数理解析研究所講究録
第2030巻 2017年 185-193

Sharp interface limit for stochastically perturbed mass conserving Allen-Cahn equation

Satoshi Yokoyama
(The University of Tokyo)

1 Introduction and the main results

We consider the solution $u = u^{\varepsilon}(t, x)$ of the following stochastic partial differential equation (1.1) in a bounded domain D in $\mathbb{R}^{n}$ having a smooth boundary ∂D:

$$(1.1)\qquad \begin{cases} \dfrac{\partial u^{\varepsilon}}{\partial t} = \Delta u^{\varepsilon} + \varepsilon^{-2}\left(f(u^{\varepsilon}) - \displaystyle\fint_{D} f(u^{\varepsilon})\right) + \alpha\dot{w}^{\varepsilon}(t), & \text{in } D \times \mathbb{R}_{+}, \\ \dfrac{\partial u^{\varepsilon}}{\partial \nu} = 0, & \text{on } \partial D \times \mathbb{R}_{+}, \\ u^{\varepsilon}(0, \cdot) = g^{\varepsilon}(\cdot), & \text{in } D, \end{cases}$$

where $\varepsilon > 0$ is a small parameter, $\alpha > 0$, ν is the inward normal vector on ∂D, $\mathbb{R}_{+} = [0, \infty)$,

$$\fint_{D} f(u^{\varepsilon}) = \frac{1}{|D|}\int_{D} f(u^{\varepsilon}(t, x))dx,$$

g^{ε} are continuous functions having the property

$$(1.2)\qquad \lim_{\varepsilon\downarrow 0} g^{\varepsilon}(x) = \chi_{\gamma_0},$$

where γ_0 is a smooth hypersurface in D without boundary with finitely many connected components and it has the form $\gamma_0 = \partial D_0$ with a smooth domain D_0 such that $\overline{D}_0 \subset D$ and $\chi_{\gamma}(x) = +1$ or -1 according to the outside or inside of the hypersurface γ. The noise $\dot{w}^{\varepsilon}(t)$ is the derivative of $w^{\varepsilon}(t) \equiv w^{\varepsilon}(t, \omega) \in C^{\infty}(\mathbb{R}_{+})$ in t defined on a certain probability space $(\Omega, \mathcal{F}, P)$ such that $w^{\varepsilon}(t)$ converges to a 1D standard Brownian motion $w(t)$ as $\varepsilon \downarrow 0$ in a suitable sense. We assume that the reaction term $f \in C^{\infty}(\mathbb{R})$ is bistable and satisfies the following three conditions:

(i) $f(\pm 1) = 0$, $f'(\pm 1) < 0$, $\displaystyle\int_{-1}^{1} f(u)du = 0$,

(ii) f has only three zeros ± 1 and one another between ± 1,

(iii) there exists $\bar{c}_1 > 0$ such that $f'(u) \leq \bar{c}_1$ for every $u \in \mathbb{R}$.

The equation (1.1) with $\alpha = 0$ and without the averaged reaction term is called the Allen-Cahn equation. When $\alpha = 0$, the mass of the solution u^{ε} of (1.1) is conserved, namely,

$$(1.3)\qquad \frac{1}{|D|}\int_{D} u^{\varepsilon}(t, x)dx = C,$$

holds for some constant $C \in \mathbb{R}$. For a mass conserving Allen-Cahn equation without noise ((1.1) with $\alpha = 0$), [3], [4] and [7] discuss the existence and uniqueness of solutions and [1] studies its sharp interface limit as $\varepsilon \downarrow 0$. On the other hand, for a stochastic case without the averaged reaction term, the sharp interface limit is discussed by [5], [6], [8] and [9].

