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Quantum scattering in a periodically pulsed magnetic field

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Abstract

We study the quantum dynamics of a charged particle in the plane in the presence of a periodically pulsed magnetic field perpendicular to the plane. We show that by controlling the cycle when the magnetic field is switched on and off appropriately, the result of the asymptotic completeness of wave operators can be obtained under the assumption that the potential V satisfies the decaying condition $|V(x)|\leq C(1+|x|)^{-\rho}$ for some $\rho>0$. The purpose of this article is to explain the reason why we can admit such a very weak decaying condition.

1 Introduction

In this article, we would like to mention the results of our paper Adachi-Kawamoto [1].

We consider a quantum system of a charged particle moving in the plane $\mathbb{R}^{2}$ in the presence of a periodically pulsed magnetic field $\mathbb{B}(t)$ which is perpendicular to the plane. We suppose that positive constants B and T_{B} are given, and that $\mathbb{B}(t)=(0,0,B(t))\in \mathbb{R}^{3}$ is given by

$$B(t)=\begin{cases}B, & t\in\cup_{n\in \mathbb{Z}}[nT, nT+T_{B})=:I_{B},\\ 0, & t\in\cup_{n\in \mathbb{Z}}[nT+T_{B}, (n+1)T)=:I_{0},\end{cases} \tag{1.1}$$

for some T with $T>T_{B}$. T is the period of $\mathbb{B}(t)$. We put

$$T_{0}:=T-T_{B}>0 \tag{1.2}$$

for simplicity. Then the free Hamiltonian under consideration is defined by

$$H_{0}(t)=\frac{1}{2m}(p-qA(t,x))^{2}$$

acting on $\mathscr{H}=L^{2}(\mathbb{R}^{2})$, where $m>0$, $q\in \mathbb{R}\backslash\{0\}$, $x=(x_{1}, x_{2})$ and $p=(p_{1},p_{2})=(-i\partial_{1}, -i\partial_{2})$ are the mass, the charge, the position, and the momentum of the charged particle, respectively, and

$$A(t,x)=\frac{B(t)}{2}(-x_{2}, x_{1})=\begin{cases}(-Bx_{2}/2, Bx_{1}/2)=:A(x), & t\in I_{B},\\ (0,0), & t\in I_{0},\end{cases}$$

is the vector potential in the symmetric gauge. Then $H_{0}(t)$ is represented as

$$H_{0}(t)=\begin{cases}H_{0}^{B}, & t\in I_{B},\\ H_{0}^{0}, & t\in I_{0},\end{cases} \tag{1.3}$$

where the free Landau Hamiltonian H_{0}^{B} and the free Schrödinger operator H_{0}^{0} are given by

$$H_{0}^{B}=\frac{D^{2}}{2m}, \quad H_{0}^{0}=\frac{p^{2}}{2m}. \tag{1.4}$$

D is the momentum of the charged particle in the presence of the constant magnetic field $\mathbb{B}=(0,0,B)$, which is given by

$$D=(D_{1}, D_{2})=\left(p_{1}+\frac{qB}{2}x_{2}, p_{2}-\frac{qB}{2}x_{1}\right)=p-qA(x). \tag{1.5}$$

Let $U_0(t,s)$ be the propagator generated by $H_0(t)$ (in the sense of Theorem 2 of Huang [2]). By (1.3) and the selfadjointness of H_0^B and H_0^0, $U_0(t,0)$ is represented as

$$U_0(t,0) = \begin{cases} e^{-i(t-nT)H_0^B} U_0(T,0)^n, & t \in [nT, nT+T_B), \\ e^{-i(t-(nT+T_B))H_0^0} e^{-iT_B H_0^B} U_0(T,0)^n, & t \in [nT+T_B, (n+1)T), \end{cases}$$

with $n \in \mathbb{Z}$, where

$$U_0(T,0) = e^{-iT_0 H_0^0} e^{-iT_B H_0^B}$$

is the monodromy operator associated with $H_0(t)$, $U_0(T,0)^0 = \mathrm{Id}$ and $U_0(T,0)^n = (U_0(T,0))^{-n}$ when $-n \in \mathbb{N}$. Put

$$\omega := \frac{qB}{m}, \quad \bar{\omega} := \frac{\omega}{2}, \quad \bar{\bar{\omega}} := \frac{\omega}{4}$$

