d_{n-1,m-1}^{[a]} + (n - akm)\, d_{n-1,m}^{[a]}, \quad 0 \leq m \leq n$$

with $d_{n-1,-1}^{[a]} = d_{n-1,n}^{[a]} = 0\,(n \geq 0)$. Then we have $d_{n0}^{[a]} = n!, d_{nn}^{[a]} = a^n$.

For $0 \leq \ell \leq \alpha_*$, we put

$$L_\ell := \sum_{K_\ell} a_i^{(\alpha)} t^{i+1+k} \left[\frac{p}{q}\delta_t + \frac{p}{q}(i+1)\right]_{\frac{p}{q}(i+1)}$$

where $K_\ell = \{(\alpha, i); i(\alpha) \le i \le i_*,\ 0 \le \alpha = p(i+1)/q - \ell\}$. We write $i(\alpha)$ by $i(\ell)$. By using Lemma 7.1, we can write the operator L_ℓ into the following form

$$\begin{aligned} L_\ell &= \sum_{K_\ell} a_i^{(\alpha)} t^{i+1+k} \sum_{m=0}^{\frac{p}{q}(i+1)} d_{\frac{p}{q}(i+1),m} t^{-km} (t^k \delta_t)^m \\ &= \sum_{m=0}^{\alpha_*} \sum_{\max\{i(\ell), qm/p-1\} \le i \le i_*} a_i^{(\frac{p}{q}(i+1)-\ell)} d_{\frac{p}{q}(i+1),m} t^{i+1+k-km} (t^k\delta_t)^m. \end{aligned}$$

Here for $\ell = 0, 1, \cdots, \alpha_*$ and $m = 0, 1, \ldots, \alpha_*$, we put

$$A_m^{[\ell]}(t) := \sum_{\max\{i(\ell), qm/p-1\} \le i \le i_*} a_i^{(\frac{p}{q}(i+1)-\ell)} d_{\frac{p}{q}(i+1),m} t^{i+1+k-km}$$

and for $\ell > \alpha_*$, we define $A_m^{[\ell]}(t) \equiv 0$ $(\forall m)$ for convinience. Then we notice that $O(A_m^{[\ell]}(t)) > 0$ if $0 \le m \le \alpha_* - 1$ and $0 \le \ell \le \alpha_*$, and when $m = \alpha_*$, we have

$$A_{\alpha_*}^{[\ell]}(t) \equiv \left(\frac{p}{q}\right)^{\alpha_*} a_{i_*}^{(\alpha_*-\ell)} \quad (0 \le \ell \le \alpha_*)$$

because of $i + 1 + k - km \ge qm/p + k - km = k(\alpha_* - m)/\alpha_*$.

Therefore we can write the differential equation of $\hat{h}_\nu(t)$ into the following form

$$\begin{aligned} \text{(7.4)}\quad \left[t^k\delta_t - \left(\frac{p}{q}\right)^{\alpha_*} a_{i_*}^{(\alpha_*)} (t^k\delta_t)^{\alpha_*}\right] \hat{h}_\nu(t) &= \sum_{\ell=0}^{\min\{\alpha_*,\nu\}} \sum_{m=0}^{\alpha_*-1} A_m^{[\ell]}(t)(t^k\delta_t)^m \hat{h}_{\nu-\ell}(t) \\ &+ \sum_{\ell=1}^{\min\{\alpha_*,\nu\}} \left(\frac{p}{q}\right)^{\alpha_*} a_{i_*}^{(\alpha_*-\ell)} (t^k\delta_t)^{\alpha_*} \hat{h}_{\nu-\ell}(t). \end{aligned}$$