Our goal is to show that the solution $u^\varepsilon(t,x)$ of (1.1) converges as $\varepsilon \downarrow 0$ to $\chi_{\gamma_t}(x)$ with certain hypersurface γ_t in D, if this holds for the initial data g^ε with a certain γ_0, and the time evolution of γ_t is governed by

$$V = \kappa - \fint_{\gamma_t} \kappa + \frac{\alpha|D|}{2|\gamma_t|} \circ \dot{w}(t), \quad t \in [0,\sigma], \tag{1.4}$$

up to a certain stopping time $\sigma > 0$ (a.s.), where V is the inward normal velocity of γ_t, κ represents the mean curvature of γ_t multiplied by $n-1$, $\fint_{\gamma_t} \kappa = \frac{1}{|\gamma_t|}\int_{\gamma_t} \kappa d\bar{s}$, $\dot{w}(t)$ is the white noise process and $\circ$ means the Stratonovich stochastic integral. When $\alpha = 0$, the equation (1.4) coincides with the limit of the mass conserving Allen-Cahn equation studied in [1]. On the other hand, in the case where the fluctuation caused by $\alpha w^\varepsilon(t)$ is added, the rigid mass conservation law is destroyed and in place of (1.3), we have the conservation law in a stochastic sense

$$\frac{1}{|D|}\int_D u^\varepsilon(t,x)dx = C + \alpha w^\varepsilon(t), \quad t \in \mathbb{R}_+, \tag{1.5}$$

which implies that the total mass per volume behaves like a Brownian motion multiplied by α as ε tends to 0. For our equation, the comparison argument does not work, so that to study the limit we adopt the asymptotic expansion method, which extends that for deterministic equations used in [1].

Let K be an integer satisfying $K > \max(n+2, 6)$ and $w^\varepsilon = w^\varepsilon(t) \equiv w^\varepsilon(t,\omega)$, $0 < \varepsilon \le 1$, $t \in \mathbb{R}_+$, $\omega \in \Omega$ be a family of $(\mathcal{F}_t)$-adapted stochastic processes defined on a probability space $(\Omega, \mathcal{F}, P)$ equipped with the filtration $(\mathcal{F}_t)_{t\ge 0}$, which satisfy that $w^\varepsilon(0) = 0$, $w^\varepsilon(\cdot) \in C^\infty(\mathbb{R}_+)$ in t a.s. ω and

$$\lim_{\varepsilon\downarrow 0} \|w^\varepsilon - w\|_{C^\theta([0,T])} = 0, \quad \text{a.s.}, \tag{1.6}$$

for every $T > 0$ and some $\theta \in (0, \frac{1}{2})$, where $w(t)$ is an $(\mathcal{F}_t)$-Brownian motion satisfying $w(0) = 0$ and

$$\|u\|_{C^\theta([0,T])} = \sup_{t\in[0,T]} |u(t)| + \sup_{\substack{0\le s,t\le T \\ s\ne t}} \frac{|u(t)-u(s)|}{|t-s|^\theta}. \tag{1.7}$$

Assumption 1.1. *For every $T > 0$, there exists $H_\varepsilon \ge 1, 0 < \varepsilon \le 1$, such that*

$$\sup_{t\in[0,T],\omega\in\Omega} \left|\frac{d^k}{dt^k} w^\varepsilon(t,\omega)\right| \le H_\varepsilon, \quad k = 1, 2, \cdots, n_1(K)+1, \tag{1.8}$$

$$\lim_{\varepsilon\downarrow 0} H_\varepsilon = \infty, \quad \lim_{\varepsilon\downarrow 0} \frac{H_\varepsilon^{2n_1(K)}}{\log\log|\log\varepsilon|} = 0, \tag{1.9}$$

where $n_1(K) \in \mathbb{N}$ is the number determined from K by Proposition 6.1 below.