$|\omega|$ is the Larmor frequency of the charged particle in the presence of the constant magnetic field $\mathbb{B}$. As is well known,

$$\sigma(H_0^B) = \sigma_{pp}(H_0^B) = \left\{ |\omega| \left(n + \frac{1}{2} \right) \Big| n \in \mathbb{N} \cup \{0\} \right\} \tag{1.6}$$

holds, and each eigenvalue of H_0^B is called a Landau level. (1.6) implies that

$$e^{-i2\pi H_0^B/|\omega|} = -\mathrm{Id}$$

holds. Taking account of this fact, we always assume $0 < T_B < 2\pi/|\omega|$, that is ,

$$0 < |\bar{\omega}| T_B < \pi \tag{1.7}$$

for the sake of simplicity.

Now we will state the assumption on the time-independent potential V:

$(V)_\rho$ V is a real-valued continuous function on $\mathbb{R}^2$ satisfying the decaying condition

$$|V(x)| \le C \langle x \rangle^{-\rho} \tag{1.8}$$

with $\rho > 0$, where $\langle x \rangle = \sqrt{1+x^2}$.

Here we introduce the time-periodic Hamiltonian $H(t)$ given by

$$H(t) := H_0(t) + V,$$

and the propagator $U(t,s)$ generated by $H(t)$. The main result of this paper is as follows:

Theorem 1.1. *Suppose* (1.7), *and that* T_0 *satisfies*

$$T_0 > T_{0,\mathrm{Cr}} := \frac{\cos(|\bar{\omega}|T_B)}{|\bar{\omega}| \sin(|\bar{\omega}|T_B)}. \tag{1.9}$$

When $\pi/2 < |\bar{\omega}|T_B < \pi$, *assume that* T_0 *satisfies*

$$T_0 \ne T_{0,\mathrm{res}} := \frac{\sin(|\bar{\bar{\omega}}|T_B)\cos(|\bar{\bar{\omega}}|T_B)}{|\bar{\bar{\omega}}|(2\sin^2(|\bar{\bar{\omega}}|T_B) - 1)} \tag{1.10}$$

additionally. Assume that V *satisfies the condition* $(V)_\rho$ *for some* $\rho > 0$. *Then the wave operators*

$$W^\pm = \text{s-}\lim_{t \to \pm\infty} U(t,0)^* U_0(t,0)$$

exist, and are asymptotically complete:

$$\mathrm{Ran}(W^\pm) = \mathscr{H}_{\mathrm{ac}}(U(T,0)).$$

Here $\mathscr{H}_{\mathrm{ac}}(U(T,0))$ *is the absolutely continuous spectral subspace associated with* $U(T,0)$.

Remark 1.2. The assumption (1.9) is the same as $D > 0$, where D is defined in (2.12). Moreover, the assumption (1.10) is the same as $L_{12} \ne 0$, where L_{12} is defined in (2.11).

2 Classical orbit of the particle

If we suppose that $B = 0$ and $T_B = 0$ in (1.1) and (1.2), respectively, hence a constant magnetic field is always absent, then it is well known that the ρ in (1.8) must be taken $\rho > 1$ in order to prove the existence of wave operators $W_0^{\pm} = \text{s-}\lim_{t\to\pm\infty} e^{it(H_0^0+V)}e^{-itH_0^0}$. On the other hand, if we suppose $B \neq 0$ and $T_0 = 0$ in (1.1) and (1.2), respectively, hence the charged particle is always influenced by a constant magnetic field. Then, even for large ρ, the existence of wave operators can not be proven. However, by switching a constant magnetic fields on and off periodically with suitable period, the existence of wave operators are proven even if $\rho < 1$. The mathematical reason of this phenomenon can be seen by analyzing the classical orbit of the charged particle governed by this system. Hence the purpose of this article is to calculate the classical orbit

$$(x(t)\phi, \phi)_{L^2(\mathbb{R}^2)}, \quad \phi \in L^2(\mathbb{R}^2), \quad x(t) = U_0(t,0)^* x U_0(t,0),$$

concretely. In the following, for simplicity, we calculate $U_0(nT,0)^* x U_0(nT,0)$, $n \in \mathbb{Z}$ only. In Adachi-Kawamoto [1], in addition to the classical orbit $x(t)$, the integral kernel of $U_0(t,0)$ can be obtained concretely. In particular, in [1], $x(t)$ and integral kernel of $U_0(t,0)$ can be obtained for every $t \in \mathbb{R}$ not only for $t = nT$ case.

