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Summability of formal solutions for $\epsilon t^{r+1}\frac{\partial}{\partial t}u = f(t, u)$

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Abstract

In this paper we consider semilinear partial differential equations. For a formal solution of the equations we give the results of the summability of the formal solution with respect to the each variable t and ϵ and the both variables.

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1 Introduction

In this paper we study the following equation:

$$\epsilon t^{r+1}\frac{\partial}{\partial t}u = f(t, u) \tag{1.1}$$

where $(t, \epsilon) \in \mathbb{C}\times\mathbb{C}$ and $f(t, u)$ is a function defined in a neighborhood of $(0, 0)$.

In this paper we assume the following conditions:

(A1) $f(t, u)$ is holomorphic in a neighborhood of $(0, 0)$,

(A2) $f(0, 0) = 0$,

(A3) $\dfrac{\partial f}{\partial u}(0, 0) \neq 0$.

Under the conditions $(A1)$, $(A2)$ and $(A3)$, we have the following expansion

$$f(t, u) = \sum_{l\geq 0} f_l(t)u^l \quad \text{with } f_0(0) = 0 \text{ and } f_1(0) \neq 0.$$

For the case $r = 0$ we have some results. In [4] Balser and Kostov studied Borel summability of formal solutions for a linear system of partial differential equations and in [9] Yamazawa and Yoshino treated a semilinear system of partial differential equations. In these papers the equation (1.1) has a formal solution $\hat{u}(t, \epsilon) = \sum_{m=0}^{\infty} u_{*m}(t)\epsilon^m$ and the formal solution is summable in a suitable direction.

For the case $r > 0$ Balser and Mozo studied a linear system of partial differential equations in [3] and got the summability of formal solutions $u(t, \epsilon)$ with respect to the respective variables and two variables. In [5] Canalis, Mozo and Schäfke treated a semilinear system of partial differential equations. By their paper we have a formal power series solution of the equation (1.1) and the solution is monomial summable (ϵt^r-summable).

In this paper we will show that the equation (1.1) has formal power series solutions in t or ϵ and the solutions are summable respectively.

In Section 2 we give formal Gevrey estimates of formal solutions $\hat{u}(t,\epsilon)=\sum_{k=1}^{\infty}u_{k*}(\epsilon)t^k$ of the equation (1.1) and Summability of the formal solution. In Section 3 we give formal Gevrey estimates of formal solutions $\tilde{u}(t,\epsilon)=\sum_{m=0}^{\infty}u_{*m}(t)\epsilon^m$ of (1.1) and Summability of the formal solution. In Section 4 we give Summability of the formal solution $\hat{u}(t,\epsilon)=\sum_{k=1}^{\infty}\sum_{m=0}^{\infty}u_{k,m}t^k\epsilon^m$ with respect to the both variables. In Section 5 we give an alternative proof of [9] for the case $r=0$.

2 Summability with respect to the variable t

In this section we will show that the equation (1.1) have formal power series solutions $\hat{u}(t,\epsilon)=\sum_{k=1}^{\infty}u_{k*}(\epsilon)t^k$ and the formal solution $\hat{u}(t,\epsilon)$ is r-summable in a direction d.

Denote the universal covering of $\mathbb{C}^*=\mathbb{C}\setminus\{0\}$ by $\widetilde{\mathbb{C}^*}$. Let us introduce s-region that is defined in [2]. Given $s=(s_1,s_2)$ with $s_1,s_2>0$, a region G is called an s-region, provided that it is an open and simply connected subset of polysecter in $\widetilde{\mathbb{C}^*}\times\widetilde{\mathbb{C}^*}$ satisfying the following condition:

- For every $(t,\epsilon)\in G$ and every real x with $0<x\le 1$, all points of the form $\zeta_s(x,t,\epsilon)=(x^{s_1}t,x^{s_2}\epsilon)$ belong to G.

We call G_∞ an s-region of infinite radius, provided that, instead of the above condition, we have the followings:

- For every $(t,\epsilon)\in G_\infty$ and every real x with $0<x<\infty$, all points of the form $\zeta_s(x,t,\epsilon)$ belong to G_∞.

Let $D_\rho=\{t\in\mathbb{C};\ |t|<\rho\}$ or $\{\epsilon\in\mathbb{C};\ |\epsilon|<\rho\}$. Set $S^t_{d,\theta}:=\{t\in\mathbb{C}\setminus\{0\};\ |\arg\xi-d|<\theta\}$ and $S^t_{d,\theta}(\rho)=S^t_{d,\theta}\cap D_\rho$, further set $S^\epsilon_{d,\theta}$ and $S^\epsilon_{d,\theta}(\rho)$ as the same rules.

Let D^ϵ be an open and bounded domain in ϵ-plane. $\mathcal{O}(D^\epsilon)[[t]]$ be the set of all formal power series $\hat{u}(t,\epsilon)=\sum_{k=0}^{\infty}u_{k*}(\epsilon)t^k$ with holomorphic coefficients in D^ϵ.

Let $\gamma>0$. By $\mathcal{O}(D^\epsilon)[[t]]_{1/\gamma}$ we denote the subset of $\mathcal{O}(D^\epsilon)[[t]]$ whose coefficients satisfy with some positive constants A, B and any proper subdomein D' of D^ϵ

$$\sup_{\epsilon\in D'}|u_{k*}(\epsilon)|\le AB^k\Gamma\Big(\frac{k}{\gamma}+1\Big)\quad\text{for}\quad k=0,1,\dots,$$

The elements of $\mathcal{O}(D^\epsilon)[[t]]_{1/\gamma}$ are called of formal series of Gevrey class $1/\gamma$.

Let $u(t,\epsilon)$ be an analytic function on $S^t_{d,\theta}(\rho)\times D^\epsilon$ for some $\rho>0$. Then $\hat{u}(t,\epsilon)\in\mathcal{O}(D^\epsilon)[[t]]_{1/\gamma}$ is called a Gevrey asymptotic expansion of $u(t,\epsilon)$ as $t\to 0$ in $S^t_{d,\theta}$, written as

$$u(t,\epsilon)\cong_{1/\gamma}\hat{u}(t,\epsilon)\quad\text{in}\quad S^t_{d,\theta}\quad\text{or}\quad u(t,\epsilon)\in A^t_{1/\gamma}(S^t_{d,\theta}(\rho)\times D^\epsilon),$$

if for any proper subdomain D' of D^ϵ there exist positive constants A, B such that $\hat{u}(t,\epsilon)\in\mathcal{O}(D^\epsilon)[[t]]_{1/\gamma}$ and

$$\sup_{\epsilon\in D'}|u(t,\epsilon)-\sum_{k=0}^{N-1}u_k(\epsilon)t^k|\le AB^N\Gamma\Big(\frac{N}{\gamma}+1\Big)|t|^N\quad\text{for } N=1,2,\dots$$

on $S^t_{d,\theta'}(\rho')$ for $0<\theta'<\theta$ and $0<\rho'<\rho$.

