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Harnack inequality and boundary Harnack principle for subordinate killed Brownian motion

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Abstract

The purpose of this note is to provide a summary of the main results of our recent paper [8], where we establish scale invariant Harnack inequality (HI) and boundary Harnack principle for subordinate killed Brownian motions. For simplicity, we only present the results in the case when the dimension is greater than or equal to 3 and the domain D is bounded.

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1 Main results

Let $W=(W_t,\mathbb{P}_x)$ be a Brownian motion in $\mathbb{R}^d$, $d\geq 3$, and let $S=(S_t)_{t\geq 0}$ be an independent subordinator with Laplace exponent ϕ. The process $X=(X_t,\mathbb{P}_x)$ defined by $X_t=W_{S_t}$, $t\geq 0$, is called a subordinate Brownian motion. It is an isotropic Lévy process with characteristic exponent $\Psi(\xi)=\phi(|\xi|^2)$. If D is an open subset of $\mathbb{R}^d$, we can kill the process X upon exiting D and obtain a process X^D known as a killed subordinate Brownian motion.

By reversing the order of subordination and killing, one obtains a process different from X^D. Let W^D be a killed Brownian motion in a domain $D\subset\mathbb{R}^d$. The process Y^D defined by $Y_t^D=W_{S_t}^D$ is called a subordinate killed Brownian motion. It is a Hunt process with infinitesimal generator $-\phi(-\Delta|_D)$, where $\Delta|_D$ is the Dirichlet Laplacian. This process is very natural and useful. For example, it was used in [5] to obtain two-sided estimates on the eigenvalues of the generator of X^D. Despite its usefulness, the potential theory of Y^D has been studied only sporadically, see [11] for a summary of some of the results. The versions of HI and BHP contained in [11] are very weak in the sense that the results are proved only for nonnegative functions which are harmonic in all of D.

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In the PDE literature, the operator $-(-\Delta|_D)^{\alpha/2}$, $\alpha \in (0,2)$, which is the generator of the subordinate killed Brownian motion via an $\alpha/2$-stable subordinator, also goes under the name of spectral fractional Laplacian, see [2] and the references therein. This operator has been of interest to quite a few people in the PDE circle. For instance, a version of Harnack inequality was also shown in [12].

In this note we will always assume that $d \geq 3$ and D is a bounded domain in $\mathbb{R}^d$. In [8] we discuss the potential theory of Y^D under the following conditions:

(A1) The potential measure U of S has a decreasing density u.

(A2) The Lévy measure of S is infinite and has a decreasing density μ that satisfies

$$\mu(r) \leq c\mu(r+1), \quad r > 1. \tag{1}$$

(A3) There exist constants $\sigma > 0$ and $\delta \in (0,1]$ such that

$$\frac{\phi'(\lambda t)}{\phi'(\lambda)} \leq \sigma t^{-\delta} \text{ for all } t \geq 1 \text{ and } \lambda \geq 1.$$

Remark 1 (1) **(A3)** is a condition on ϕ near ∞.
(2) **(A1)**–**(A3)** hold if ϕ is a complete Bernstein function satisfying the following weak scaling condition near ∞: There exist $a_1, a_2 > 0$ and $\delta_1, \delta_2 \in (0,1)$ satisfying

$$a_1 \lambda^{\delta_1} \phi(t) \leq \phi(\lambda t) \leq a_2 \lambda^{\delta_2} \phi(t), \qquad \lambda \geq 1, t \geq 1. \tag{2}$$

In this case, $\phi(\lambda) \asymp \lambda \phi'(\lambda)$, $\lambda > 0$.

The following are examples satisfying **(A1)**–**(A3)** (and **(A4)** below). Note that examples (6)–(7) do not satisfy (2).

(1) Stable subordinator: $\phi(\lambda) = \lambda^\alpha$, $0 < \alpha < 1$, with $\delta = 1 - \alpha$.

(2) Sum of two stable subordinators: $\phi(\lambda) = \lambda^\beta + \lambda^\alpha$, $0 < \beta < \alpha < 1$, with $\delta = 1 - \alpha$.

(3) Stable with logarithmic correction: $\phi(\lambda) = \lambda^\alpha(\log(1+\lambda))^\beta$, $0 < \alpha < 1$, $0 < \beta < 1 - \alpha$, with $\delta = 1 - \alpha - \epsilon$ for every $\epsilon > 0$.

(4) Stable with logarithmic correction: $\phi(\lambda) = \lambda^\alpha(\log(1+\lambda))^{-\beta}$, $0 < \alpha < 1$, $0 < \beta < \alpha$, with $\delta = 1 - \alpha$.