In this section we will use the following notation

$$\begin{cases} x_0^0(t) &= e^{itH_0^0} x e^{-itH_0^0}, \\ p_0^0(t) &= e^{itH_0^0} p e^{-itH_0^0}, \end{cases} \quad \begin{cases} x_0^B(t) = e^{itH_0^B} x e^{-itH_0^B}, \\ p_0^B(t) = e^{itH_0^B} p e^{-itH_0^B} \end{cases}$$

for simplicity.

2.1 Free motion

At first, we consider the case where $B = 0$ and $T_B = 0$ in (1.1) and (1.2), respectively. Then we notice that, for this case, $U_0(t,0)$ can be rewritten as $e^{-itH_0^0}$. Straightforward calculation shows

$$\frac{d}{dt} p_0^0(t) = e^{itH_0^0} i[H_0^0, p] e^{-itH_0^0} = 0, \tag{2.1}$$

$$\frac{d}{dt} x_0^0(t) = e^{itH_0^0} i[H_0^0, x] e^{-itH_0^0} = e^{itH_0^0} (p/m) e^{-itH_0^0}, \tag{2.2}$$

and we notice that (2.1) yields $p_0^0(t) = p$ and we also notice that $p_0^0(t) = p$ and (2.2) yield

$$x_0^0(t) = tp/m + x. \tag{2.3}$$

Above equation yields that, for the case $B = 0$ and $T_B = 0$,

$$(x(t)\phi, \phi)_{L^2(\mathbb{R}^2)} = (x_0^0(t)\phi, \phi)_{L^2(\mathbb{R}^2)} = t((p/m)\phi, \phi)_{L^2(\mathbb{R}^2)} + (x\phi, \phi)_{L^2(\mathbb{R}^2)}.$$

As is well known that, in the sense of quantum dynamics, $((p/m)\phi, \phi)_{L^2(\mathbb{R}^2)}$ stands for the initial velocity of a quantum particle and $(x\phi, \phi)_{L^2(\mathbb{R}^2)}$ stands for the initial position of the quantum particle. Thus one can understand that the particle behaves in uniformly liner motion with the average velocity $((p/m)\phi, \phi)_{L^2(\mathbb{R}^2)}$.

2.2 Classical orbit associated with Landau level

Next, we consider the case where $B \neq 0$ and $T_0 = 0$ in (1.1) and (1.2), respectively. In this case $U_0(t,0)$ can be rewritten as $e^{-itH_0^B}$, and the classical orbit $(x(t)\phi, \phi)_{L^2(\mathbb{R}^2)}$ can be rewritten as $(x_0^B(t)\phi, \phi)_{L^2(\mathbb{R}^2)}$. Saying from conclusion, the charged particle behaves in circular motion with center $\|x_c\phi\|_{L^2(\mathbb{R}^2)}$, $x_c = (x_{c,1}, x_{c,2})$, $x_{c,1} = D_2/m\omega + x_1$, $x_{c,2} = -D_1/m\omega + x_2$ and radius $r = \|D\phi\|_{L^2(\mathbb{R}^2)}/|m\omega|$. Moreover, the period of circular motion is $2\pi/|\omega|$. By this we notice that the charged particle can not move out to the some compact region because of the influence of the constant magnetic field. This fact and (1.6) are closely related.