Definition 2.1 *We say that* $\hat{u}(t,\epsilon) \in \mathcal{O}(D^\epsilon)[[t]]_{1/\gamma}$ *is* γ*-summable with respect to the variable* t *in a direction* $d \in \mathbb{R}$ *if there exist a sector* $S^t_{d,\theta}(\rho)$ *with* $\theta > \pi/(2\gamma)$ *and a function* $u(t,\epsilon)$ *analytic on* $S^t_{d,\theta}(\rho) \times D^\epsilon$ *such that* $u(t,\epsilon) \cong_{1/\gamma} \hat{u}(t,\epsilon)$ *in* $S^t_{d,\theta}$.

Remark 2.2 *Let us remark that the function* $u(t,\epsilon)$ *is unique if it exists, in that case* $u(t,\epsilon)$ *is called the* γ*-sum of* $\hat{u}(t,\epsilon)$ *with respect to the variable* t.

Here let us give our theorem of the summability of formal solutions $\hat{u}(t,\epsilon) = \sum_{k=1}^\infty u_{k*}(\epsilon)t^k \in \mathcal{O}(D_\rho)[[t]]_{1/r}$ for the equation (1.1).

Set $d_j = \arg \partial f/\partial u(0,0) + 2\pi j$ for $j \in \mathbb{Z}$.

Theorem 2.3 *Assume the conditions (A1), (A2) and (A3). Then the equation (1.1) has a formal power solution* $\hat{u}(t,\epsilon) = \sum_{k=1}^\infty u_{k*}(\epsilon)t^k$ *and the solution* $\hat{u}(t,\epsilon)$ *is in* $\mathcal{O}(D_\rho)[[t]]_{1/r}$. *Further the formal solution* $\hat{u}(t,\epsilon)$ *is* r*-summable with respect to the variable* t *in any direction* d *for any* d *and* ϵ *with* $d_j < \arg\epsilon + rd < d_{j+1}$ *and* $|\epsilon| < \rho$ *for a suitable constant* $\rho > 0$.

Remark 2.4 *For* (τ,ϵ) *with* $d_j < \arg\epsilon + r\arg\tau < d_{j+1}$ *the following holds*

$$r\epsilon\tau^r - \frac{\partial f}{\partial u}(0,0) \neq 0.$$

Further the set described by $d_j < \arg\epsilon + r\arg\tau < d_{j+1}$ *is an s-region.*

We prove the following proposition in order to show Theorem 2.3.

Proposition 2.5 *Assume the conditions (A1), (A2) and (A3). Then the equation (1.1) has a formal power solution* $\hat{u}(t,\epsilon) = \sum_{k=1}^\infty u_{k*}(\epsilon)t^k$ *and there exists constants* $U_{k*} \geq 0$ *such that for* $0 < \rho' < \rho$

$$\sup_{\epsilon \in D_{\rho'}} |u_{k*}(\epsilon)| \leq U_{k*}(k-1)!^{1/r} \quad \text{for } k \geq 1 \tag{2.1}$$

and a series $\sum_{k\geq 1} U_{k*}t^k$ *converges in a neighborhood of* $t = 0$.

Proof. Set $f_0(t) = \sum_{k=1}^\infty f_{0,k}t^k$ and $f_l(t) = \sum_{k=0}^\infty f_{l,k}t^k$ for $l \geq 1$. By substituting $\hat{u}(t,\epsilon) = \sum_{k=1}^\infty u_{k*}(\epsilon)t^k$ into the equation (1.1) we have

$$\begin{aligned} 0 &= f_{0,1} + f_{1,0}u_{1*}(\epsilon) \\ (k-r)\epsilon u_{k-r*}(\epsilon) &= f_{0,k} + f_{1,0}u_{k*}(\epsilon) + \sum_{\substack{k_0+k_1=k \\ k_0,k_1\geq 1}} f_{1,k_0}u_{k_1*}(\epsilon) \\ &\quad + \sum_{l=2}^{k} \sum_{\substack{k(l)=k \\ k_i\geq 1, 1\leq i\leq l}} f_{l,k_0} \prod_{i=1}^{l} u_{k_i*}(\epsilon) \quad \text{for } k \geq 2 \end{aligned} \tag{2.2}$$

where $k(l) = k_0 + k_1 + \cdots + k_l$ and $u_{k*}(\epsilon) \equiv 0$ for $k \leq 0$. By the condition (A3) we have $\partial f/\partial u(0,0) \neq 0$. Then we obtain a formal power series solution $\hat{u}(t,\epsilon) = \sum_{k=1}^\infty u_{k*}(\epsilon)t^k$ by the recurrence formula (2.2).

Let us give estimates for the coefficients $u_{k*}(\epsilon)$. By the first equation in (2.2) we have $u_{1*}(\epsilon) = -f_{0,1}/f_{1,0}$. Then set

$$U_{1*} = \left|\frac{f_{0,1}}{f_{1,0}}\right|.$$

Let show the estimates for $k \geq 2$ on induction. By the induction's assumption and

$$\prod_{i=1}^{l}(k_i-1)! \leq (k_1+\cdots+k_l-l)! \leq (k-l)!$$

for $k(l)=k$ and $k_i \geq 1$ $(i \geq 1)$, we obtain

$$\begin{aligned}|f_{1,0}U_{k*}(\epsilon)| \leq &\Big\{|f_{0,k}|+\rho_0 U_{k-r*}+\sum_{\substack{k_0+k_1=k\\k_0,k_1\geq 1}}|f_{1,k_0}|U_{k_1*}+\sum_{l=2}^{k}\sum_{\substack{k(l)=k\\k_i\geq 1, i\geq 1}}|f_{l,k_0}|\prod_{i=1}^{l}U_{k_i*}\Big\}\\ &\times (k-1)!^{1/r}\end{aligned} \tag{2.3}$$

for $\epsilon \in D_{\rho'}$. Set

$$U_{k*} := |f_{1,0}|^{-1}\Big\{|f_{0,k}|+\rho' U_{k-r*}+\sum_{\substack{k_0+k_1=k\\k_0,k_1\geq 1}}|f_{1,k_0}|U_{k_1*}+\sum_{l=2}^{k}\sum_{\substack{k(l)=k\\k_i\geq 1, i\geq 1}}|f_{l,k_0}|\prod_{i=1}^{l}U_{k_i*}\Big\}. \tag{2.4}$$

Then we obtain the estimate (2.1).