(5) Relativistic stable subordinator: $\phi(\lambda) = (\lambda + m^{1/\alpha})^\alpha - m$, $0 < \alpha < 1$ and $m > 0$, with $\delta = 1 - \alpha$.

(6) Geometric stable subordinator: $\phi(\lambda) = \log(1 + \lambda^\alpha)$, $0 < \alpha < 1$, with $\delta = 1$.

(7) Gamma subordinator: $\phi(\lambda) = \log(1 + \lambda)$, with $\delta = 1$.

We need some geometric conditions for D. These conditions are related to the heat kernel $p^D(t,x,y)$ of the killed Brownian motion W^D and its tail function $t \mapsto \mathbb{P}_x(t < \tau_D^W)$.

(B1) The function $t \mapsto \mathbb{P}_x(t < \tau_D^W)$ satisfies the doubling property (with a doubling constant independent of $x \in D$), i.e., for every $T > 0$, there exists a constant $c > 0$ such that

$$\mathbb{P}_x(t < \tau_D^W) \le c\mathbb{P}_x(2t < \tau_D^W), \qquad \text{for all } x \in D \text{ and } t \in (0,T].$$

(B2) There exist constants $c \ge 1$ and $M \ge 1$ such that for all $t \le 1$ and $x, y \in D$,

$$\begin{aligned} &c^{-1}\,\mathbb{P}_x(t < \tau_D^W)\,\mathbb{P}_y(t < \tau_D^W)\,t^{-d/2}e^{-\frac{M|x-y|^2}{t}} \\ &\le p^D(t,x,y) \le c\,\mathbb{P}_x(t < \tau_D^W)\,\mathbb{P}_y(t < \tau_D^W)\,t^{-d/2}e^{-\frac{|x-y|^2}{Mt}}. \end{aligned}$$

For any Borel $B \subset D$, let $\tau_B = \inf\{t > 0 : Y_t^D \notin B\}$ be the exit time of Y^D from B.

Definition 2 A real-valued function f defined on D is said to be *harmonic* in an open set $V \subset D$ with respect to Y^D if for every open set $U \subset \overline{U} \subset V$,

$$\mathbb{E}_x\left[|f(Y^D_{\tau_U})|\right] < \infty \quad \text{and} \quad f(x) = \mathbb{E}_x\left[f(Y^D_{\tau_U})\right] \qquad \text{for all } x \in U. \tag{3}$$

The first main results of [8] is the following scale invariant Harnack inequality, which extends the Harnack inequalities in [11, 12].

Theorem 3 (Harnack inequality) *Assume that* **(A1)**–**(A3)** *hold and that* $D \subset \mathbb{R}^d$ *is a domain satisfying* **(B1)**–**(B2)**. *There exists a constant* $C > 0$ *such that for any* $r \in (0,1]$ *and* $B(x_0,r) \subset D$ *and any function* f *which is non-negative in* D *and harmonic in* $B(x_0,r)$ *with respect to* Y^D, *we have*

$$f(x) \le Cf(y), \qquad \text{for all } x, y \in B(x_0, r/2).$$

A very successful technique for proving Harnack inequality for stable-like Markov jump processes was developed in [1]. The proof relied on an estimate of Krylov and Safonov type:

$$\mathbb{P}_x(\tau_{A^c} < \tau_{B(0,r)}) \ge c\frac{|A|}{|B(0,r)|}, \qquad r \in (0,1), x \in B(0,r/2).$$

Although this technique is quite general and can be applied to a much larger class of Markov jump processes, there are situations when it is not applicable even to a rotationally invariant Lévy process. For example, for a geometric stable process it is possible (see [10]) to find a sequence of radii r_n and closed sets $A_n \subset B(0,r_n)$ such that $r_n \to 0$, $\frac{|A_n|}{|B(0,r_n)|} \ge 1/4$ and

$$\mathbb{P}_0(\tau_{A_n^c} < \tau_{B(0,r_n)}) \to 0, \quad \text{as } n \to \infty.$$

Our proof of the Harnack inequality is modeled after the powerful method developed in [6], which uses the following maximum principle: If $(\mathcal{U}_r f)(x_0) < 0$ for some $x_0 \in D$ and $r > 0$, then $f(x_0) > \inf_{x \in D} f(x)$, where

$$(\mathcal{U}_r f)(x) = \frac{\mathbb{E}_x[f(Y^D(\tau_{B(x,r)}))] - f(x)}{\mathbb{E}_x \tau^{Y^D}_{B(x,r)}}.$$