We first define

$$D(t) = (D_1(t), D_2(t)) := e^{itH_0^B} D e^{-itH_0^B},$$
$$k(t) = (k_1(t), k_2(t)) := e^{itH_0^B} k e^{-itH_0^B},$$

where k is called pseudomomentum of the charged particle, and it is defined as

$$k = (k_1, k_2) = \left(p_1 - \frac{qB}{2}x_2, p_2 + \frac{qB}{2}x_1\right) = p + qA(x). \tag{2.4}$$

Then, $D(t)$ and $k(t)$ satisfy

$$\frac{d}{dt}D(t) = e^{itH_0^B} i[H_0^B, D] e^{-itH_0^B},$$
$$\frac{d}{dt}k(t) = e^{itH_0^B} i[H_0^B, k] e^{-itH_0^B}.$$

Here, noting the commutation relations

$$i[D_1, D_2] = -qB, \quad i[D_i, k_j] = 0, \ i, j \in \{1, 2\},$$

we have

$$i[H_0^B, D_1] = \omega D_2, \quad i[H_0^B, D_2] = -\omega D_1, \quad i[H_0^B, k_1] = i[H_0^B, k_2] = 0 \tag{2.5}$$

and (2.5) yields, for $D(t) = (D_1(t), D_2(t)) = (e^{itH_0^B} D_1 e^{-itH_0^B}, e^{itH_0^B} D_2 e^{-itH_0^B})$ and $k(t)$,

$$\begin{cases} D_1'(t) - \omega D_2(t) = 0, \\ D_2'(t) + \omega D_1(t) = 0, \end{cases} \qquad k(t) = k, \quad D_j'(t) = \frac{d}{dt}D_j(t).$$

Thus we have

$$\begin{pmatrix} D_1(t) \\ D_2(t) \end{pmatrix} = \begin{pmatrix} \cos(\omega t) & \sin(\omega t) \\ -\sin(\omega t) & \cos(\omega t) \end{pmatrix} \begin{pmatrix} D_1 \\ D_2 \end{pmatrix}, \tag{2.6}$$

$$k(t) = k. \tag{2.7}$$

Noting (2.6) and (2.7) and that x can be rewritten as

$$\begin{pmatrix} qBx_1 \\ qBx_2 \end{pmatrix} = \begin{pmatrix} k_2 - D_2 \\ D_1 - k_1 \end{pmatrix},$$

we can deduce that, for $x_0^B(t) = (x_{0,1}^B(t), x_{0,2}^B(t))$,

$$x_{0,1}^B(t) = \frac{\sin(\omega t)}{m\omega}D_1 + \frac{1 - \cos(\omega t)}{m\omega}D_2 + x_1, \tag{2.8}$$

$$x_{0,2}^B(t) = \frac{\cos(\omega t) - 1}{m\omega}D_1 + \frac{\sin(\omega t)}{m\omega}D_2 + x_2 \tag{2.9}$$

hold. Here we subtract the term

$$x_c = (x_{c,1}, x_{c,2}), \ x_{c,1} = \frac{D_2}{m\omega} + x_1, \ x_{c,2} = -\frac{D_1}{m\omega} + x_2,$$

from $x_0^B(t)$, then we obtain

$$\|x_0^B(t)\phi - x_c\phi\|^2_{L^2(\mathbb{R}^2)^2} = \|D\phi\|^2_{L^2(\mathbb{R}^2)^2} / (m\omega)^2.$$

This equation implies that the constant magnetic field makes the orbit of the particle circular.

2.3 Calculation of $x(T)$ and $x(nT)$

At last, we consider the case where $B \neq 0$ and $T_0 \neq 0$ in (1.1) and (1.2), respectively. By the virtue of (2.3) (2.8) and (2.9), we can calculate $x(T)$ concretely since $x(T)$ is denoted by $x(T) = e^{iT_B H_0^B} e^{iT_0 H_0^0} x e^{-iT_0 H_0^0} e^{-iT_B H_0^B}$. Noting (2.6), (2.7) and that

$$p = (D+k)/2,$$

one has

$$p_0^B(t) = e^{itH_0^B} p e^{-itH_0^B} = \frac{1}{2}\left\{\begin{pmatrix} k_1 \\ k_2 \end{pmatrix} + \begin{pmatrix} \cos(\omega t) & \sin(\omega t) \\ -\sin(\omega t) & \cos(\omega t) \end{pmatrix}\begin{pmatrix} D_1 \\ D_2 \end{pmatrix}\right\}.$$

