数理解析研究所講究録
第2020巻 2017年 86-91

Functional equations with solutions of irregular singular type

Sunao ŌUCHI [1]
Sophia University

Let

$$L(z,u(z)) = \sum_{i=1}^{m} a_i(z)u(\varphi_i(z)) \tag{0.1}$$

be a linear functional operator, where $\{\varphi_i(z)\}_{i=1}^m$ are different holmorphic functions at $z=0$ such that $\varphi_i(0)=0$, $\varphi_1'(0)=\varphi_2'(0)=\cdots=\varphi_m'(0)\neq 0$. The set of all formal power series is denoted by $\mathbb{C}[[z]]$ and $\mathbb{Z}_+ = \{n\in \mathbb{Z}; n\geq 0\}$. A Functional equation

$$L(z,u(z)) = f(z) \tag{0.2}$$

is treated, where $\{a_i(z)\}_{i=1}^m$ $(m\geq 2)$ and $f(z)$ are holomorphic in a sector with vertex $z=0$ or formal power series of z. The following problems (1),(2) and (3) are considered in this paper. [2]

(1) Existence of solutions of formal power series.

(2) Existence of formal homogeneous solutions ($f(z)\equiv 0$) taking the form of
$$\tilde{v}(z) = e^{\psi(1/z)}z^{\alpha}\tilde{w}(z),$$
where $\psi(t)$ is a polynomial of t and $\tilde{w}(z)\in\mathbb{C}[[z]]$.

(3) Asymptotic analysis. Existence of genuine solutions on a sector whose asymptotic behaviors are same as formal solutions.

The problems (1) and (3) are studied in [4] under the condition that the coefficients $\{a_i(z)\}_{i=1}^m$ are constants. The details of this paper will be published elsewhere.

We may assume $\varphi_i'(0)=1$ and $\varphi_1(z)=z$. Hence

$$\varphi_i(z) = z(1+\boldsymbol{b}_{i,p_i}z^p+\cdots). \tag{0.3}$$

for $i\geq 2$, and put $p=\min\{p_i; i\geq 2\}$. We introduce a subclass of $\mathbb{C}[[z]]$.

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[1]Sophia University, Tokyo, Japan. e-mail s_ouchi@sophia.ac.jp

[2]2010 Mathematical Classification Numbers. 30D05; Secondary 39B32, 44A10

Definition 0.1. *Let $s \geq 0$. A formal power series $f(z) = \sum_{n=0}^{\infty} f_n z^n$ is said to be of Gevrey order s, if there exist constants $A, C \geq 0$ such that*

$$|f_n| \leq AC^n\Gamma(sn+1) \quad \textit{for} \quad n \in \mathbb{Z}_+. \tag{0.4}$$

$\mathbb{C}_{\{s\}}[[z]]$: *the totality of all formal power series of Gevrey order s.*

If $s = 0$, $f(z)$ converges and is holomorphic at $z = 0$. We have the following existence Theorems in $\mathbb{C}[[z]]$. Let

$$\mathscr{A}(z) = \sum_{i=1}^{m} a_i(z). \tag{0.5}$$

Theorem 0.2. *Suppose $a_i(z) \in \mathbb{C}[[z]](1 \leq i \leq m)$ and $f(z) = \sum_{n=0}^{\infty} f_n z^n \in \mathbb{C}[[z]]$ and one of the following (1)-(2) holds.*

(1) $\mathscr{A}(0) \neq 0$.

(2) $\mathscr{A}(0) = \cdots = \mathscr{A}^{(p-1)}(0) = 0, \quad \sum_{i=1}^{m} a_i(0)b_{i,p} \neq 0,$
$\frac{\mathscr{A}^{(p)}(0)}{p!} + n\sum_{i=1}^{m} a_i(0)b_{i,p} \neq 0$ *for* $n \in \mathbb{Z}_+$
and $f_0 = f_1 = \cdots = f_{p-1} = 0$.