Let $Q \in \partial D$. We say that D is $C^{1,1}$ near Q if there exist a localization radius $R > 0$, a $C^{1,1}$-function $\varphi = \varphi_Q : \mathbb{R}^{d-1} \to \mathbb{R}$ satisfying $\varphi(0) = 0$, $\nabla\varphi(0) = (0, \dots, 0)$, $\|\nabla\varphi\|_\infty \le \Lambda$, $|\nabla\varphi(z) - \nabla\varphi(w)| \le \Lambda|z - w|$, and an orthonormal coordinate system CS_Q with its origin at Q such that

$$B(Q, R) \cap D = \{y = (\widetilde{y}, y_d) \in B(0, R) \text{ in } CS_Q : y_d > \varphi(\widetilde{y})\},$$

where $\widetilde{y} := (y_1, \dots, y_{d-1})$. The pair (R, Λ) will be called the $C^{1,1}$ characteristics of D at Q. D is said to be (uniform) $C^{1,1}$ with characteristics (R, Λ) if it is $C^{1,1}$ with characteristics (R, Λ) near every boundary point $Q \in \partial D$.

Recently, a BHP for general discontinuous Feller processes in metric measure spaces has been proved in [3] and [9] under some comparability assumptions on the jumping kernel. These can not be applied to subordinate killed Brownian motions even in the case of a stable subordinator. The other two main results of [8] are two different type scale invariant boundary Harnack principles with explicit decay rates for non-negative harmonic functions of Y^D. The first boundary Harnack principle deals with a $C^{1,1}$ domain D and non-negative functions which are harmonic near the boundary of D.

For any open set $U \subset \mathbb{R}^d$ and $x \in \mathbb{R}^d$, we use $\delta_U(x)$ to denote the distance between x and the boundary ∂U.

Theorem 4 *Suppose that* **(A1)**–**(A3)** *hold. Let D be a bounded $C^{1,1}$ domain with $C^{1,1}$ characteristics (R, Λ). There exists a constant $C = C(d, \Lambda, R, \phi) > 0$ such that for any $r \in (0, R]$, $Q \in \partial D$, and any non-negative function f in D which is harmonic in $D \cap B(Q, r)$ with respect to Y^D and vanishes continuously on $\partial D \cap B(Q, r)$, we have*

$$\frac{f(x)}{\delta_D(x)} \le C \frac{f(y)}{\delta_D(y)} \qquad \text{for all } x, y \in D \cap B(Q, r/2).$$

It follows from the theorem above that if a non-negative function which is harmonic with respect to Y^D vanishes near the boundary of D, then its rate of decay is proportional to the distance to the boundary. This shows that near the boundary of D, Y^D behaves like the killed Brownian motion W^D.

The second BHP is for a more general domain D and non-negative functions which are harmonic near the boundary of an interior open subset of D. We need one additional assumption.

(A4) If the constant δ in **(A3)** satisfies $0 < \delta \le 1/2$, then we assume that there exist $\sigma_2 > 0$ and $\gamma \in [\delta, 1)$ such that

$$\frac{\phi(\lambda t)}{\phi(\lambda)} \ge \sigma_2 t^{1-\gamma} \quad \text{for all } t \ge 1 \text{ and } \lambda \ge 1.$$

Theorem 5 *Suppose that* ***(A1)-(A4)*** *hold. Let $D \subset \mathbb{R}^d$ be a domain satisfying* ***(B1)*** *and* ***(B2)***. *There exists a constant $b = b(\phi, d) > 0$ such that, for every open set $E \subset D$ and every $Q \in \partial E \cap D$ such that E is $C^{1,1}$ near Q with characteristics $(\delta_D(Q) \wedge 1, \Lambda)$, the* *following holds: There exists a constant* $C = C(\delta_D(Q) \wedge 1, \Lambda, \phi, d) > 0$ *such that for every* $r \le b(\delta_D(Q) \wedge 1)$ *and every non-negative function* f *on* D *which is regular harmonic in* $E \cap B(Q, r)$ *with respect to* Y^D *and vanishes on* $E^c \cap B(Q, r)$*, we have*

$$\frac{f(x)}{\phi(\delta_E(x)^{-2})^{-1/2}} \le C \frac{f(y)}{\phi(\delta_E(y)^{-2})^{-1/2}}, \qquad x, y \in E \cap B(Q, \tilde{c}r),$$

where $\tilde{c} = 2^{-6}(1 + (1 + \Lambda)^2)^{-2}$.

When $\phi(\lambda) = \lambda^{\alpha/2}$, we have $\delta_E(x)^{\alpha/2}$.

