数理解析研究所講究録
第2023巻 2017年 77-85

Distributions of the determinants of Gaussian beta ensembles*

Trinh Khanh Duy
Institute of Mathematics for Industry, Kyushu University

Abstract

A central limit theorem for the log-determinants of Gaussian beta ensembles is established. Moreover, the log-determinants of Gaussian unitary ensembles and Gaussian orthogonal ensembles are shown to be equal to a sum of independent random variables, from which the central limit theorem also follows.

1 Tridiagonal matrix models

Gaussian beta ensembles are referred to as ensembles of 'particles' on the real line distributed according to a distribution with the following joint probability density function

$$p_{n,\beta}(\lambda_1, \dots, \lambda_n) = \frac{1}{Z_{n,\beta}} |\Delta(\lambda)|^\beta e^{-\frac{\beta}{4}(\lambda_1^2 + \cdots + \lambda_n^2)}, \quad \beta > 0. \tag{1}$$

Here $\Delta(\lambda) = \prod_{i<j}(\lambda_j - \lambda_i)$ denotes the Vandermonde determinant and $Z_{n,\beta}$ is a normalizing constant. Three specific values of beta, $\beta = 1, 2, 4$, correspond to Gaussian orthogonal ensembles (GOE), Gaussian unitary ensembles (GUE) and Gaussian symplectic ensembles (GSE), respectively. These three ensembles are related to certain classes of invariant random matrices whose eigenvalues are distributed as $p_{n,\beta}, (\beta = 1, 2, 4)$. For instance, GOE is the ensemble of random real symmetric matrices with independent Gaussian components,

$$GOE_n = (\xi_{ij})_{i,j=1}^n, \text{ where } \begin{cases} \xi_{ij} = \xi_{ji} \sim \mathcal{N}(0,1), & i \neq j, \\ \xi_{ii} \sim \mathcal{N}(0,2). \end{cases}$$

These matrices are invariant under orthogonal conjugation, that is, $H(GOE_n)H^t$ has the same distribution with GOE_n for any orthogonal matrix H, hence the name.

By tridiagonalizing a GOE matrix, we get the following tridiagonal form,

$$T_{n,1} = \begin{pmatrix} \mathcal{N}(0,2) & \chi_{n-1} & & \\ \chi_{n-1} & \mathcal{N}(0,2) & \chi_{n-2} & \\ & \ddots & \ddots & \ddots \\ & & \chi_1 & \mathcal{N}(0,2) \end{pmatrix}.$$

This means that there is a random orthogonal matrix H of order $n-1$ such that

$$\begin{pmatrix} 1 & 0 \\ 0 & H \end{pmatrix} (GOE_n) \begin{pmatrix} 1 & 0 \\ 0 & H^t \end{pmatrix} = T_{n,1}.$$

*This work is supported by JSPS Grant-in-Aid for Young Scientists (B) no. 16K17616

Here the matrix $T_{n,1}$ is a random real symmetric tridiagonal matrix, called a random Jacobi matrix, with i.i.d. (independent identically distributed) Gaussian random variables $\mathcal{N}(0,2)$ on the diagonal and independent chi distributed random variables off the diagonal.

The new matrix model for GOE is not invariant under orthogonal conjugation. However,if the parameters of the chi distributions off the diagonal are replaced by $(n-1)\beta, (n-2)\beta, \dots, \beta$, respectively, we get a matrix model for (scaled) Gaussian beta ensembles. To be more precise, the eigenvalues of the following random Jacobi matrix,

$$T_{n,\beta} = \frac{1}{\sqrt{\beta}} \begin{pmatrix} \mathcal{N}(0,2) & \chi_{(n-1)\beta} & & \\ \chi_{(n-1)\beta} & \mathcal{N}(0,2) & \chi_{(n-2)\beta} & \\ & \ddots & \ddots & \ddots \\ & & \chi_\beta & \mathcal{N}(0,2) \end{pmatrix},$$

are distributed as Gaussian beta ensembles. This fact was discovered in [3].