Let us show that $\sum_{k=1}^{\infty}U_{k*}t^k$ converges in a neighborhood of $t=0$. We consider the following equation:

$$|f_{1,0}|U(t)=\sum_{k\geq 1}|f_{0,k}|t^k+\rho_0 t^r U(t)+\sum_{l\geq 2}\sum_{k\geq 0}|f_{l,k}|t^k\{U(t)\}^l. \tag{2.5}$$

By $r>0$ and Implicit function theorem, the equation (2.5) has a holomorphic solution $U(t)=\sum_{k=1}^{\infty}U_{k*}t^k$ in a neighborhood of $t=0$ and U_{k*} satisfies the relation (2.4). Q.E.D.

Definition 2.6 *For $\hat{u}(t,\epsilon)=\sum_{k=1}^{\infty}u_{k*}(\epsilon)t^k$ we define the formal Borel transform $(\hat{\mathcal{B}}_\gamma\hat{u})(\tau,\epsilon)$ by*

$$(\hat{\mathcal{B}}_\gamma\hat{u})(\tau,\epsilon):=\sum_{k=1}^{\infty}u_{k*}(\epsilon)\frac{\tau^{k-\gamma}}{\Gamma(k/\gamma)}.$$

Then γ-summability of $\hat{u}(t,x)\in\mathcal{O}(S^{\epsilon}_{d,\theta}(\rho))[[t]]_{1/\gamma}$ can be characterized by the following Proposition.

Proposition 2.7 (L.M.S. [7]) *The formal power series $\hat{u}(t,\epsilon)\in\mathcal{O}(S^{\epsilon}_{d,\theta}(\rho))[[t]]_\gamma$ is γ-summable with respect to the variable t in a direction d if the following two properties hold:*

1. *The power series $\tau^\gamma u_B(\tau,\epsilon):=\tau^\gamma(\hat{\mathcal{B}}_\gamma\hat{u})(\tau,\epsilon)$ converges on $D_{\rho'}\times S^{\epsilon}_{d,\theta}(\rho)$.*
2. *Let S^ϵ be any proper subdomain of $S^{\epsilon}_{d,\theta}(\rho)$. There exists a $\theta>0$ such that for any $\epsilon\in\overline{S^\epsilon}$ the function $u_B(\tau,\epsilon)$ can be continued with respect to τ into the sector $S^{\tau}_{d,\theta}$. Moreover for any $0<\theta'<\theta$ there exist constants C, $K>0$ such that*

$$\sup_{\epsilon\in S^\epsilon}|u_B(\tau,\epsilon)|\leq Ce^{K|\tau|^\gamma}\quad \textit{for}\quad \tau\in S^{\tau}_{d,\theta'}.$$

Then $(\mathcal{L}_{\gamma,d}u_B)(t,\epsilon)$ is γ-sum with respect to the variable t in a direction d of $\hat{u}(t,\epsilon)$, where $\mathcal{L}_{\gamma,d}$ is the Laplace transform that is defined by

$$(\mathcal{L}_{\gamma,d}\phi)(t,\epsilon) := \int_0^{\infty e^{id}} \exp\big(-(\tfrac{\tau}{t})^\gamma\big)\phi(\tau,\epsilon)d\tau^\gamma.$$

Let us seek for the equation that is satisfied with $u_B(\tau,\epsilon)$.

Definition 2.8 *Let $\phi_i(\tau,\epsilon) \in \mathcal{O}(S^\tau_{d,\theta}\times D)$, $i=1,2$, satisfy $|\phi_i(\tau,\epsilon)| \le C|\tau|^{\delta-\gamma}$ for $\delta>0$ where D is an open domain. Then γ-convolution of $\phi_1(\tau,\epsilon)$ and $\phi_2(\tau,\epsilon)$ is defined by*

$$(\phi_1 *_\gamma \phi_2)(\tau,\epsilon) = \int_0^\tau \phi_1((\tau^\gamma-\eta^\gamma)^{1/\gamma},\epsilon)\phi_2(\eta,\epsilon)d\eta^\gamma.$$

Set $u_B(\tau,\epsilon)^{l*_r} = \underbrace{u_B(\tau,\epsilon) *_r \cdots *_r u_B(\tau,\epsilon)}_{l}$. By operating $\hat{\mathcal{B}}_r$ to the equation (1.1), we get the following convolution equation:

$$(\epsilon r\tau^r - f_{1,0})u_B(\tau,\epsilon) = f_{0,B}(\tau) + \sum_{l\ge1} f_{l,B}(\tau) *_r u_B(\tau,\epsilon)^{l*_r} \tag{2.6}$$

where $f_{1,B}(\tau) = (\hat{\mathcal{B}}_r(f_1-f_{1,0}))(\tau)$ and $f_{l,B}(\tau) = (\hat{\mathcal{B}}_r f_l)(\tau)$ for $l \ne 1$.

Let us solve the equation (2.6). We construct $u_B = \sum_{k=1}^\infty u_{B,k}$ as follows;

$$(2.7)\qquad \begin{aligned} &(\epsilon r\tau^r - f_{1,0})u_{B,1} = f_{0,B}(\tau) \quad \text{and for } k\ge 2\\ &(\epsilon r\tau^r - f_{1,0})u_{B,k} = f_{1,B}(\tau) *_r u_{B,k-1} + \sum_{l=2}^{k}\sum_{k^*(l)=k} f_{l,B}(\tau) *_r u_{B,k_1} *_r \cdots *_r u_{B,k_l}\end{aligned}$$

where $k^*(l) = k_1+\cdots+k_l$.

Set $G_j = \{d_j < \arg\epsilon + r\arg\tau < d_{j+1} \text{ and } |\epsilon|<\rho\}$. Then we have;

Proposition 2.9 *There exist constants $U_{B,k}\ge 0$ such that*

$$|u_{B,k}| \le U_{B,k}\frac{|\tau|^{k-r}}{\Gamma(k/r)}e^{c|\tau|^r} \quad on \;\; G_j \tag{2.8}$$

for some $c>0$ and a series $\sum_{k=1}^\infty U_{B,k}t^k$ converges in a neighborhood of $t=0$.

In order to show Proposition 2.9 we will use the following lemma:

Lemma 2.10 ([8], Lemma 1.4, p.516) *Assume that the functions $\phi_i(\tau,\epsilon)\in\mathcal{O}(G_j)$, $i=1,2$, satisfy*

$$|\phi_i(\tau,\epsilon)| \le C_i\frac{|\tau|^{s_i-\gamma}}{\Gamma(s_i/\gamma)}e^{c|\tau|^\gamma} \quad on \quad G_j$$

for $i=1,2$. Then convolution $(\phi_1 *_\gamma \phi_2)(\tau,\epsilon)$ satisfies

$$|(\phi_1 * \phi_2)(\tau,\epsilon)| \le C_1C_2\frac{|\tau|^{s_1+s_2-\gamma}}{\Gamma((s_1+s_2)/\gamma)}e^{c|\tau|^\gamma} \quad on \quad G_j.$$