The decay rates in the two theorems above are not the same, reflecting different boundary and interior behaviors of Y^D. The two theorems above are new even in the case of a stable subordinator. The method of proof of Theorem 5 is quite different from that of Theorem 4. It relies on a comparison of the Green functions of subprocesses of Y^D and X for small interior subsets of D, and on some already available potential-theoretic results for X obtained in [7].

2 Sketch of the proof of Theorem 4

One of the key ingredients of the proof of Theorem 4 is a Carleson type estimate. Choose a $C^{1,1}$-function $\varphi : \mathbb{R}^{d-1} \to \mathbb{R}$ satisfying $\varphi(\widetilde{0}) = 0$, $\nabla\varphi(\widetilde{0}) = (0, \dots, 0)$, $\|\nabla\varphi\|_\infty \le \Lambda$, $|\nabla\varphi(\widetilde{y}) - \nabla\varphi(\widetilde{w})| \le \Lambda|\widetilde{y} - \widetilde{w}|$, and an orthonormal coordinate system CS_z with its origin at $z \in \partial D$ such that

$$B(z, R) \cap D = \{y = (\widetilde{y}, y_d) \in B(0, R) \text{ in } CS_z : y_d > \varphi(\widetilde{y})\}.$$

Define $\rho_z(x) := x_d - \varphi(\widetilde{x})$, where $(\widetilde{x}, x_d)$ are the coordinates of x in CS_z.

Theorem 6 (Carleson estimate) *There exists a constant* $C = C(R, \Lambda) > 0$ *such that for every* $z \in \partial D$, $0 < r < R/2$, *and every non-negative function* f *in* D *that is harmonic in* $D \cap B(z, r)$ *with respect to* Y^D *and vanishes continuously on* $\partial D \cap B(z, r)$*, we have*

$$f(x) \le C f(x_0) \qquad \text{for } x \in D \cap B(z, r/2),$$

where $x_0 \in D \cap B(z, r)$ *with* $\rho_z(x_0) = r/2$.

Our proof is probabilistic and uses "the box method". Let $\kappa = \kappa(\Lambda) := (1 + (1 + \Lambda)^2)^{-1/2}$. For $x \in B(Q, 2^{-7}\kappa r)$, let $Q_x \in \partial D$ be such that $|x - Q_x| = \delta_D(x)$ and let CS be the coordinate system with origin at Q_x such that

$$B(Q_x, R) \cap D = \{y = (\widetilde{y}, y_d) \in B(0, R) \text{ in } CS : y_d > \varphi(\widetilde{y})\}.$$

For any $a, b > 0$, define "the box"

$$D(a, b) := \left\{y = (\widetilde{y}, y_d) \text{ in } CS : 0 < y_d - \varphi(\widetilde{y}) < 2^{-2}\kappa r a,\ |\widetilde{y}| < 2^{-2}\kappa r b\right\}.$$

Let $V(1)$ be a $C^{1,1}$ subset of D such that $D(1/2, 1/2) \subset V(1) \subset D(1, 1)$. It is easy to see that $V(1) \subset D(1, 1) \subset D \cap B(Q_x, r/4) \subset D \cap B(Q, r/2)$. Thus if f is non-negative in D and harmonic in $D \cap B(Q, r)$, then

$$f(x) = \mathbb{E}_x[f(Y^D(\tau_{V(1)}))].$$

Our key estimates are

$$\mathbb{P}_x\Big(Y^D(\tau_{V(1)}) \in D(3,1) \setminus D(2,1)\Big) \geq c\frac{\delta_D(x)\phi'(r^{-2})}{r^3\phi(r^{-2})},$$

and

$$\mathbb{P}_x\left(Y^D(\tau_{V(1)}) \in D(2,2)\right) \leq c\frac{\delta_D(x)\phi'(r^{-2})}{r^3\phi(r^{-2})}.$$

Using these key estimates, HI, BHP and Carleson estimate, we can get

$$\begin{aligned}
f(x) &\geq \mathbb{E}_x\Big[f\big(Y^D(\tau_{V(1)})\big); Y^D_{\tau_{V(1)}} \in D(3,1) \setminus D(2,1)\Big] \\
&\geq c_1 f(x_0)\mathbb{P}_x\Big(Y^D(\tau_{V(1)}) \in D(3,1) \setminus D(2,1)\Big) \geq c_2 f(x_0)\frac{\delta_D(x)\phi'(r^{-2})}{r^3\phi(r^{-2})},
\end{aligned}$$