Global spectral properties of Gaussian beta ensembles have been well studied. Among them, Wigner's semicircle law states that the empirical distribution

$$L_{n,\beta} = \frac{1}{n}\sum_{j=1}^{n} \delta_{\lambda_j/\sqrt{n}},$$

converges weakly to the semicircle distribution. Here δ_x denotes the Dirac measure at x. This means that for any bounded continuous function $f\colon \mathbb{R} \to \mathbb{R}$,

$$\langle L_{n,\beta}, f\rangle = \frac{1}{n}\sum_{j=1}^{n} f(\frac{\lambda_j}{\sqrt{n}}) \to \int_{-2}^{2} f(x)\frac{\sqrt{4-x^2}}{2\pi}dx, \text{ almost surely as } n \to \infty,$$

where $\sqrt{4-x^2}/(2\pi)$ is the density of the semicircle distribution. The Gaussian fluctuation around the limit was studied by analysing the joint probability density function [8]. It was shown that for a 'nice' function f, the linear statistics

$$\sum_{j=1}^{n} f(\frac{\lambda_j}{\sqrt{n}}),$$

after centering, converges weakly to a Gaussian distribution whose variance can be written as a quadratic functional of f. For polynomials f, the problem was reconsidered in [4] by exploiting the tridiagonal matrix model. An analogous result for spectral measures of Gaussian beta ensembles was also investigated [5].

The purpose of this paper is to study the limiting behaviour of the determinants of Gaussian beta ensemble matrices. The log-determinants (more precisely, the logarithms of the absolute values of the determinants) are linear statistics of eigenvalues with respect to $f(x) = \log|x|$. Since the test function $\log|x|$ is not a 'nice' one, the Gaussian fluctuation mentioned above is not applicable. However, with different normalizing factors, we will show that the log-determinants converge to a Gaussian limit as the matrix size tends to infinity. Furthermore, for the GUE and GOE cases, the log-determinants can be written as a sum of independent random variables. This interesting result also implies the central limit theorem for log-determinants.

2 The determinants of Gaussian beta ensemble matrices

Let $\{a_n\}_{n\ge1}$ be an i.i.d. sequence of Gaussian random variables $\mathcal{N}(0, \frac{2}{\beta})$ and let $\{b_n\}_{n\ge1}$ be a sequence of independent random variables, which is also independent of $\{a_n\}_{n\ge1}$, with $b_n \sim$

$\chi_{n\beta}/\sqrt{\beta}$. Then the sequence of Gaussian beta ensemble matrices can be coupled in the following way,

$$T_{n,\beta} = \begin{pmatrix} a_n & b_{n-1} & & & \\ b_{n-1} & a_{n-1} & b_{n-2} & & \\ & \ddots & \ddots & \ddots & \\ & & & b_1 & a_1 \end{pmatrix}. \tag{2}$$

Let $D_n = \det T_{n,\beta}$ be the determinant of $T_{n,\beta}$. By expressing the determinant in terms of its minors along the first row, we obtain a recursion relation

$$D_n = a_n D_{n-1} - b_{n-1}^2 D_{n-2}.$$

If we write

$$b_n^2 (= \frac{\chi_{n\beta}^2}{\beta}) = n + \sqrt{n} c_n,$$

then $\{c_n\}_{n\ge1}$ becomes a sequence of independent random variables with mean 0 and variance $\frac{2}{\beta}$. Therefore, the same argument as in [12] yields a central limit theorem for log-determinants of Gaussian beta ensembles.

Theorem 2.1. *For fixed $\beta > 0$, as $n \to \infty$,*

$$\frac{\log|\det(G\beta E_n)| - \frac{1}{2}\log n! + \frac{1}{4}\log n}{\sqrt{\frac{1}{\beta}\log n}} \xrightarrow{d} \mathcal{N}(0,1).$$

Here "$\xrightarrow{d}$" denotes the convergence in distribution of random variables.