Proof of Proposition 2.9. We have that the following estimate holds

$$|\epsilon r \tau^r - f_{1,0}| \geq K_1^{-1} \quad \text{on } G_j \tag{2.9}$$

and

$$|f_{l,B}| \leq \begin{cases} F_{l,B}|\tau|^{1-r}e^{c|\tau|^r} & l=0,1 \\ F_{l,B}e^{c|\tau|^r} & l \geq 2 \end{cases} \tag{2.10}$$

where $\sum_{l\geq 0} F_{l,B}t^l$ converges in a neighborhood of $t=0$. Let us give estimates on $u_{B,k}$. By the recurrence formula (2.7) and the estimate (2.9) we have

$$|u_{B,1}| \leq U_{B,1}|\tau|^{1-r}e^{c|\tau|^r} \quad \text{on } G_j$$

where $U_{B,1} = K_1 F_{0,B}$. For $k \geq 2$ we show the estimate (2.8) on induction. By the induction's assumptions and Lemma 2.10 we have

$$|\epsilon r \tau^r - f_{1,0}||u_{B,k}| \leq F_{1,B}U_{B,k-1}\frac{|\tau|^{k-r}}{\Gamma(k/r)}e^{c|\tau|^r} + \sum_{l=2}^{k}\sum_{k^*(l)=k} F_{l,B}\prod_{i=1}^{l} U_{B,k_i}\frac{|\tau|^{k-r}}{\Gamma(k/r)}e^{c|\tau|^r}. \tag{2.11}$$

Then by setting

$$U_{B,k} = K_1\Big\{F_{1,B}U_{B,k-1} + \sum_{l=2}^{k}\sum_{k^*(l)=k} F_{l,B}\prod_{i=1}^{l} U_{B,k_i}\Big\} \tag{2.12}$$

we get the estimates

$$|u_{B,k}| \leq U_{B,k}\frac{|\tau|^{k-r}}{\Gamma(k/r)}e^{c|\tau|^r} \quad \text{on } G_j. \tag{2.13}$$

Let us show that $\sum_{k=1}^{\infty} U_{B,k}t^k$ converges in a neighborhood of $t=0$. We consider the following equation:

$$U(t) = tU_{B,1} + K_1\Big\{tF_{1,B}U(t) + \sum_{l=2}^{\infty} F_{l,B}\{U(t)\}^l\Big\}. \tag{2.14}$$

By Implicit function theorem, the equation (2.14) has a holomorphic solution $U(t) = \sum_{k=1}^{\infty} U_{B,k}t^k$ in a neighborhood of $t=0$ and $U_{B,k}$ satisfies the relation (2.12). Q.E.D.

We will show the uniqueness of solution near $\tau = 0$ for the equation (2.6). Let u_B and v_B be solutions of the convolution equation (2.6). Then $w_B := u_B - v_B$ satisfies the following convolution equation:

$$(\epsilon r \tau^r - f_{1,0})w_B = f_{1,B} *_r w_B + \sum_{l=2}^{\infty} f_{l,B} *_r w_B^{l*_r}. \tag{2.15}$$

We can get that there exist positive constants A and B such that

$$|w_B| \leq AB^n\frac{|\tau|^{n-r}}{\Gamma(n/r)}e^{c|\tau|^r} \quad \text{for } n \geq 1 \tag{2.16}$$

as the same way as in the proof of Proposition 2.9. By letting $n \to \infty$ we obtain $u_B = v_B$. Q.E.D.

3 Summability with respect to the variable ϵ

In this section we will show that the equation (1.1) has a formal solution $\tilde{u}(t,\epsilon) = \sum_{m=0}^{\infty} u_{*m}(t)\epsilon^m$ and the formal solution is 1-summable in a direction d. We will use the same notations with respect to the variable ϵ as those with respect to the variable t in Section 2.

Theorem 3.1 *Assume the conditions (A1), (A2) and (A3). Then the equation (1.1) has a formal power series solution* $\tilde{u}(t,\epsilon) = \sum_{m=0}^{\infty} u_{*m}(t)\epsilon^m$ *and the solution* $\tilde{u}(t,\epsilon)$ *is in* $\mathcal{O}(D_\rho)[[\epsilon]]_1$. *Further the formal solution* $\tilde{u}(t,\epsilon)$ *is 1-summable with respect to the variable* ϵ *in a direction* d *for any* t *with* $d_j < d + r\arg t < d_{j+1}$ *and* $|t| < \rho$.

Proof. Let us show that (1.1) has a formal power series solution. By substituting $\tilde{u}(t,\epsilon) = \sum_{m=0}^{\infty} u_{*m}(t)\epsilon^m$ into (1.1) we have

$$0 = f(t, u_{*0}(t))$$

$$(3.1)\qquad t^{r+1}\frac{\partial}{\partial t}u_{*m-1}(t) = f_1(t)u_{*m}(t) + \sum_{l=2}^{m} \sum_{\substack{m^*(l)=m \\ m_i \geq 1}} f_l(t)\prod_{i=1}^{l} u_{*m_i}(t).$$

By the condition (A3) we have $f_1(t) \neq 0$ for $|t| \ll 1$. By the conditions (A2), (A3) and Implicit function theorem for the first relation in (3.1), we get a holomorphic function $u_{*0}(t)$ in a neighborhood of $t = 0$ with $u_{*0}(0) = 0$. For $m \geq 1$ we can get $u_{*m}(t)$ from the second relation in (3.1) by $f_1(t) \neq 0$. Then we have a formal solution $\tilde{u}(t,\epsilon) = \sum_{m=0}^{\infty} u_{*m}(t)\epsilon^m$

Let us give estimates to the coefficients $u_{*m}(t)$. Set $a(t) = u_{*0}(t)$ and $\tilde{u}_1(t,\epsilon) = \tilde{u}(t,\epsilon) - u_{*0}(t)$. Then $\tilde{u}_1(t,\epsilon)$ is a solution of the following equation:

$$(3.2)\qquad f_1(t)u_1 = \epsilon t^{r+1}\frac{\partial}{\partial t}a(t) + \epsilon t^{r+1}\frac{\partial}{\partial t}u_1 - u_1\int_0^1 \frac{\partial f_2}{\partial u}(t, su_1 + a(t))ds$$

where $f_2(t,u) = \sum_{l=2}^{\infty} f_l(t)u^l$.

By substituting $\tilde{u}_1(t,\epsilon) = \sum_{m=1}^{\infty} u_{*m}(t)\epsilon^m$ into the equation (3.2) and setting $u_{*m}(t)/m! = v_m(t)$ we get

(3.3)

$$f_1(t)v_1(t) = t^{r+1}\frac{\partial}{\partial t}a(t) \quad \text{for } m \geq 2$$

$$m!f_1(t)v_m(t) = (m-1)!t^{r+1}\frac{\partial}{\partial t}v_{m-1}(t) - \sum_{l=2}^{m} f_l(t)\sum_{l'=1}^{l}\frac{l!}{l'!(l-l')!}\sum_{m^*(l')=m}\prod_{i=1}^{l'} m_i!v_{m_i}(t)a^{l-l'}$$

where $m(l'+1) = m_1 + \cdots + m_{l'+1}$.