$$\begin{aligned}
&\mathbb{E}_x\left[f\left(Y^D(\tau_{V(1)})\right); Y^D(\tau_{V(1)}) \notin D(2,2)\right] \\
&\asymp \frac{\delta_D(x)}{\phi(r^{-2})}\int_{\mathbb{R}^d\setminus D(2,2)} f(y)\frac{1}{|y|}\left(\frac{\delta_D(y)}{|y|}\wedge 1\right)\frac{\mu(|y|^2)}{|y|^{d-2}}dy =: \frac{\delta_D(x)}{\phi(r^{-2})}I(f)
\end{aligned}$$

and

$$\begin{aligned}
&\mathbb{E}_x\left[f\left(Y^D(\tau_{V(1)})\right); Y^D(\tau_{V(1)}) \in D(2,2)\right] \\
&\leq c_3\, f(x_0)\mathbb{P}_x\left(Y^D(\tau_{V(1)}) \in D(2,2)\right) \leq c_4\, f(x_0)\frac{\delta_D(x)\phi'(r^{-2})}{r^3\phi(r^{-2})}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&f(x) \\
=&\mathbb{E}_x\left[f(Y^D(\tau_{V(1)})); Y^D(\tau_{V(1)}) \in D(2,2)\right] + \mathbb{E}_x\left[f(Y^D(\tau_{V(1)})); Y^D(\tau_{V(1)}) \notin D(2,2)\right] \\
\leq& c_5\delta_D(x)\left(\frac{\phi'(r^{-2})}{r^3\phi(r^{-2})}f(x_0) + \frac{1}{\phi(r^{-2})}I(f)\right)
\end{aligned}$$

and

$$\begin{aligned}
f(x) &= \frac{1}{2}f(x) + \frac{1}{2}f(x) \\
&\geq \frac{1}{2}\mathbb{E}_x\Big[f\big(Y^D(\tau_{V(1)})\big); Y^D_{\tau_{V(1)}} \in D(3,1) \setminus D(2,1)\Big] \\
&\quad + \frac{1}{2}\mathbb{E}_x\left[f(Y^D(\tau_{V(1)})); Y^D(\tau_{V(1)}) \notin D(2,2)\right] \\
&\geq c_6\delta_D(x)\left(\frac{\phi'(r^{-2})}{r^3\phi(r^{-2})}f(x_0) + \frac{1}{\phi(r^{-2})}I(f)\right).
\end{aligned}$$

3 Sketch of the proof of Theorem 5

Let J^X and J^{Y^D} be the jumping kernels of X and Y^D respectively. Define

$$F(x,y) := \frac{J^{Y^D}(x,y)}{J^X(x,y)} - 1 = \frac{J^{Y^D}(x,y) - J^X(x,y)}{J^X(x,y)} \in (-1, 0].$$

One can show that there is $b > 4$ such that for any open $U \subset D$ with $\mathrm{diam}(U) < r$ and $\mathrm{dist}(U, \partial D) \geq br$, we have

$$|F(x,y)| < \frac{1}{2}, \quad x, y \in U.$$

Define a non-local multiplicative functional

$$K_t^U := \exp \sum_{0<s\leq t} \log(1 + F(X_{s-}^U, X_s^U)),$$

and the non-local Feynman-Kac semigroup

$$T_t^U f(x) := \mathbb{E}_x[K_t^U f(X_t^U)].$$

The quadratic form $(\mathcal{Q}, \mathcal{D}(\mathcal{E}^{X^U}))$ of $(T_t^U)_{t\geq 0}$ was computed in [4]. We show that $(\mathcal{Q}, \mathcal{D}(\mathcal{E}^{X^U}))$ is equal to $(\mathcal{E}^{Y^{D,U}}, \mathcal{D}(\mathcal{E}^{Y^{D,U}}))$, the Dirichlet form of Y^D killed upon exiting U. Consequently, if V^U denotes the Green function of the semigroup $(T_t^U)_{t\geq 0}$, then $V^U = G_U^{Y^D}$ - the Green function of Y^D killed upon exiting U.

On the other hand,

$$V^U(x,y) = u^U(x,y) G_U^X(x,y), \quad x, y \in U,$$

where

$$u^U(x,y) := \mathbb{E}_x^y[K_{\tau_X^U}^U] \leq 1$$

is the conditional gauge for K_t^U. The main effort is to show that there exists $c > 0$ (independent of U) such that

$$u^U(x,y) \geq c, \quad x, y \in U.$$

With this we have that

$$G_U^{Y^D}(x,y) \asymp G_U^X(x,y), \quad x, y \in U.$$

Now the proof of the BHP uses the corresponding result for X, comparison of Green functions above and properties of J^{Y^D}.

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