Assumption 1.2. *There exist stopping times σ^ε and σ such that $V^{\alpha\dot{w}^\varepsilon}$ (resp. V), the solution of* (2.2) *below with $v = \alpha\dot{w}^\varepsilon$ (resp.* (1.4)*), exists uniquely in $[0, \sigma^\varepsilon]$ (resp. $[0, \sigma]$). In addition, $\sigma^\varepsilon > 0$ and $\sigma > 0$ hold* a.s. *Furthermore, for every $T > 0$ and $m \in \mathbb{N}$, the joint variable $(\sigma^\varepsilon, d^\varepsilon(t \wedge \sigma^\varepsilon)) \in \mathbb{R}_+ \times C([0,T], C^m(D))$ converges in this space to $(\sigma, d(t \wedge \sigma))$ as $\varepsilon \downarrow 0$ in* a.s.*-sense, where $d^\varepsilon(t)$ (resp. $d(t)$) is the signed distance determined by the hypersurface $\gamma_t^{\alpha\dot{w}^\varepsilon}$ (resp. γ_t), which is negative inside $\gamma_t^{\alpha\dot{w}^\varepsilon}$ (resp. γ_t).*

We state the main results.

Theorem 1.1. *Let γ_0 be a smooth hypersurface in D without boundary with finitely many connected components and it has the form $\gamma_0 = \partial D_0$ with a smooth domain D_0 such that $\overline{D}_0 \subset D$. Suppose that a local solution $\Gamma = \cup_{0 \le t < \sigma}(\gamma_t \times \{t\})$ of* (1.4) *up to the stopping time $\sigma > 0$ (a.s.) satisfying $\gamma_t \subset D$ for all $t \in [0, \sigma]$ uniquely exists (a.s.). Furthermore, let us assume three Assumptions 1.1, 1.2 and 5.1. Then, one can find a family of continuous functions $\{g^\varepsilon(\cdot)\}_{\varepsilon \in (0,1]}$ satisfying*

$$\lim_{\varepsilon \downarrow 0} g^\varepsilon(x) = \chi_{\gamma_0}, \tag{1.10}$$

such that $(\sigma^\varepsilon, u^\varepsilon(t \wedge \sigma^\varepsilon \wedge \tau, \cdot))$ converges to $(\sigma, \chi_{\gamma_{t \wedge \sigma \wedge \tau}}(\cdot))$ in $\mathbb{R}_+ \times C(\mathbb{R}_+, L^2(D))$ as $\varepsilon \downarrow 0$ in a.s.*-sense, where u^ε is the solution of* (1.1) *with initial value g^ε and $\tau = \tau(\omega) > 0$ is that given Assumption 5.1.*

Assumption 1.2 holds in law sense when the limit curve γ stays convex. Indeed,

Theorem 1.2. *Let D be a two-dimensional bounded domain and γ_0 be a closed convex curve given such that $\gamma \Subset D$. Then, the dynamics* (1.4) *has a unique solution for $0 \le t < \sigma$ for some stopping time $\sigma > 0$ (a.s.).*

2 Signed distance from γ_t and parametrization of γ_t

The expansion of the solution $u^\varepsilon(t, x)$ of (1.1) in ε will be given only in ε appearing in the reaction term and not that in the noise term. To make this clear, we consider the following equation with an external force $v(t)$, which is deterministic (non-random) such that $v \in C^\infty(\mathbb{R}_+)$:

$$\begin{cases} \dfrac{\partial u^\varepsilon}{\partial t} = \Delta u^\varepsilon + \varepsilon^{-2}\left(f(u^\varepsilon) - \displaystyle\fint_D f(u^\varepsilon)\right) + v(t), & \text{in } D \times \mathbb{R}_+, \\ \dfrac{\partial u^\varepsilon}{\partial \nu} = 0, & \text{on } \partial D \times \mathbb{R}_+, \\ u^\varepsilon(\cdot, 0) = g^\varepsilon(\cdot), & \text{in } D. \end{cases} \tag{2.1}$$

Clearly, the solution of (1.1) is the same as that of (2.1) with $v = \alpha\dot{w}^\varepsilon$. In addition, we consider the hypersurface $\{\gamma_t^v\}$ whose evolution is governed by

$$V^v = \kappa - \fint_{\gamma_t^v} \kappa + \frac{|D|}{2|\gamma_t^v|} v(t), \tag{2.2}$$

where V^v is the inward normal velocity of γ_t^v. Suppose that (2.2) has a unique solution for $t \le T^v$ with some $T^v > 0$. Under these settings, we will first expand the solution $u^\varepsilon = u^{\varepsilon,v}$ of (2.1) in ε based on the solution $\gamma_t = \gamma_t^v$ of (2.2). Next, we will estimate each term appearing in the expansion by a suitable norm of v.