Here, by noting (2.8), (2.9), (1.5) and (2.4), $x_0^B(T_B)$ and $p_0^B(T_B)$ can be decomposed into

$$\begin{aligned} x_0^B(T_B) = e^{iT_B H_0^B} x e^{-iT_B H_0^B} &= \frac{1}{2}\begin{pmatrix} 1+\cos(\omega T_B) & \sin(\omega T_B) \\ -\sin(\omega T_B) & 1+\cos(\omega T_B) \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\ &+ \frac{1}{qB}\begin{pmatrix} \sin(\omega T_B) & 1-\cos(\omega T_B) \\ -(1-\cos(\omega T_B)) & \sin(\omega T_B) \end{pmatrix}\begin{pmatrix} p_1 \\ p_2 \end{pmatrix}, \end{aligned}$$

and

$$\begin{aligned} p_0^B(T_B) = e^{iT_B H_0^B} p e^{-iT_B H_0^B} &= -\frac{qB}{4}\begin{pmatrix} \sin(\omega T_B) & 1-\cos(\omega T_B) \\ -(1-\cos(\omega T_B)) & \sin(\omega T_B) \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\ &+ \frac{1}{2}\begin{pmatrix} 1+\cos(\omega T_B) & \sin(\omega T_B) \\ -\sin(\omega T_B) & 1+\cos(\omega T_B) \end{pmatrix}\begin{pmatrix} p_1 \\ p_2 \end{pmatrix}. \end{aligned}$$

Hence, by the straightforward calculation, one can also obtain

$$\begin{aligned} x(T) &= e^{iT_B H_0^B} e^{iT_0 H_0^0} x e^{-iT_0 H_0^0} e^{-iT_B H_0^B} = e^{iT_B H_0^B}(x + T_0 p/m) e^{-iT_B H_0^B} = x_0^B(T_B) + (T_0 p_0^B(T_B)/m) \\ &= \left[\frac{1}{2}\begin{pmatrix} 1+\cos(\omega T_B) & \sin(\omega T_B) \\ -\sin(\omega T_B) & 1+\cos(\omega T_B) \end{pmatrix} - \frac{qBT_0}{4m}\begin{pmatrix} \sin(\omega T_B) & 1-\cos(\omega T_B) \\ -(1-\cos(\omega T_B)) & \sin(\omega T_B) \end{pmatrix}\right]\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\ &+ \left[\frac{1}{qB}\begin{pmatrix} \sin(\omega T_B) & 1-\cos(\omega T_B) \\ -(1-\cos(\omega T_B)) & \sin(\omega T_B) \end{pmatrix} + \frac{T_0}{2m}\begin{pmatrix} 1+\cos(\omega T_B) & \sin(\omega T_B) \\ -\sin(\omega T_B) & 1+\cos(\omega T_B) \end{pmatrix}\right]\begin{pmatrix} p_1 \\ p_2 \end{pmatrix} \end{aligned}$$

Consequently, one has

$$\begin{aligned} x(T) &= \frac{1}{2}\begin{pmatrix} 1+\cos(\omega T_B) - \omega T_0 \sin(\omega T_B)/2 & \sin(\omega T_B) - \omega T_0(1-\cos(\omega T_B))/2 \\ -\sin(\omega T_B) + \omega T_0(1-\cos(\omega T_B))/2 & 1+\cos(\omega T_B) - \omega T_0 \sin(\omega T_B)/2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\ &+ \frac{1}{qB}\begin{pmatrix} \sin(\omega T_B) + \omega T_0(1+\cos(\omega T_B))/2 & 1-\cos(\omega T_B) + \omega T_0 \sin(\omega T_B)/2 \\ -(1-\cos(\omega T_B)) - \omega T_0 \sin(\omega T_B)/2 & \sin(\omega T_B) + \omega T_0(1+\cos(\omega T_B))/2 \end{pmatrix}\begin{pmatrix} p_1 \\ p_2 \end{pmatrix}. \end{aligned}$$