Then there exists a unique formal solution $u(z) \in \mathbb{C}[[z]]$ of (0.2). *Moreover, if $a_i(z) \in \mathbb{C}[[z]]_{\{1/p\}}(1 \leq i \leq m)$ and $f(z) \in \mathbb{C}[[z]]_{\{1/p\}}$, then $u(z) \in \mathbb{C}[[z]]_{\{1/p\}}$.*

Theorem 0.3. *Suppose $a_i(z) \in \mathbb{C}[[z]](1 \leq i \leq m)$ and $f(z) = \sum_{n=0}^{\infty} f_n z^n \in \mathbb{C}[[z]]$. Further assume*

$$\mathscr{A}(0) = \cdots = \mathscr{A}^{(p)}(0) = 0,$$

$$\sum_{i=1}^{m} a_i(0)b_{i,p} \neq 0$$

and $f_0 = f_1 = \cdots = f_p = 0$. Then for given c_0 there exists a unique formal solution $u(z) \in \mathbb{C}[[z]]$ of (0.2) with $u(0) = c_0$. Moreover, if $a_i(z) \in \mathbb{C}[[z]]_{\{1/p\}}(1 \leq i \leq m)$ and $f(z) \in \mathbb{C}[[z]]_{\{1/p\}}$, then $u(z) \in \mathbb{C}[[z]]_{\{1/p\}}$.

From Theorem 0.3 we get existence non trivial homogeneous solutions in $\mathbb{C}[[z]]$.

Corollary 0.4. *Suppose* $a_i(z) \in \mathbb{C}[[z]](1 \leq i \leq m)$,

$$\mathscr{A}(0) = \cdots = \mathscr{A}^{(p)}(0) = 0$$

and

$$\sum_{i=1}^{m} a_i(0) b_{i,p} \neq 0.$$

Then there exists $u(z) \in \mathbb{C}[[z]]$ *with* $u(0) = 1$ *satisfying* $L(z, u(z)) = 0$. *Moreover, if* $a_i(z) \in \mathbb{C}[[z]]_{\{1/p\}}(1 \leq i \leq m)$, *then* $u(z) \in \mathbb{C}[[z]]_{\{1/p\}}$.

We try to find homogeneous solutions wider class than $\mathbb{C}[[z]]$. We obtain formal solutions with exponential factor.

Theorem 0.5. *Suppose that there exists* ξ_0 *such that* $\sum_{i=1}^{m} a_i(0) e^{-pb_{i,p}\xi_0} = 0$ *with* $\sum_{i=1}^{m} a_i(0) b_{i,p} e^{-pb_{i,p}\xi_0} \neq 0$. *Then there exists a formal solution of* $L(z, u(z)) = 0$ *in the form*

$$u(z) = \exp(\frac{C_p}{z^p} + \frac{C_{p-1}}{z^{p-1}} + \cdots + \frac{C_1}{z}) z^{\alpha} w(z), \tag{0.6}$$

where $C_p = \xi_0$ *and* $w(z) \in \mathbb{C}[[z]]$ *with* $w(0) = 1$. *Moreover, if* $a_i(z) \in \mathbb{C}[[z]]_{\{1/p\}}(1 \leq i \leq m)$, *then* $w(z) \in \mathbb{C}[[z]]_{\{1/p\}}$.

In the previous part we find formal solutions. The next aim is to give analytical meanings to them. We define $S(\theta, \delta, r) = \{z; |\arg z - \theta| < \delta, |z| < r\}$ which is a sector in z-space.

Definition 0.6. *We say that* $\tilde{w}(z) = \sum_{n=0}^{\infty} c_n z^n \in \mathbb{C}[[z]]$ *is* γ*-Borel summable in a direction* θ, *if there exists a holomorphic function* $w(z)$ *on* $S(\theta, \delta, r)$, $\delta > \pi/(2\gamma)$, *such that*

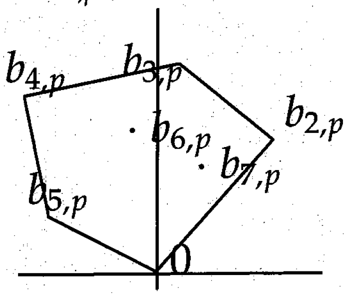
$$|w(z) - \sum_{n=0}^{N-1} c_n z^n| \leq AC^N \Gamma(\frac{N}{\gamma} + 1)|z|^N \tag{0.7}$$

holds for all $N \in \mathbb{Z}_+$. *We denote (0.7) by* $w(z) \underset{\gamma}{\sim} \sum_{n=0}^{\infty} c_n z^n$.

We have $\tilde{w}(z) \in \mathbb{C}[[z]]_{\{1/\gamma\}}$. Since $\delta > \pi/(2\gamma)$, $w(z)$ is uniquely determined for $\tilde{w}(z)$. Hence we may identify $\tilde{w}(z)$ with $w(z)$. As for the Borel summability (multi-summability) of functions we refer to [1] and [2].