We will study distributions of the determinants of GUE and GOE in detail in the next two subsections. The purpose is to show that the distributions of the (absolute values of) determinants coincide with the distribution of a product of independent random variables. Then the above central limit theorem follows as a consequence.

2.1 GUE

The Mellin transform of the determinant of GUE was calculated in [9], (cf. Eq. (2.18)),

$$\mathbb{E}[|\det(GUE_n)|^{s-1}] = \prod_{j=1}^{n} 2^{\frac{s-1}{2}} \frac{\Gamma(\frac{s}{2} + \lfloor \frac{j}{2} \rfloor)}{\Gamma(\frac{1}{2} + \lfloor \frac{j}{2} \rfloor)}, \quad s \in \mathbb{C}, \operatorname{Re} s > 0. \tag{3}$$

Here Γ denotes the gamma function. Each factor on the right hand side of the above expression is exactly the Mellin transform of a chi distribution. Indeed, the chi distribution with k degrees of freedom, χ_k, is a distribution on $(0,\infty)$ with the probability density function

$$f(x;k) = \frac{x^{k-1} e^{-\frac{x^2}{2}}}{2^{\frac{k}{2}-1}\Gamma(\frac{k}{2})}, \quad x > 0,$$

for which

$$\mathbb{E}[|\chi_k|^{s-1}] = \int_0^\infty x^{s-1} \frac{x^{k-1} e^{-\frac{x^2}{2}}}{2^{\frac{k}{2}-1}\Gamma(\frac{k}{2})} = 2^{\frac{s-1}{2}} \frac{\Gamma(\frac{s+k-1}{2})}{\Gamma(\frac{k}{2})},$$

provided that $\operatorname{Re} s > 1-k$. Consequently, the absolute value of the determinant of GUE has the same distribution with a product of n independent chi-distributed random variables, namely,

Theorem 2.2.
$$|\det(GUE_n)| \stackrel{d}{=} \underbrace{\chi_1\chi_3\chi_3\cdots\chi_{2\lfloor\frac{n}{2}\rfloor+1}}_{n}.$$

Let us now prove the formula (3). The argument here is taken from [9]. First of all, the Mellin transform of the determinant of GUE can be written as,

$$\mathbb{E}[|\det(GUE_n)|^{s-1}] = \int\cdots\int|\lambda_1\cdots\lambda_n|^{s-1}\frac{1}{Z_{n,2}}|\Delta(\lambda)|^2e^{-\frac{1}{2}(\lambda_1^2+\cdots+\lambda_n^2)}d\lambda_1\cdots d\lambda_n.$$

Then the desired formula is a direct consequence of the following result.

Lemma 2.3. *For $\alpha>0$, (or for $\alpha\in\mathbb{C}$ with $\operatorname{Re}\alpha>0$), it holds that*

$$\begin{aligned}\frac{1}{(2n)!}\int\cdots\int|x_1\cdots x_{2n}|^{2\alpha-1}|\Delta(x)|^2e^{-(x_1^2+\cdots+x_{2n}^2)}dx_1\cdots dx_{2n}\\ =\Gamma(2)^2\Gamma(3)^2\cdots\Gamma(n)^2\Gamma(\alpha)\Gamma(\alpha+1)^2\cdots\Gamma(\alpha+n-1)^2\Gamma(\alpha+n),\end{aligned} \quad (4)$$

$$\begin{aligned}\frac{1}{(2n+1)!}\int\cdots\int|x_1\cdots x_{2n+1}|^{2\alpha-1}|\Delta(x)|^2e^{-(x_1^2+\cdots+x_{2n+1}^2)}dx_1\cdots dx_{2n+1}\\ =\Gamma(2)^2\Gamma(3)^2\cdots\Gamma(n)^2\Gamma(n+1)\Gamma(\alpha)\Gamma(\alpha+1)^2\cdots\Gamma(\alpha+n-1)^2\Gamma(\alpha+n)^2.\end{aligned} \quad (5)$$