When $\nu \le \alpha_*$, we substitute $\hat{h}_0(t) = 1 + \tilde{h}_0(t)$ into the above equation. After some calculations we replace $\tilde{h}_0$ by $\hat{h}_0$. Then we obtain the following canonical differential equation of $\hat{h}_\nu(t)$ for all ν

$$\begin{aligned} \text{(7.5)}\quad \left[t^k\delta_t - \left(\frac{p}{q}\right)^{\alpha_*} a_{i_*}^{(\alpha_*)} (t^k\delta_t)^{\alpha_*}\right] \hat{h}_\nu(t) &= \sum_{\ell=0}^{\min\{\alpha_*,\nu\}} \sum_{m=0}^{\alpha_*-1} A_m^{[\ell]}(t)(t^k\delta_t)^m \hat{h}_{\nu-\ell}(t) \\ &+ A_0^{[\nu]}(t) + \sum_{\ell=1}^{\min\{\alpha_*,\nu\}} \left(\frac{p}{q}\right)^{\alpha_*} a_{i_*}^{(\alpha_*-\ell)} (t^k\delta_t)^{\alpha_*} \hat{h}_{\nu-\ell}(t). \end{aligned}$$

§ 7.3. Convolution equations

We shall obtain the convolution equations by operating the Borel transform to the canonical differential equations which are obtained in the previous subsection.

After operating the formal k-Borel transformation to the equation (7.5) and differentiating the both sides, we substitute $D_s h_{\nu B}(s) = w_\nu(s)$ or $h_{\nu B}(s) = D_s^{-1} w_\nu(s)$, where $D_s = d/(ds)$ and $D_s^{-1} = \int_0^s$. Then by noticing $\hat{\mathcal{B}}_k(t^k\delta_t) = kD_s^{-1}s^kD_s$ the convolution equations for $w_\nu(s)$ are given by the following expressions.

$$\begin{aligned} &\left[ks^k - a_{i_*}^{(\alpha_*)}\left((p/q)ks^k\right)^{\alpha_*}\right] w_\nu(s) \\ &= D_s \left[\sum_{\ell=0}^{\min\{\alpha_*,\nu\}} \sum_{m=0}^{\alpha_*-1} A_{mB}^{[\ell]}(s) *_k D_s^{-1} k^m s^{km} w_{\nu-\ell}(s) + A_{0B}^{[\nu]}(s) \right. \\ &\left. + D_s^{-1} \sum_{\ell=1}^{\min\{\alpha_*,\nu\}} a_{i_*}^{(\alpha_*-\ell)}((p/q)ks^k)^{\alpha_*} w_{\nu-\ell}(s) \right], \end{aligned} \tag{7.6}$$

where $A_{mB}^{[\ell]}(s) = (\hat{\mathcal{B}}_k A_m^{[\ell]})(s)$ for $0 \le m \le \alpha_* - 1$ and $0 \le \ell \le \alpha_*$, and $A_{mB}^{[\ell]}(s) \equiv 0$ for $\ell > \alpha_*$ and all m.

Here the k-convolution $a(s) *_k b(s)$ with $a(0) = b(0) = 0$ is defined by the following integral

$$(a *_k b)(s) = \int_0^s a\left((s^k - u^k)^{1/k}\right) \frac{d}{du} b(u) du. \tag{7.7}$$

We remark that if $a(0) = b(0) = 0$, the convolution is commutative. Note that this formula is same with that in [1, Sec 5.3] although the expression is a little different from it.

We put

$$A_*(s) := \left[ks^k - a_{i_*}^{(\alpha_*)}\left((p/q)ks^k\right)^{\alpha_*}\right] = ks^k \left[1 - a_{i_*}^{(\alpha_*)}\left((p/q)^{p/q} k^{1/k} s\right)^{i_*+1}\right]$$

and put

$$T_\nu(w_\nu)(s) := \frac{1}{A_*(s)}\Big(\text{the right hand side of (7.6)}\Big).$$

Then we remark that for each ν

$$T_\nu : \mathbb{C}[[s]] \to \mathbb{C}[[s]],$$

where $\mathbb{C}[[s]]$ denotes the set of formal power series, and we can prove that $w = T_\nu(w)$ has a unique formal power series solution. Therefore for each ν, the function $w_\nu(s) = D_s h_{\nu B}(s) = D_s \sum_{n\ge 0} A_\nu(n)m(n)s^n/\Gamma(1 + n/k)$ is a unique holomorphic solution in a neighborhood of the origin for the convolution equation $w = T_\nu(w)$. Let us prove this.