We can suppose $|f_1(t)| \geq K_1^{-1}$ for $|t| \ll 1$. Then we have

$$|v_1(t)| \leq K_1|t^{r+1}\frac{\partial}{\partial t}a(t)| \quad \text{for } m \geq 2$$

$$(3.4)\qquad m|v_m(t)| \leq K_1\Big\{|t^{r+1}\frac{\partial}{\partial t}v_{m-1}(t)| + \sum_{l=2}^{m}|f_l(t)|\sum_{l'=1}^{l}\frac{l!}{l'!(l-l')!}\sum_{m(l')=m}\prod_{i=1}^{l'} m_i|v_{m_i}(t)a^{l-l'}|\Big\}.$$

Let us take T, Y_0 and Y_1 with $0 < T \ll 1$,

$$Y_0 = \sup_{t \in D_T} |a(t)| \quad \text{and} \quad Y_1 = \max\{ \sup_{t \in D_T} |v_1(t)|, \sup_{t \in D_T} |(\partial/\partial t) v_1(t)|\}$$

and consider the following equation:

$$Y = Y_1 \epsilon + \frac{K_1}{T-r}\Big\{ e\epsilon Y + Y \int_0^1 \frac{\partial F_2}{\partial u}(sY + Y_0) ds \Big\}, \tag{3.5}$$

where $F_2 = \sum_{l=2}^{\infty} \{F_l/(T-r)^l\} u^l$ and $F_l = \sup_{t \in D_T} |f_l(t)|$ for $0 < r < T$. By Implicit function theorem, the equation (3.5) has a holomorphic solution $Y = \sum_{m=1}^{\infty} Y_m \epsilon^m$ and with a form

$$Y_m = \frac{C_m}{(T-r)^{m-1}} \quad (C_1 = Y_1,\ C_m > 0).$$

Then we have;

Proposition 3.2 *For any $m \geq 1$ we have*

$$m \sup_{t \in D_r} |v_m(t)| \leq Y_m \quad \textit{and} \quad \sup_{t \in D_r} \Big| \frac{\partial}{\partial t} v_m(t) \Big| \leq e Y_m. \tag{3.6}$$

In order to show Proposition 3.2 we will use the following lemma:

Lemma 3.3 (Nagumo's lemma) *If a holomorphic function $u(t)$ in D_T satisfies*

$$\sup_{t \in D_r} |u| \leq \frac{C}{(T-r)^p} \quad \textit{for } 0 < r < T$$

then we have

$$\sup_{t \in D_r} \Big| \frac{\partial}{\partial t} u \Big| \leq \frac{Ce(p+1)}{(T-r)^{p+1}} \quad \textit{for } 0 < r < T.$$

For the proof, see Hörmander ([6], lemma 5.1.3).

Proof of Proposition 3.2. For $m = 1$ the estimate (3.6) holds by the rule to take Y_1. Let us show the estimate (3.6) for $m \geq 2$ on induction. By substituting $Y = \sum_{m=1}^{\infty} Y_m \epsilon^m$ into the equation 3.5 we have

$$Y_m = \frac{K_1}{T-r}\Big\{ e Y_{m-1} + \sum_{l=2}^{m} \frac{F_l}{(T-r)^l} \sum_{l'=1}^{l} \frac{l!}{l'!(l-l')!} \sum_{m^*(l')=m} \prod_{i=1}^{l'} Y_{m_i} Y_0^{l-l'} \Big\} \tag{3.7}$$

for $m \geq 2$. By the induction's assumptions, the recurrence formulas (3.4) and (3.7), the following holds

$$m \sup_{t \in D_r} |v_m(t)| \leq (T-r) Y_m \leq Y_m.$$

Therefore we get the first estimate in the estimate 3.6.

Let us show the second estimate. By

$$m \sup_{t \in D_r} |v_m(t)| \leq (T-r) Y_m = \frac{C_m}{(T-r)^{m-2}}$$

and Lemma 3.3 we have

$$\sup_{t\in D_r}\left|\frac{\partial}{\partial t}v_m(t)\right| \le \frac{1}{m}\frac{e(m-1)C_m}{(T-r)^{m-1}} \le eY_m.$$

Hence we can get the second estimate in the estimate 3.6. Q.E.D.

Here we will show that the formal solution $\hat{u}(t,\epsilon)$ is 1-summable. Let us give one definition and one proposition in [1] that are needed in order to show Theorem 3.1.

Definition 3.4 *Let $\gamma > 0$, and G be a bounded s-region. $A^{\epsilon}_{\gamma,0}(G)$ is the set of all function $f(t,\epsilon) \in \mathcal{O}(G)$ such that for any proper subdomains $S^t \times S^\epsilon$ of G*

$$\sup_{\epsilon\in S^t}|f(t,\epsilon)| \le C\exp\left(-c_0|\epsilon|^{-\gamma}\right) \tag{3.8}$$

where c_0 depends on S^ϵ where S^t and S^ϵ are sectors.

Proposition 3.5 ([1], Proposition 18, p.121) *Let $\gamma > 0$, any function u, holomorphic in S, be given. Then $u(\epsilon) \in A_\gamma(S)$ is equivalent to the existence of a normal covering $S_0, \ldots, S_m$, with $S_0 = S$, and function u_j, holomorphic in S_j, $0 \le j \le m$, with $u_0 = u$ and $u_m(\epsilon e^{-2\pi i})$, $\epsilon \in S_m$, so that all u_j are bounded at the origin, and*

$$u_{j-1}(\epsilon) - u_j(\epsilon) \in A_{\gamma,0}(S_{j-1}\cap S_j) \quad \textit{for } 1 \le j \le m.$$

Proof of Theorem 3.1. Let us take a number d_j^* with

$$d_j < \arg\epsilon + rd_j^* < d_{j+1}$$

and set

$$G_j^* = \{d_j - \pi/2 < \arg\epsilon + r\arg t < d_{j+1} + \pi/2\}$$

Then we define the r-sum $u_j(t,\epsilon)$ of $\hat{u}(t,\epsilon)$ with respect to the variable t in a direction d_j^* in Section 2 by

$$u_j(t,\epsilon) = \int_0^{\infty e^{id_j^*}} u_B(\tau,\epsilon)e^{-(\tau/t)^r}d\tau^r. \tag{3.9}$$

Remark 3.6 *By changing a direction d_j^* with $d_j < \arg\epsilon + rd_j^* < d_{j+1}$, $u_j(t,\epsilon)$ is holomorphic on G_j with $|t| < \rho$ and $|\epsilon| < \rho$.*