Proof of the formula (3). We give here a proof for the even case. The odd case is similar. From the formula (4), if we use the change of variables $x_i=\lambda_i/\sqrt{2}, i=1,\dots,2n$, then for $s(=2\alpha)\in\mathbb{C}$ with $\operatorname{Re}s>0$,

$$\begin{aligned}I_{2n}(s) &:= \int\cdots\int|\lambda_1\cdots\lambda_{2n}|^{s-1}|\Delta(\lambda)|^2e^{-\frac{1}{2}(\lambda_1^2+\cdots+\lambda_{2n}^2)}d\lambda_1\cdots d\lambda_{2n}\\ &= C_{2n}2^{ns}\Gamma(\frac{s}{2})\Gamma(\frac{s}{2}+1)^2\cdots\Gamma(\frac{s}{2}+n-1)^2\Gamma(\frac{s}{2}+n)\\ &= C_{2n}\prod_{j=1}^{2n}2^{\frac{s}{2}}\Gamma(\frac{s}{2}+\lfloor\frac{j}{2}\rfloor),\end{aligned}$$

where C_{2n} is a constant which does not depend on s. Note that the normalizing constant $Z_{2n,2}$ is nothing but $I_{2n}(1)$. Thus we have

$$\mathbb{E}[|\det(GUE_{2n})|^{s-1}] = \frac{I_{2n}(s)}{I_{2n}(1)} = \prod_{j=1}^{2n}2^{\frac{s-1}{2}}\frac{\Gamma(\frac{s}{2}+\lfloor\frac{j}{2}\rfloor)}{\Gamma(\frac{1}{2}+\lfloor\frac{j}{2}\rfloor)},$$

which completes the proof. □

Proof of Lemma 2.3. We begin with the following formula (called Heine's formula)

$$\frac{1}{n!}\int_{\mathbb{R}}\cdots\int_{\mathbb{R}}|\Delta(x)|^2d\mu(x_1)\cdots d\mu(x_n) = \det(c_{i+j})_{i,j=0}^{n-1}, \quad (6)$$

where μ is a measure on $\mathbb{R}$ and $c_n=\int x^nd\mu(x)$ is the nth moment of μ. The proof of that formula can be found in many books or papers, for example, in [11]. Our task is now to calculate the determinant in the right hand side of (6) for the measure μ_α

$$\mu_\alpha(x) = |x|^{2\alpha-1}e^{-x^2}, \quad x\in\mathbb{R}, (\alpha>0).$$

The moments of μ_α are given by

$$c_k = \begin{cases} \Gamma(n+\alpha), & \text{if } k = 2n, \\ 0, & \text{if } k = 2n+1. \end{cases}$$

Thus by rearranging rows and columns of the matrix $(c_{i+j})_{i,j=0}^{n-1}$, we have

$$\det(c_{i+j})_{i,j=0}^{n-1} = \det(c_{2i+2j})_{i,j=0}^{\lfloor (n-1)/2 \rfloor} \det(c_{2i+2j+2})_{i,j=0}^{\lfloor (n-2)/2 \rfloor}.$$

The proof is complete by showing

$$\det\left(H_n(\alpha) := \left(\Gamma(\alpha+i+j)\right)_{i,j=0}^{n-1}\right) = \prod_{j=0}^{n-1} \Gamma(j+1)\Gamma(\alpha+j). \tag{7}$$

Now let us prove the above formula. By subtracting $(\alpha+i-1)$ times the $(i-1)$th row from the ith row for $i = n-1, n-2, \dots, 1$, the ith row, for $1 \le i \le n-1$, changes to

$$\Gamma(\alpha+i+j) - (\alpha+i-1)\Gamma(\alpha+i+j-1) = j\Gamma(\alpha+i+j-1).$$