Let $0 < \sigma < \sigma_0$ for some σ_0 and $W(\sigma)$ be a Banach space of holomorphic functions on $0 < |s| \le \sigma$ with the norm $\|w\|_\sigma := \sup_{0<|s|\le\sigma} |w(s)| < \infty$ for $w(s) \in W(\sigma)$. We assume that for $0 < |s| \le \sigma_0$

$$\left|\frac{1}{A_*(s)}\right| \le \frac{B_1}{|s|^k}, \quad |D_s A_{mB}^{[\ell]}(s)| \le B_2 |s|^{\frac{q}{p}m+k-km-1} \ (m \ne 0) \tag{7.8}$$
$$|D_s A_{0B}^{[\ell]}(s)| \le B_2|s|^k, \quad |a_{i_*}^{(\alpha_*-\ell)}((p/q)k)^{\alpha_*}| \le B_2$$

with positive constants $B_j (j = 1,2)$ for $\ell = 0, 1, \ldots, \alpha_*$ and all m.

Now, we shall prove that for each ν, the convolution equation $w = T_\nu(w)$ has a unique holomorphic solution in a neighborhood except $s = 0$ by induction.

We notice that for $A(s)$ and $w(s)$ with $A(0) = 0$, we have

$$D_s\Big(A(s) *_k D_s^{-1} w(s)\Big) = D_s \int_0^s A\Big((s^k - u^k)^{1/k}\Big) w(u)du \tag{7.9}$$
$$= s^{k-1} \int_0^s D_s A\Big((s^k - u^k)^{1/k}\Big)(s^k - u^k)^{1/k-1} w(u)du$$
$$= s \int_0^1 D_s A\Big(s(1-t^k)^{1/k}\Big)(1-t^k)^{1/k-1} w(st)dt \quad (u = st).$$

We first prove that $T_\nu : W(\sigma) \to W(\sigma)$ is well-defined for each ν.

The case $\nu = 0$. Let $w_0(s) \in W(\sigma)$. Then we have

$$|T_0(w_0)(s)| \le \left|\frac{1}{A_*(s)} D_s \left[A_{0B}^{[0]}(s) + \sum_{m=0}^{\alpha_*-1} A_{mB}^{[0]}(s) *_k D_s^{-1} k^m s^{km} w_0(s)\right]\right|$$
$$\le \frac{B_1 B_2}{|s|^k}\Bigg[|s|^k + \|w_0\|_\sigma |s|^{1+k} \int_0^1 (1-t^k)^{\frac{1}{k}} dt$$
$$+ \sum_{m=1}^{\alpha_*-1} k^m \|w_0\|_\sigma |s|^{\frac{q}{p}m+k} \int_0^1 (1-t^k)^{\frac{\frac{q}{p}m+k-km}{k}-1} t^{km} dt\Bigg].$$

Therefore we have

$$|T_0(w_0)(s)| \le B\left[1 + \|w_0\|_\sigma |s| + \mathcal{K}_1 \|w_0\|_\sigma |s|^{\frac{q}{p}}\right],$$

where we put $B := B_1 B_2$ and $\mathcal{K}_1 := \sum_{m=1}^{\alpha_*-1} k^m \sigma_0^{\frac{q}{p}(m-1)} \int_0^1 (1-t^k)^{-m/\alpha_*} t^{km} dt$. This implies that $T_0 : W(\sigma) \to W(\sigma)$ is well-defined.