Proposition 3.7 *$u_j(t,\epsilon) - u_{j-1}(t,\epsilon) \in A^{\epsilon}_{1,0}(G_j^*\cap G_{j-1}^*)$ holds, that is, there exist positive constants K and C such that*

$$|u_j(t,\epsilon) - u_{j-1}(t,\epsilon)| \le Ke^{-c|\epsilon|^{-1}} \quad \textit{for } (t,\epsilon) \in G_j^*\cap G_{j-1}^*. \tag{3.10}$$

Remark 3.8

$$G_j^*\cap G_{j-1}^* = \{d_j - \pi/2 < \arg\epsilon + r\arg t < d_j + \pi/2\} \Rightarrow \Re(f_{1,0}/(r\epsilon t^r)) > 0$$

Proof. We have

$$u_j(t,\epsilon) - u_{j-1}(t,\epsilon) = \int_C u_B(\tau,\epsilon)e^{-(\tau/t)^r}d\tau^r \tag{3.11}$$

where a path C is a circle with centered at $f_{1,0}$ and any positive radius in $\eta = r\epsilon\tau^r$ plane. By a change variable $\eta = r\epsilon\tau^r$ we get

$$u_j(t,\epsilon) - u_{j-1}(t,\epsilon) = \frac{1}{\epsilon r}\int_C u_B\Big(\Big(\frac{\eta}{\epsilon r}\Big)^{1/r},\epsilon\Big)e^{-\eta/(\epsilon r t^r)}d\eta. \tag{3.12}$$

By Residue theorem and Lebesgue's dominated convergence theorem, we have

$$\int_C u_B\Big(\Big(\frac{\eta}{\epsilon r}\Big)^{1/r},\epsilon\Big)e^{-\eta/(t\epsilon t^r)}d\eta = 2\pi\sqrt{-1}f_{0,B}\Big(\Big(\frac{f_{1,0}}{\epsilon r}\Big)^{1/r}\Big)e^{-f_{1,0}/(r\epsilon t^r)}. \tag{3.13}$$

By (3.13) and Remark 3.8 we obtain Proposition 3.7. Q.E.D.

By Proposition 3.7 and Proposition 3.5 we have $u = u_0(t,\epsilon) \in A_1(G_0)$ in ϵ. We can get an opening of $\arg\epsilon$ bigger than π. By Definition 2.1 we have that $u_0(t,\epsilon)$ is 1-sum of $\tilde{u}(t,\epsilon)$. Q.E.D.

4 Summability with respect to the both variables

In this section we will study the summability for the formal solution $\bar{u}(t,\epsilon) = \sum_{k\geq 1}\sum_{m\geq 0} u_{k,m}t^k\epsilon^m$ of the equation (1.1) with respect to the both variables (t,ϵ).

Let us introduce the definition of the summability of the both variables by Balser ([2]).

We define $\mathcal{H}^{(s)}(G_\infty)$ to be the set of all holomorphic function $f(t,\epsilon)$ on G_∞ and have the following property: For every element of $\mathcal{O} := \{(t,\epsilon) \in G_\infty \text{ with } |t|^2 + |\epsilon|^2 = 1\}$ there exist constants $c, K > 0$ such that

$$|f(\zeta_s(x,t,\epsilon))| \leq ce^{Kx} \quad \text{for } x > 0. \tag{4.1}$$

Let $s = (s_1, s_2)$ be $s_1, s_2 > 0$ and set $k = (1/s_1, 1/s_2)$ and

$$\hat{\mathcal{B}}_s\bar{u}(t,\epsilon) = \sum_{k\geq 1, m\geq 0} u_{k,m}\frac{t^{k-r}\epsilon^{m-1}}{\Gamma(s_1k + s_2m)}.$$

Definition 4.1 *We say that $\bar{u}(t,\epsilon) = \sum_{k\geq 1,m\geq 0} u_{k,m}t^k\epsilon^m$ is k-summable with direction $\mathcal{O}$ if the following two statements hold;*

1. *$t^r\epsilon\hat{\mathcal{B}}_s\bar{u}(t,\epsilon)$ converges in a neighborhood of $(t,\epsilon) = (0,0)$.*
2. *$\hat{\mathcal{B}}_s\bar{u}(t,\epsilon)$ can be continued into the region G_∞, and its continuation is in $\mathcal{H}^{(s)}(G_\infty)$.*

Then the following $u(t,\epsilon)$ is k-sum of $\bar{u}(t,\epsilon)$ is direction $\mathcal{O}$:

$$u(t,\epsilon) = t^r\epsilon\int_0^\infty e^{-\eta}v(\eta^{s_1}t, \eta^{s_2}\epsilon)d\eta = \mathcal{L}_s v(t,\epsilon).$$

4.1 Formal solution

Here let us show that equation (1.1) has a formal solution $\bar{u}(t,\epsilon) = \sum_{k=1}^{\infty}\sum_{m=0}^{\infty} u_{k,m}t^k\epsilon^m$ and give estimates for the solution.

Theorem 4.2 *Assume the conditions (A1), (A2) and (A3). Then (1.1) has a formal power solution $\bar{u}(t,\epsilon) = \sum_{k=1}^{\infty}\sum_{m=0}^{\infty} u_{k,m}t^k\epsilon^m$ and the solution satisfies $t^r\epsilon y(t,\epsilon) = t^r\epsilon\hat{\mathcal{B}}_s\bar{u}(t,\epsilon)$ converges in a neighborhood of $(t,\epsilon) = (0,0)$ with $s_1r + s_2 = 1$ and $s_1, s_2 > 0$.*

Proof. By Proposition 2.1 and 3.4 we can prove this theorem.

4.2 k-summability

Here we will show that the formal solution $\bar{u}(t,\epsilon)$ is k-summable in a direction. The proof follows that in Balser and Mozo [3].

Theorem 4.3 *Assume the conditions (A1), (A2) and (A3). Then the formal solution $\bar{u}(t,\epsilon)$ in Theorem 4.2 is k-summable in direction $\mathcal{O}$.*

Let $u_j(t,\epsilon)$ be defined by (3.9) and $s = (s_1, s_2)$ with $s_1, s_2 > 0$ and $s_1r + s_2 = 1$. Set

$$y_1(t,\epsilon) := \frac{t^{-r}\epsilon^{-1}}{2\pi i}\int_{\Gamma} e^{\tau}u_j(\tau^{-s_1}t, \tau^{-s_2}\epsilon)d\tau \tag{4.2}$$

where the path $\Gamma = \Gamma(\delta, R)$ is as follows; form infinity along a ray $\arg\tau = -(\pi+\delta)/2$ to a circle of radius $R > 0$ about the origin, along the circle to the ray $\arg\tau = (\pi+\delta)/2$, and along that ray back to infinity.