We give a condition to study the relation between formal solutions and genuine solutions. Let $b_{1,p} = 0$ and $B = \{b_{i,p}; 1 \leq i \leq m\}$ which is a finite set in $\mathbb{C}$.

Condition B. $a_1(0) \neq 0$ and $b_{i,p} \neq 0$ for $2 \leq i \leq m$ and $\{0\}$ is a vertex of the convex hull $\widehat{B}$ of B.



Let us remember $\mathscr{A}(z) = \sum_{i=1}^m a_i(z)$ and $f(z) = \sum_{n=0}^\infty f_n z^n$.

Theorem 0.7. *Suppose that Condition B and one of the following (1)-(3) hold.*

(1) $\mathscr{A}(0) \neq 0$.

(2) $\mathscr{A}(0) = \cdots = \mathscr{A}^{(p)}(0) = 0, \quad \sum_{i=1}^m a_i(0) b_{i,p} \neq 0$
and $f_0 = f_1 = \cdots = f_p = 0$.

(3) $\mathscr{A}(0) = \cdots = \mathscr{A}^{(p-1)}(0) = 0, \quad \sum_{i=1}^m a_i(0) b_{i,p} \neq 0,$
$\frac{\mathscr{A}^{(p)}(0)}{p!} + n(\sum_{i=1}^m a_i(0) b_{i,p}) \neq 0$ *for* $n \in \mathbb{Z}_+$
and $f_0 = f_1 = \cdots = f_{p-1} = 0$.

Then there exists a direction θ_0 such that the formal solution $u(z) \in \mathbb{C}[[z]]$ of (0.2) *is p-Borel summable in the direction θ_0, provided $a_i(z)(1 \leq i \leq m)$ and $f(z)$ are p-Borel summable in the direction θ_0.*

As for homogeneous formal solutions in Theorem 0.5 we have

Theorem 0.8. *Suppose that Condition B holds. Then there exists a direction θ_0 such that if $a_i(z)$ is p-Borel summable in the direction θ_0, $w(z) \in \mathbb{C}[[z]]$ of* (0.6) *in Theorem 0.5 is also p-Borel summable in the direction θ_0.*

The direction θ_0 is determined by the location of zeros of

$$h(\zeta) = \sum_{i=1}^m a_i(0) e^{p b_{i,p} \zeta^p}.$$

We sum up the obtained results.

(1) It follows from Theorems 0.2, 0.3 and 0.7 that there exist solutions of formal power series with Gevrey order.

(2) Theorem 0.5 is the existence of formal solutions of the homogeneous equation, which are represented with exponential factors.

These facts are similar to the properties of solutions of ordinary differential equations with an irregular singular point at $z = 0$. As for the properties of solutions of irregular singular ordinary differential equations we refer to [3] and [5] and papers cited there.

Finally we give a simple example to understand the results more concretely.

Example.

$$\begin{cases} u(z) + u(z/(1-z)) = \dfrac{z}{2-z} \\ u(z) = \displaystyle\int_0^{e^{i\theta}\infty} \frac{e^{-\frac{\zeta}{z}+\frac{\zeta}{2}}}{2(1+e^{\zeta})} d\zeta \end{cases}$$

$$\hat{u}(\zeta) = \frac{\exp(\frac{\zeta}{2})}{2(1+\exp\zeta)} = \sum_{n=0}^{\infty} c_n \zeta^n \quad (|\zeta| < \pi)$$

$$u(z) \underset{1}{\sim} \sum_{n=1}^{\infty} c_{n-1}\Gamma(n) z^n \qquad |\arg z| < \pi - \varepsilon$$

$\{\exp \frac{(2n+1)\pi i}{z};\ n \in \mathbb{Z}\}$ are homogeneous solutions.

References

[1] W. Balser, From Divergent Power Series to Analytic Functions, vol.1582, Lecture Notes in Math., Springer, 1994.

[2] W. Balser, Formal power series and linear systems of meromorphic ordinary differential equations, Universitext, Springer, 1999.

[3] W. Balser, B.L.J. Braaksma, J.-P. Ramis and Y. Sibuya, Multisummabilty of formal power series solutions of linear ordinary differential equations, Asymptotic Analysis, **5** (1991), 27-45.

[4] S. Ōuchi. On some functional equation with Borel summable solutions. Funkcialaj Ekvacioj **58** (2015), 223-251.

[5] W. Wasow, Asymptotic Expansions of Ordinary Differential Equations, Dover Publications, New York, 1987.