Note that the determinant does not change by the transformation and the first column $j = 0$ now has only one non-zero element. Thus we get the recurrence relation

$$\det(H_n(\alpha)) = \Gamma(\alpha)\Gamma(n)\det(H_{n-1}(\alpha+1)),$$

from which the formula (7) is derived. □

Remark 2.4. Let μ be a probability measure on $\mathbb{R}$ with all finite moments. Assume that μ is not supported on finitely many points. Then the set of monomials $\{1, x, x^2, \dots, \}$ is independent in $L^2(\mathbb{R}, \mu)$. By the Gram–Schmidt orthogonalization, we obtain a sequence of orthogonal monic polynomials $\{P_n(x)\}_{n\ge 0}$,

$$P_n(x) = x^n + \text{lower order terms}; P_0 = 1,$$
$$\int_{\mathbb{R}} P_n(x)P_m(x)\mu(dx) = 0, \text{ if } m \ne n.$$

If we let $p_n := P_n/\|P_n\|_{L^2(\mathbb{R},\mu)}$, then $\{p_n\}_{n\ge 0}$ becomes an orthonormal system. Moreover, the sequence $\{p_n\}_{n\ge 0}$ satisfies the famous three-term recurrence relation

$$\begin{cases} xp_n &= b_n p_{n-1} + a_{n+1} p_n + b_{n+1} p_{n+1}, \\ xp_0 &= a_1 p_0 + b_1 p_1, \end{cases}$$

where $\{a_n\}_{n\ge 1}$ are real numbers and $\{b_n\}_{n\ge 1}$ are positive real numbers. In other words, multiplication by x in the orthonormal set $\{p_n\}_{n\ge 0}$ has the transformation matrix

$$J = \begin{pmatrix} a_1 & b_1 & & \\ b_1 & a_2 & b_2 & \\ & \ddots & \ddots & \ddots \end{pmatrix}.$$

J is called the Jacobi matrix of μ and their relation is given by

$$J^k(1,1) = \langle J^k e_1, e_1 \rangle = \int_{\mathbb{R}} x^k \mu(dx),$$

where $e_1 = (1, 0, \dots)^t$. Denote by $H_n = \det(c_{i+j})_{i,j=0}^n$. Then the coefficients $\{b_n\}_{n\ge1}$ can be calculated from $\{H_n\}$,

$$b_n = \frac{H_{n-2}H_n}{H_{n-1}^2}, (H_{-1} := 0). \tag{8}$$

See [11] for more details about Jacobi matrices.

Now for the symmetric probability measure $\mu_\alpha/\Gamma(\alpha)$, it follows from Lemma 2.3 that

$$b_n = \begin{cases} \sqrt{\alpha + k}, & \text{if } n = 2k+1, \\ \sqrt{k}, & \text{if } n = 2k. \end{cases}$$

Note that $a_n = 0$ because the measure is symmetric. Thus, we get the Jacobi matrix of the probability measure $\mu_\alpha/\Gamma(\alpha)$,

$$J_\alpha = \begin{pmatrix} 0 & \sqrt{\alpha} & & & & & \\ \sqrt{\alpha} & 0 & \sqrt{1} & & & & \\ & \sqrt{1} & 0 & \sqrt{\alpha+1} & & & \\ & & \sqrt{\alpha+1} & 0 & \sqrt{2} & & \\ & & & \sqrt{2} & 0 & \sqrt{\alpha+2} & \\ & & & & \ddots & \ddots & \ddots \end{pmatrix}.$$

It is worth mentioning that from the relation (8), the determinant $H_n = \det(c_{i+j})_{i,j=0}^n$ can be expressed in terms of $\{b_n\}_{n\ge1}$ as

$$H_n = b_1^{2n} b_2^{2(n-1)} \cdots b_n^2, \quad (n = 1, 2, \dots),$$

which gives an easy way to remember the statements in Lemma 2.3.

2.2 GOE

The Mellin transform of the determinant of GOE was calculated explicitly in [2] as follows.