Moreover, by noting $e^{iT_0 p^2/(2m)} p e^{-iT_0 p^2/(2m)} = p$, one also has

$$\begin{aligned} p(T) = p_0^B(T_B) &= \frac{1}{2}\begin{pmatrix} 1+\cos(\omega T_B) & \sin(\omega T_B) \\ -\sin(\omega T_B) & 1+\cos(\omega T_B) \end{pmatrix}\begin{pmatrix} p_1 \\ p_2 \end{pmatrix} \\ &- \frac{qB}{4}\begin{pmatrix} \sin(\omega T_B) & 1-\cos(\omega T_B) \\ -(1-\cos(\omega T_B)) & \sin(\omega T_B) \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}. \end{aligned}$$

Then we notice that by taking

$$A_1 = \frac{1}{2}\begin{pmatrix} 1+\cos(\omega T_B) & \sin(\omega T_B) \\ -\sin(\omega T_B) & 1+\cos(\omega T_B) \end{pmatrix}, \qquad A_2 = \frac{1}{2}\begin{pmatrix} -\sin(\omega T_B) & -(1-\cos(\omega T_B)) \\ 1-\cos(\omega T_B) & -\sin(\omega T_B) \end{pmatrix},$$

the vector ${}^t(x(T), p(T))$ can be written as

$$\begin{pmatrix} x(T) \\ p(T) \end{pmatrix} = \begin{pmatrix} A_1 + \omega T_0 A_2/2 & 2/(qB)(-A_2 + \omega T_0 A_1/2) \\ qBA_2/2 & A_1 \end{pmatrix}\begin{pmatrix} x \\ p \end{pmatrix}.$$

By the simple calculation, A_1 and A_2 can be rewritten as

$$A_1 = \begin{pmatrix} \cos(\omega T_B/2)^2 & \sin(\omega T_B/2)\cos(\omega T_B/2) \\ -\sin(\omega T_B/2)\cos(\omega T_B/2) & \cos(\omega T_B)^2 \end{pmatrix}$$
$$= \cos(\omega T_B/2)\begin{pmatrix} \cos(\omega T_B/2) & \sin(\omega T_B/2) \\ -\sin(\omega T_B/2) & \cos(\omega T_B) \end{pmatrix},$$
$$A_2 = \begin{pmatrix} -\sin(\omega T_B/2)\cos(\omega T_B/2) & -\sin(\omega T_B/2)^2 \\ \sin(\omega T_B/2)^2 & -\sin(\omega T_B/2)\cos(\omega T_B/2) \end{pmatrix}$$
$$= -\sin(\omega T_B/2)\begin{pmatrix} \cos(\omega T_B/2) & \sin(\omega T_B/2) \\ -\sin(\omega T_B/2) & \cos(\omega T_B/2) \end{pmatrix}.$$

Here recall the notation $\omega = qB/m$ and $\bar{\omega} = \omega/2$, and put $R(\bar{\omega}T_B)$

$$R(\bar{\omega}T_B) = \begin{pmatrix} \cos(\bar{\omega}T_B) & \sin(\bar{\omega}T_B) \\ -\sin(\bar{\omega}T_B) & \cos(\bar{\omega}T_B) \end{pmatrix}.$$

Then one can obtain

$$\begin{pmatrix} x(T) \\ p(T) \end{pmatrix}$$
$$= \begin{pmatrix} (\cos(\bar{\omega}T_B) - \bar{\omega}T_0\sin(\bar{\omega}T_B))R(\bar{\omega}T_B) & (2/(qB))(\sin(\bar{\omega}T_B) + \bar{\omega}T_0\cos(\bar{\omega}T_B))R(\bar{\omega}T_B) \\ -(qB/2)\sin(\bar{\omega}T_B)R(\bar{\omega}T_B) & \cos(\bar{\omega}T_B)R(\bar{\omega}T_B) \end{pmatrix}\begin{pmatrix} x \\ p \end{pmatrix}$$
$$= \begin{pmatrix} \cos(\bar{\omega}T_B) - \bar{\omega}T_0\sin(\bar{\omega}T_B) & (2/(qB))(\sin(\bar{\omega}T_B) + \bar{\omega}T_0\cos(\bar{\omega}T_B)) \\ -(qB/2)\sin(\bar{\omega}T_B) & \cos(\bar{\omega}T_B) \end{pmatrix}\begin{pmatrix} R(\bar{\omega}T_B)x \\ R(\bar{\omega}T_B)p \end{pmatrix}$$
$$\equiv L\begin{pmatrix} R(\bar{\omega}T_B)x \\ R(\bar{\omega}T_B)p \end{pmatrix}, \tag{2.10}$$