Remark 4.4 *We have that the function $y_1(t,\epsilon)$ is holomorphic in $d_j + \delta/2 < r\arg t + \arg\epsilon < d_{j+1} - \delta/2$ since the function $u_j(t,\epsilon)$ is holomorphic on G_j.*

Proposition 4.5 *Assume the conditions (A1), (A2) and (A3). We have $y_1(t,\epsilon) = y(t,\epsilon)$ on $d_j + \delta/2 < r\arg t + \arg\epsilon < d_{j+1} - \delta/2$ with $|t^r\epsilon| < T$ for a sufficiently small $T > 0$ where $y(t,\epsilon)$ is in Theorem 4.2.*

Let $u_i(t,\epsilon)$ be holomorphic on $d_j + \delta/2 < r\arg t + \arg\epsilon < d_{j+1} - \delta/2$ with $|t^r\epsilon| < T$ for a sufficiently small $T > 0$ for $i = 1, 2$. Then we define s-convolution $u_1 *_s u_2$ by

$$u_1(t,\epsilon) *_s u_2(t,\epsilon) := t^r\epsilon\int_0^1 u_1(\tau^{s_1}t, \tau^{s_2}\epsilon)u_2((1-\tau)^{s_1}t, (1-\tau)^{s_2}\epsilon)d\tau.$$

Then we have the following lemma:

Lemma 4.6

$$\mathcal{L}_su_1\mathcal{L}_su_2 = \mathcal{L}_s(u_1 *_s u_2).$$

Let seek out the equation that $y_1(t,\epsilon)$ satisfies.

Lemma 4.7 *Set*

$$u(t,\epsilon) = t^r\epsilon\int_0^{\infty} e^{-\eta}v(\eta^{s_1}t, \eta^{s_2}\epsilon)d\eta.$$

Then we have

$$\begin{aligned}\epsilon t^{r+1}\frac{\partial}{\partial t}u(t,\epsilon) &= \epsilon t^r\int_0^{\infty} e^{-\eta}\frac{1}{s_1}(t\eta^{s_1})^r(\epsilon\eta^{s_2})v(\eta^{s_1}t, \eta^{s_2}\epsilon)d\eta\\ &- \epsilon t^r\epsilon t^r\int_0^{\infty}\frac{s_2}{s_1}e^{-\eta}\frac{\partial}{\partial\epsilon}(\epsilon v(\eta^{s_1}t, \eta^{s_2}\epsilon))d\eta.\end{aligned}$$

Proof. By

$$\frac{\partial}{\partial \eta}(v(t\eta^{s_1}, \epsilon\eta^{s_2})) = s_1 t\eta^{s_1-1}\frac{\partial}{\partial t}v + s_2\epsilon\eta^{s_2-1}\frac{\partial}{\partial \epsilon}v$$

we can prove Lemma 4.7. Q.E.D.

Set $f_{1,s}(\tau) = (\hat{\mathcal{B}}_s(f_1 - f_{1,0}))(\tau)$ and $f_{l,s}(\tau) = (\hat{\mathcal{B}}_s f_l)(\tau)$ for $l \neq 1$.
By Lemma 4.7 $y_1(t, \epsilon)$ satisfies the following equation:

$$\left(\frac{1}{s_1}\epsilon t^r - f_{1,0}\right)y_1(t,\epsilon) = f_{0,s}(t) + \sum_{l\geq 1} f_{l,s}(t) *_s y_1(t,\epsilon)^{l*_s} + \frac{s_2}{s_1}(1) *_s \left(\frac{\partial}{\partial \epsilon}\epsilon y_1(t,\epsilon)\right), \tag{4.3}$$

where $y_1(t,\epsilon)^{l*_s} = \underbrace{y_1(\tau,\epsilon) *_s \cdots *_s y_1(\tau,\epsilon)}_{l}$. Further $y(t,\epsilon)$ also satisfies (4.3).

Proof of Proposition 4.5.

Set $y_0(t,\epsilon) = y_1(t,\epsilon) - y(t,\epsilon)$. Then $y_0(t,\epsilon)$ satisfies

$$\begin{aligned}\left(\frac{1}{s_1}\epsilon t^r - f_{1,0}\right)y_0(t,\epsilon) &= \frac{s_1}{s_2}(1) *_s \left(\frac{\partial}{\partial \epsilon}\epsilon y_0(t,\epsilon)\right) + f_{1,s}(t) *_s y_0(t,\epsilon) \\ &\quad + \sum_{l\geq 2} f_{l,s}(t) *_s y_0(t,\epsilon) *_s \int_0^1 (y_0\tau + y)^{(l-1)*_s}.\end{aligned} \tag{4.4}$$

By the equation (4.4) there exist positive constant

$$|y_0(t,\epsilon)| \leq AB^n |t^r\epsilon|^n \quad \text{for any } n = 0, 1, \cdots$$

on $d_j + \delta/2 < r\arg t + \arg\epsilon < d_{j+1} - \delta/2$ with $|t^r\epsilon| < T$. By letting $k \to \infty$ we get $y_1(t,\epsilon) = y(t,\epsilon)$.
Q.E.D

Proof of Theorem 4.3.
Let us show $|y_1(\eta^{s_1}t, \eta^{s_2}\epsilon)| \leq ce^{K|\eta|}$. By a transform $\tau = z\nu$ with $\nu = rt^r\epsilon$, we have

$$y_1(t,\epsilon) := \frac{1}{2\pi i}\int_{\Gamma'} e^{z\nu} u_j((z\nu)^{-s_1}t, (z\nu)^{-s_2}\epsilon)\frac{dz}{z} \tag{4.5}$$

where the new path Γ' is of the same shape as Γ, with two radial pieces along rays $\arg z = \alpha$, $\arg z = \beta$, and $\beta - \alpha > \pi$. By $((z\nu)^{-s_1}t)^r(z\nu)^{-s_2}\epsilon = (rz)^{-1}$ the integrand (4.5) is well defined for the radius R of the circular section of Γ' is sufficiently large and

$$d_j - \frac{\pi}{2} < -\beta < -\alpha < d_{j+1} + \frac{\pi}{2}.$$

By a transform $\tau \mapsto \eta\tau$ with $\eta > 0$ (4.5) is changed into

$$y_1(\eta^{s_1}t, \eta^{s_2}\epsilon) := \frac{1}{2\pi i}\int_{\Gamma'} e^{\eta\tau} u_j(\tau^{-s_1}t, \tau^{-s_2}\epsilon)\frac{d\tau}{\tau}.$$

By Proposition 3.7, we have

$$|y_1(\eta^{s_1}t, \eta^{s_2}\epsilon)| \leq ce^{R(t,\epsilon)\eta}$$

with $R(t,\epsilon) = \max\{(|t|/\rho)^{1/s_1},\ (|\epsilon|/\rho)^{1/s_2}\}$. Then

$$u(t,\epsilon) = \int_0^\infty e^{-\eta} y_1(\eta^{s_1}t, \eta^{s_2}\epsilon)d\eta \tag{4.6}$$

converges on G_j. Q.E.D.