The case $\nu \ge 1$. We assume that it holds up to $\nu - 1$. Let $w_n(s) \in W(\sigma)$ for

$n = 0, 1, \ldots, \nu$. Then similarly to the case $\nu = 0$, we have

$$|T_\nu(w_\nu)(s)| \leq \left| \frac{1}{A_*(s)} D_s \left[A_{0B}^{[\nu]}(s) + \sum_{\ell=0}^{\min\{\alpha_*,\nu\}} \sum_{m=0}^{\alpha_*-1} A_{mB}^{[\ell]}(s) *_k D_s^{-1} k^m s^{km} w_{\nu-\ell}(s) \right.\right.$$

$$\left.\left. + D_s^{-1} \sum_{\ell=1}^{\min\{\alpha_*,\nu\}} a_{i_*}^{(\alpha_*-\ell)} \left(\frac{p}{q} k s^k\right)^{\alpha_*} w_{\nu-\ell}(s) \right] \right|$$

$$\leq B \left[1 + \sum_{\ell=0}^{\min\{\alpha_*,\nu\}} \|w_{\nu-\ell}\|_\sigma |s| + \mathcal{K}_1 | \sum_{\ell=0}^{\min\{\alpha_*,\nu\}} \|w_{\nu-\ell}\|_\sigma |s|^{\frac{q}{p}} + \mathcal{K}_2 \sum_{\ell=1}^{\min\{\alpha_*,\nu\}} \|w_{\nu-\ell}\|_\sigma |s| \right],$$

where we put $\mathcal{K}_2 := \sigma_0^{i_*}$. This implies that $T_\nu : W(\sigma) \to W(\sigma)$ is well-defined.

Let $M > B = B_1 B_2$. Then we define that $W(\sigma, M) := \{w \in W(\sigma); \|w\|_\sigma \leq M\}$.

We next prove that $T_\nu : W(\sigma, M) \to W(\sigma, M)$ is well-defined for each ν. In the following, we assume $\frac{q}{p} \geq 1$ for the simplicity of the proof.

The case $\nu = 0$. We put $\sigma_1 := (M/B - 1)/M(1 + \mathcal{K}_1 \sigma_0^{q/p-1})$. Then we see that for σ with $\sigma < \min\{\sigma_0, \sigma_1\}$, T_0 is well-defined on $W(\sigma, M)$.

The case $\nu \geq 1$. We put $\sigma_2 := (M/B - 1)/\left[M\{(\alpha_* + 1)(1 + \mathcal{K}_1 \sigma_0^{q/p-1}) + \alpha_* \mathcal{K}_2\}\right]$. Then we see that for σ with $\sigma < \min\{\sigma_0, \sigma_1, \sigma_2\}$, T_ν is well-defined on $W(\sigma, M)$.

Finally, we prove that T_ν becomes a contraction map on $W(\sigma, M)$ for each ν.

The case $\nu = 0$. Let $w_0, v_0 \in W(\sigma, M)$. Then we have

$$|T_0(w_0)(s) - T_0(v_0)(s)| \leq B(1 + \mathcal{K}_1 \sigma_0^{q/p-1}) \|w_0 - v_0\|_\sigma |s|.$$

We put $\sigma_3 := 1/B(1 + \mathcal{K}_1 \sigma_0^{q/p-1})$. Then we see that for $\sigma_* < \min\{\sigma_i (i = 0, 1, 2, 3)\}$, $T_0(w_0) = w_0$ has a unique solution in $W(\sigma_*, M)$.

The case $\nu \geq 1$. Let $w_n (n \leq \nu), v_\nu \in W(\sigma, M)$. Then we have

$$|T_\nu(w_\nu)(s) - T_\nu(v_\nu)(s)| \leq B(1 + \mathcal{K}_1 \sigma_0^{q/p-1}) \|w_\nu - v_\nu\|_\sigma |s|.$$

Therefore we see that for $\sigma_* < \min\{\sigma_i (i = 0, 1, 2, 3)\}$, $T_\nu(w_\nu) = w_\nu$ has a unique solution in $W(\sigma_*, M)$.