where L can be written as

$$L := \begin{pmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{pmatrix}$$
$$= \begin{pmatrix} \cos(|\bar{\omega}|T_B) - |\bar{\omega}|T_0\sin(|\bar{\omega}|T_B) & (1/(m|\bar{\omega}|))(\sin(|\bar{\omega}|T_B) + |\bar{\omega}|T_0\cos(|\bar{\omega}|T_B)) \\ -(|m\bar{\omega}|)\sin(|\bar{\omega}|T_B) & \cos(|\bar{\omega}|T_B) \end{pmatrix}. \tag{2.11}$$

By (2.10) we have ${}^t(x(nT), p(nT)) = L^n \times {}^t(R(n\bar{\omega}T_B)x, R(n\bar{\omega}T_B)p)$, $n \in \mathbb{N}$. Thus the asymptotic behavior of $x(NT)$ as $n \to \infty$ can be seen by analyzing L^n. Here we calculate L^n. Take $\lambda_\pm$ as the eigenvalues of L. Then, we have

$$\lambda_\pm = \lambda_0 \pm \sqrt{D/4}, \quad \lambda_0 = \mathrm{Tr}(L)/2, \quad D/4 = \lambda_0^2 - 1. \tag{2.12}$$

In the case of $D \neq 0$, it can be calculated that

$$L^n = \frac{1}{\mu_1}\begin{pmatrix} L_{11}\mu_n - \mu_{n-1} & L_{12}\mu_{12} \\ L_{21}\mu_n & L_{22}\mu_n - \mu_{n-1} \end{pmatrix}, \quad \mu_n = \lambda_+^n - \lambda_-^n.$$

Moreover, in the case of $D = 0$, it can be calculated that

$$L^n = \begin{pmatrix} nL_{11}\lambda_0^{n-1} - (n-1)\lambda_0^{n-2} & nL_{12}\lambda_0^{n-1} \\ nL_{21}\lambda_0^{n-1} & nL_{22}\lambda_0^{n-1} - (n-1)\lambda_0^{n-2} \end{pmatrix}.$$

In particular, in the case of $D > 0$, $|\lambda_-| > 1$ holds and which implies $|\mu_n| = \mathcal{O}(e^{\delta n})$ holds for some $\delta > 0$. Here, in addition to $D > 0$, we assume $L_{12} \neq 0$. Then, for all $\phi \in C_0^\infty(\mathbb{R}^2)$, we can prove the following equation

$$\|x^2 U_0(nT, 0)\phi\|_{L^2(\mathbb{R}^2)} = \|((x(nT))^2\phi\|_{L^2(\mathbb{R}^n)} = \mathcal{O}(e^{2\delta n}). \tag{2.13}$$

By using (2.13), for large n, $U_0(nT, 0)$ can be decomposed into

$$U_0(nT, 0)\phi = \chi_{|x| \geq e^{\tilde{\delta}n}} U_0(nT, 0)\phi + \mathcal{O}(e^{-\tilde{\delta}n}), \quad 0 < \tilde{\delta} < \delta$$

holds, where χ is a cut-off function be such that $\chi_{s \geq \tau} = 0$ for $s \leq \tau$ and $\chi_{s \geq \tau} = 1$ for $s \geq \tau$. Thus, for the case of $D > 0$ and $L_{12} \neq 0$, we can prove the existence and completeness of wave operators under the condition $(V)_\rho$ with some $\rho > 0$ since $(1 + |x|)^{-\tau}\chi_{|x| \geq e^{\tilde{\delta}n}} \in l^1(\mathbb{R}_n)$ holds for every $\tau > 0$.

References

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