(i) n odd. For $s \in \mathbb{C}$ with $\operatorname{Re} s > 0$, (cf. Eq. (19)),

$$\mathbb{E}[|\det(GOE_n)|^{s-1}] = 2^{s-1}\frac{\Gamma(\frac{s}{2})}{\Gamma(\frac{1}{2})} \prod_{j=1}^{(n-1)/2} 2^{s-1}\frac{\Gamma(s+j-\frac{1}{2})}{\Gamma(j+\frac{1}{2})}.$$

(ii) n even. For $s \in \mathbb{C}$ with $\operatorname{Re} s > 0$, (cf. Eq. (26)),

$$\mathbb{E}[|\det(GOE_n)|^{s-1}]$$
$$= 2^{(s-1)/2}\frac{\Gamma(\frac{s}{2})\,\Gamma(\frac{n+1}{2})}{\Gamma(\frac{1}{2})\,\Gamma(\frac{s+n}{2})} F(\frac{s}{2}, \frac{1-s}{2}; \frac{s+n}{2}; \frac{1}{2}) \left(\prod_{j=1}^{n/2} 2^{s-1}\frac{\Gamma(s+j-\frac{1}{2})}{\Gamma(j+\frac{1}{2})} \right).$$

Here $F(\cdot)$ is a hypergeometric function.

The factor $2^{s-1}\frac{\Gamma(s+j-\frac{1}{2})}{\Gamma(j+\frac{1}{2})}$ is the Mellin transform of the chi-squared distribution with $2j+1$ degrees of freedom. Thus, for odd n,

$$|\det(GOE_n)| \overset{d}{=} (\sqrt{2}\chi_1)\underbrace{\chi_3^2\chi_5^2\cdots\chi_n^2}_{(n-1)/2}. \tag{9}$$

For even n, by the following result, the (absolute value of) determinant can be also expressed as a product of independent random variables.

Lemma 2.5. *Let ξ be the beta distribution with parameters $\frac{1}{2}$ and $\frac{n}{2}$. Let*

$$f(\xi) = \xi^{1/2}(2-\xi)^{1/2}.$$

Then

$$\mathbb{E}[f(\xi)^{s-1}] = 2^{(s-1)/2}\frac{\Gamma(\frac{s}{2})\,\Gamma(\frac{n+1}{2})}{\Gamma(\frac{1}{2})\,\Gamma(\frac{s+n}{2})}F(\frac{s}{2},\frac{1-s}{2};\frac{s+n}{2};\frac{1}{2}).$$

Proof. This proof is for more general case. Let ξ be the beta distribution with parameters $\alpha \geq \frac{1}{2}$ and $\beta > 0$. Let z be a real number in $(0,1)$. We will calculate the Mellin transform of

$$g(\xi) = \xi^{1/2}(1-z\xi)^{1/2},$$

by using Formula 3.197 in [7],

$$\int_0^1 x^{\lambda-1}(1-x)^{\mu-1}(1-\beta x)^{-\nu}dx = B(\lambda,\mu)F(\nu,\lambda;\lambda+\mu;\beta),$$

$$(\text{for } \operatorname{Re}\lambda > 0, \operatorname{Re}\mu > 0, |\beta| < 1).$$

Here $B(\lambda,\mu)$ denotes the beta function. The calculation is straight forward as follows. Note that the probability density function of ξ is given by

$$p_\xi(x) = \frac{1}{B(\alpha,\beta)}x^{\alpha-1}(1-x)^{\beta-1}, \quad 0 < x < 1.$$