We have to remark that $s = 0$ is a removable singularity for solutions of the convolution equations $w = T_\nu(w)$. Moreover, we have to remark that for each ν, the solution w_ν may be continued analytically on $S(\theta, \varepsilon_0)$ with $\theta \not\equiv (-\arg a_{i_*}^{(\alpha_*)} + 2\pi m)/(i_* + 1) \pmod{2\pi}$ for $m = 0, 1, \cdots, i_*$ and $\varepsilon_0 > 0$, because the roots of $A_*(s)$ are the only singular points of the analytic convolution equations $w_\nu = T_\nu(w_\nu)$.

§ 7.4. Outline of a proof of Lemma 5.2

We shall give an outline of a proof for fact that $w_\nu(s)$ has the exponential growth estimate of order at most k in a sector with infinite radius. As the consequence, we see that $h_{\nu B}(s) = D_s^{-1} w_\nu(s)$ also has the same exponential growth estimate as that of $w_\nu(s)$. For the estimate of w_ν, we consider the convolution equations $w = T_\nu(w)$ on $S_1 := \{s \in S(\theta, \varepsilon_0); |s| \geq \sigma_*/2\}$, where σ_* appears in the previous subsection (cf. [3]).

Let $s_0 \in S_1$ with $|s_0| = \sigma_*/2$. We modify the operator T_ν by $\tilde{T}_\nu$ on S_1 by replacing in $w = T_\nu(w)$ the convolutions $a *_k b$ by $a \tilde{*} b$, where

$$(a \tilde{*} b)(s) := \int_{s_0}^{s} a\left((s^k - u^k)^{1/k}\right) \frac{d}{du} b(u) du, \quad s \in S_1.$$

Then we obtain the convolution equations $w_\nu = \tilde{T}_\nu(w_\nu)$ on S_1, where

$$\tilde{T}_\nu = F_\nu(s) + \frac{1}{A_*(s)} \left\{ \sum_{\ell=0}^{\min\{\alpha_*,\nu\}} \sum_{m=1}^{\alpha_*-1} k^m D_s (A_{mB}^{[\ell]} \tilde{*} D_s^{-1} s^{km} w_{\nu-\ell})(s) \right.$$
$$\left. + \sum_{\ell=1}^{\min\{\alpha_*,\nu\}} a_{i_*}^{(\alpha_*-\ell)} \left(\tfrac{p}{q} k s^k\right)^{\alpha_*} w_{\nu-\ell}(s) \right\},$$

$$F_\nu(s) = \frac{1}{A_*(s)} D_s \left\{ A_{0B}^{[\nu]}(s) + \sum_{\ell=0}^{\min\{\alpha_*,\nu\}} \sum_{m=1}^{\alpha_*-1} k^m \int_0^{s_0} A_{mB}^{[\ell]} \left((s^k - u^k)^{\frac{1}{k}}\right) u^{km} w_{\nu-\ell}(u) du \right\}.$$

We assume that for $s \in S_1$,

$$\left| \frac{1}{A_*(s)} \right| \leq \frac{B_1}{(1 + |s|^{i_*+1})|s|^k}, \quad \left| D_s A_{mB}^{[\ell]}(s) \right| \leq B_2 |s|^{i_*+k-km},$$
$$\frac{|s|^{i_*}}{1 + |s|^{i_*+1}} \leq B_3, \quad \frac{|s|^{i_*+1}}{1 + |s|^{i_*+1}} \leq B_3$$

with some positive constants B_1, B_2 and B_3 for $0 \leq m \leq \alpha_* - 1$ and $0 \leq \ell \leq \alpha_*$. Then we notice that $F_\nu(s)$ are bounded in S_1, because $w_{\nu-\ell}(s)$ are bounded in $|s| \leq |s_0|$.

Finally, by employing the method of successive approximation for the convolution equation $w_\nu = \tilde{T}_\nu(w_\nu)$ on S_1 for each ν, we obtain the desired exponential estimate for $w_\nu(s)$ (for the detail, see a forthcoming paper [6]).

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