Remark 4.8 *In fact the formula (4.6) can be defined by*

$$u(t,\epsilon) = \int_0^{\infty e^{i\alpha}} e^{-\eta} y_1(\eta^{s_1} t, \eta^{s_2}\epsilon) d\eta$$

with $d_i < \alpha + r\arg t + \arg\epsilon < d_{j+1}$ *and* $\cos\alpha > R(t,\epsilon)$.

5 Alternative proof of the case $r = 0$

In this section we will give an alternative proof of the result of the summability of formal solutions for the following equation.

$$\epsilon t \frac{\partial}{\partial t} u = f(t,u). \tag{5.1}$$

Let us consider a formal solution $\hat{u}(t,\epsilon) = \sum_{m=0}^{\infty} u_{*,m}(t)\epsilon^m$ of (5.1).

Theorem 5.1 *Assume the conditions (A1), (A2) and (A3). Then the equation (5.1) has a formal solution* $\hat{u}(t,\epsilon) = \sum_{m=0}^{\infty} u_{*,m}(t)\epsilon^m \in \mathcal{O}(D_R)[[\epsilon]]_1$ *Further the formal solution* $\hat{u}(t,\epsilon)$ *is Borel summable in a direction* $d \neq \arg(\partial f/\partial u)(0,0)$.

Proof. We will show that the equation (5.1) has a formal solution $\tilde{u}(t,\epsilon) = \sum_{k=1}^{\infty} u_{k,*}(\epsilon)t^k$ and the solution $\tilde{u}(t,\epsilon)$ is holomorphic in a suitable domain.

Set

$$G = \left\{\epsilon \in \mathbb{C}\setminus\{0\} : \left|\arg \frac{\partial f}{\partial u}(0,0) + \pi - \arg\epsilon\right| < \pi - \delta\right\}.$$

For $\tilde{u}(t,\epsilon)$ we have the following proposition.

Proposition 5.2 *Assume the conditions (A1), (A2) and (A3). Then (5.1) has a formal solution* $\tilde{u}(t,\epsilon)$. *Further the solution* $\tilde{u}(t,\epsilon)$ *is holomorphic on* $D_R \times G$ *for some* $R > 0$. *Then set* $u_G(t,\epsilon) := \tilde{u}(t,\epsilon)$.

Proof. Set $f_l(t) = \sum_k f_{l,k}t^k$. By substituting $\tilde{u}(t,\epsilon) = \sum_{k=1}^{\infty} u_{k,*}(\epsilon)t^k$ into (5.1) we have

$$\begin{aligned} &(\epsilon - f_{1,0})u_{1,*}(\epsilon) = f_{0,1} \\ &(\epsilon k - f_{1,0})u_{k,*}(\epsilon) = f_{0,k} + \sum_{k_0+k_1=k} f_{1,k_0}u_{k_1,*}(\epsilon) + \sum_{k(l)=k} f_{l,k_0}\prod_{i=1}^{l} u_{k_i,*}(\epsilon). \end{aligned} \tag{5.2}$$

Remark 5.3 *We have*

$$|(\epsilon k - f_{1,0})^{-1}| \le C \quad \text{on } G.$$

Then we can show Theorem 5.2. Q.E.D.

It is sufficient to show that the following proposition holds in order to prove Theorem 5.1.

Proposition 5.4 *We have*

$$\left|\left(\frac{\partial}{\partial\epsilon}\right)^n u_G(t,\epsilon)\right| \le AB^n(n!)^2 \quad \text{on } D_R \times G.$$

Proof. By Proposition 5.4 and the argument of G is greater than π we can show that $u_G(t,\epsilon)$ is the sum of the formal solution $\hat{u}(t,\epsilon)$ in Theorem 5.1.

Let us show Proposition 5.4. The proof is similar to that of Balser and Kostov [4]. Set $u_n(t,\epsilon) = \frac{1}{n!}\Big(\frac{\partial}{\partial\epsilon}\Big)^n u_G(t,\epsilon)$. Then we have

$$\begin{aligned}(5.3)\qquad &\epsilon t\frac{\partial}{\partial t}u_n(t,\epsilon) - f_1(t)u_n(t,\epsilon)\\ &= -t\frac{\partial}{\partial t}u_{n-1}(t,\epsilon) + \sum_{n(l)=n} f_l(t)u_{n_1}(t,\epsilon)\times\cdots\times u_{n_l}(t,\epsilon).\end{aligned}$$

For $u_n(t,\epsilon) = \sum_{k=1}^{\infty} u_{n,k}(\epsilon)t^k$ we define the following norm

$$\begin{aligned}(5.4)\qquad &U_{n,k} := \sup_{\epsilon\in G}|u_{n,k}(\epsilon)|\\ &||u_n||_{n,R_1} := \sup_{t\in D_{R_1}} (R_1-|t|)^n\sum_{k=1}^{\infty} U_{n,k}|t|^k\end{aligned}$$

By introducing the norm we get

$$\begin{aligned}(5.5)\qquad &||u_n||_{n,R_1} \le C\Big\{en||u_{n-1}||_{n-1,R_1}\\ &\quad + \sum_{n(l)=n} ||f_l||_{n_0,R_1}||u_{n_1}||_{n_1,R_1}\times\cdots\times||u_{n_l}||_{n_l,R_1}\Big\}.\end{aligned}$$

By (5.5) we can show Proposition 5.4. Q.E.D.

References

[1] W. Balser, Formal power series and linear systems of mermorphic ordinary differential equations, *Universitext, Springer-Verlag, New York, Berlin Heiderburg* (1999).

[2] W. Balser, Summability of power series in several variables, with applications to singular perturbation problems and partial differential equations, *Annales de la Faculté des Sci. de Toulouse* Vol.14 No. 4 (2005), 593–608

[3] W. Balser and J. Mozo, Multisummability of formal solutions of singular perturbation problems *J. Differential Equations* **183**(2002), 525–545.

[4] W. Balser and V. Kostov, Singular perturbation of linear systems with a regular singularity, *J. Dynam. Control. Syst* **8** No. 3 (2002) 313–322.

[5] S. Canalis, J. Mozo and R. Schäfke, Monomial summability and doubly singular differential equations, *J. Differential Equations* **233** (2007), 485–511.

[6] L. Hörmander, *Linear partial differential operators*, Springer, 1963.

[7] D. A. Lutz, M. Miyake and R. Schäfke, On the Borel sommability of divergent solutions of the heat equation, *Nagoya Math. J.* **154** (1999), 1–29.

[8] S. Ōuchi, Multisummability of Formal Solutions of Some Linear Partial Differential Equations, *J. Differential Equations* **185** (2002), 513–549.

[9] H. Yamazawa and M. Yoshino, Borel summability of some semilinear system of partial differential equations, *Opuscula Mathematica* **35** No. 5 (2015), 825–845