Thus, for $s \in \mathbb{C}$ with $\operatorname{Re} s > 0$,

$$\begin{aligned}
\mathbb{E}[g(\xi)^{s-1}] &= \int_0^1 x^{(s-1)/2}(1-zx)^{(s-1)/2}\frac{1}{B(\alpha,\beta)}x^{\alpha-1}(1-x)^{\beta-1}dx \\
&= \frac{1}{B(\alpha,\beta)}\int_0^1 x^{(\alpha+(s-1)/2-1)}(1-x)^{\beta-1}(1-zx)^{(s-1)/2}dx \\
&= \frac{B(\alpha+\frac{s}{2}-\frac{1}{2},\beta)}{B(\alpha,\beta)}F(\frac{1}{2}-\frac{s}{2},\frac{s}{2}+\alpha-\frac{1}{2};\frac{s}{2}+\alpha+\beta-\frac{1}{2};z) \\
&= \frac{\Gamma(\alpha+\frac{s}{2}-\frac{1}{2})}{\Gamma(\alpha)}\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha+\beta+\frac{s}{2}-\frac{1}{2})}F(\frac{1}{2}-\frac{s}{2},\frac{s}{2}+\alpha-\frac{1}{2};\frac{s}{2}+\alpha+\beta-\frac{1}{2};z).
\end{aligned}$$

Here we have used the relationship between beta function and gamma function,

$$B(\alpha,\beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}.$$

Note that our problem is a special case in which $\alpha = \frac{1}{2}, \beta = \frac{n}{2}$ and $z = \frac{1}{2}$. The proof is complete. □

Consequently, for even n,

$$|\det(GOE_n)| \overset{d}{=} \xi^{1/2}(2-\xi)^{1/2}\underbrace{\chi_3^2\chi_5^2\cdots\chi_{n+1}^2}_{n/2}, \tag{10}$$

where $\xi \sim \text{Beta}(\frac{1}{2},\frac{n}{2})$. Recently, it was shown in [1] that for even n,

$$|\det(GOE_n)| \overset{d}{=} \chi_1(\chi_1^2+\chi_n^2)^{1/2}\underbrace{\chi_3^2\chi_5^2\cdots\chi_{n-1}^2}_{(n-2)/2}. \tag{11}$$

The two identities are equivalent by observing that

Lemma 2.6. *Let $\xi \sim \mathrm{Beta}(\frac{1}{2}, \frac{n}{2})$ be independent of χ^2_{n+1}. Then*

$$\xi^{1/2}(2-\xi)^{1/2}\chi^2_{n+1} \stackrel{d}{=} \chi_1(\chi_1^2 + 2\chi_n^2)^{1/2}.$$

We need the following lemma.

Lemma 2.7. *Let X and Y be independent random variables having χ_1^2, χ_n^2 distributions.*

(i) *Then $\xi := X/(X+Y) \sim \mathrm{Beta}(\frac{1}{2}, \frac{n}{2})$.*

(ii) *Let*

$$(U, V) = \left(\frac{X(X+2Y)}{(X+Y)^2}, X+Y\right).$$

Then V has χ^2_{n+1} distribution and is independent of U.

Proof. (i) is a well-known relation. (ii) follows by calculating the joint probability density function of U and V. □

Proof of Lemma 2.6. Let $X \sim \chi_1^2$ and $Y \sim \chi_n^2$ be independent random variables. Then

$$\xi := \frac{X}{X+Y} \sim \mathrm{Beta}(\frac{1}{2}, \frac{n}{2}).$$

Thus,

$$\xi^{1/2}(2-\xi)^{1/2} = \frac{X^{1/2}(X+2Y)^{1/2}}{(X+Y)} = U^{1/2}.$$

Moreover, $V = X + Y \sim \chi^2_{n+1}$ is independent of U. Therefore,

$$\xi^{1/2}(2-\xi)^{1/2}\chi^2_{n+1} \stackrel{d}{=} U^{1/2}V = X^{1/2}(X+2Y)^{1/2} = \chi_1(\chi_1^2 + 2\chi_n^2)^{1/2}.$$

The lemma is proved. □

Combining (9) and (11), we have the following.

Theorem 2.8.

$$|\det(GOE_n)| \stackrel{d}{=} \begin{cases} (\sqrt{2}\chi_1)\underbrace{\chi_3^2\chi_5^2\cdots\chi_n^2}_{(n-1)/2}, & \text{if } n \text{ is odd,} \\ \chi_1(\chi_1^2+\chi_n^2)^{1/2}\underbrace{\chi_3^2\chi_5^2\cdots\chi_{n-1}^2}_{(n-2)/2}, & \text{if } n \text{ is even.} \end{cases}$$

Here the right hand side is a product of independent random variables.

Remark 2.9. (i) The above results for GUE and GOE were derived in [6], [1], respectively by investigating the singular values.

(ii) Since the determinants of GUE and GOE can be written as a product of independent random variables, the central limit theorem for log-determinants can be derived as a consequence of the following result.

Theorem (cf. [10, Theorem 1]). *Let $\{X_{k,i}\}_{i=1,\dots,n_k}; k = 1, 2, \dots$ be a triangular array of i.i.d. positive random variables with finite moment of order $p > 2$. Let $S_k := X_1 + \cdots + X_{n_k}$. Suppose that*

(i) $c_n^2 = \sum_{k=1}^n \frac{1}{n_k} \to \infty$ *as* $n \to \infty$;

(ii) $\sum_{k=1}^\infty \frac{1}{n_k^\alpha} < \infty$ *for all* $\alpha > 1$.

Then as $n \to \infty$,

$$\frac{1}{\gamma c_n}\left(\sum_{k=1}^{n}(\log S_k - \log n_k\mu) + \frac{\gamma^2 c_n^2}{2}\right) \xrightarrow{d} \mathcal{N}(0,1).$$

Here $\gamma = \sigma/\mu$, $\mu = \mathbb{E}[X_1] > 0$, *and* $\sigma^2 = \mathrm{Var}[X_1]$.

References

[1] F. Bornemann and M. La Croix: *The singular values of the GOE.* arXiv:1502.05946v3.

[2] R. Delannay and G. Le Caër: *Distribution of the determinant of a random real-symmetric matrix from the Gaussian orthogonal ensemble*, Phys. Rev. E (3) **62** (2000), no. 2, part A, 1526–1536.

[3] I. Dumitriu and A. Edelman: *Matrix models for beta ensembles*, J. Math. Phys. **43** (2002), no. 11, 5830–5847.

[4] I. Dumitriu and A. Edelman: *Global spectrum fluctuations for the β-Hermite and β-Laguerre ensembles via matrix models*, J. Math. Phys. **47** (2006), 063302.

[5] T. K. Duy: *On spectral measures of random Jacobi matrices.* arXiv:1601.01146

[6] A. Edelman and M. La Croix: *The singular values of the GUE (less is more).* arXiv:1410.7065v2.

[7] I. S. Gradshteyn and I. M. Ryzhik: Table of integrals, series, and products, Elsevier/Academic Press, Amsterdam, seventh edition, 2007. Translated from the Russian, Translation edited and with a preface by Alan Jeffrey and Daniel Zwillinger, With one CD-ROM (Windows, Macintosh and UNIX).

[8] K. Johansson: *On fluctuations of eigenvalues of random Hermitian matrices*, Duke Math. J. **91** (1998), no. 1, 151–204.

[9] M. L. Mehta and J. M. Normand: *Probability density of the determinant of a random Hermitian matrix*, J. Phys. A **31** (1998), no. 23, 5377–5391.

[10] G. Rempala and J. Wesolowski: *Asymptotics for products of independent sums with an application to Wishart determinants*, Statist. Probab. Lett. **74** (2005), no. 2, 129–138.

[11] B. Simon: *The classical moment problem as a self-adjoint finite difference operator*, Adv. Math. **137** (1998), no. 1, 82–203.

[12] T. Tao and V. Vu: *A central limit theorem for the determinant of a Wigner matrix*, Adv. Math. **231** (2012), no. 1, 74–101.

Trinh Khanh Duy
Institute of Mathematics for Industry
Kyushu University
Fukuoka 819-0395
JAPAN
E-mail address: trinh@imi.kyushu-u.ac.jp