Let $d = d^v(t,x)$ be the signed distance of $x \in D$ to the hypersurface γ_t, which is negative inside γ_t. Let $\mathcal{S} \subset \mathbb{R}^n$ be an oriented compact $(n-1)$-dimensional submanifold without boundary and with finitely many connected components being smoothly embedded in $\mathbb{R}^n$. For each $s = (s^l)_{l=1}^n \in \mathcal{S}$, except some singular points, s^n is represented by other coordinates such that $s^n = s^n(s^1, \dots, s^{n-1})$ and thus we can take $s = (s^l)_{l=1}^{n-1}$ as a local coordinate of $\mathcal{S}$. We parametrize $\gamma_t, t \in [0,T]$ as $x = X_0(t,s)$ by $s = (s^l)_{l=1}^{n-1} \in \mathcal{S}$ such that $X_0 \in C^\infty([0,T] \times \mathcal{S}, \mathbb{R}^n)$ and the map $X_0(t,\cdot) : \mathcal{S} \to \gamma_t$ is homeomorphic for every $t \in [0,T]$. In particular, $(\frac{\partial X_0(t,s)}{\partial s^1}, \dots, \frac{\partial X_0(t,s)}{\partial s^{n-1}})$ forms a basis of the tangent space to γ_t at $x = X_0(t,s)$ for each $s \in \mathcal{S}$.

We denote by $\mathbf{n}(t,s)$ the unit outer normal vector on γ_t so that

$$\mathbf{n}(t,s) = \nabla d(t, X_0(t,s)). \tag{2.3}$$

Let $\delta > 0$ be small enough such that the signed distance function $d(t,x)$ from γ_t is smooth in the 3δ-neighborhood of γ_t and the distance between γ_t and ∂D is larger than 3δ for every $t \in [0, T^v]$. A local coordinate $(r,s) \in (-3\delta, 3\delta) \times \mathcal{S}$ of x in a tubular neighborhood of γ_t is defined by

$$x = X_0(t,s) + r\mathbf{n}(t,s) =: X(t,r,s). \tag{2.4}$$

Its inverse function is given by

$$r = d(t,x), \quad s = \mathbf{S}(t,x) = (S^1(t,x), \dots, S^{n-1}(t,x)).$$

Changing coordinates from (t,x) to (t,r,s) for a function $\phi = \phi(t,x)$, we associate another function $\tilde{\phi} = \tilde{\phi}(t,r,s)$ as

$$\tilde{\phi}(t,r,s) = \phi(t, X_0(t,s) + r\mathbf{n}(t,s)).$$

Then, we have

$$\begin{aligned}
\partial_t \phi(t,x) &= (V\partial_r + \partial_t^\Gamma)\tilde{\phi}(t, d(t,x), \mathbf{S}(t,x)), \\
\nabla \phi(t,x) &= (\mathbf{n}(t, \mathbf{S}(t,x))\partial_r + \nabla^\Gamma)\tilde{\phi}(t, d(t,x), \mathbf{S}(t,x)), \\
\Delta \phi(t,x) &= (\partial_r^2 + \Delta d(t,x)\partial_r + \Delta^\Gamma)\tilde{\phi}(t, d(t,x), \mathbf{S}(t,x)),
\end{aligned}$$

where the superscripts Γ mean the derivatives tangential to the hypersurface γ^v seen under the coordinate $s \in \mathcal{S}$:

$$\begin{aligned}
\partial_t^\Gamma \tilde{\phi} &= (\partial_t + \sum_{i=1}^{n-1} S_t^i \partial_{s^i})\tilde{\phi}, \\
\nabla^\Gamma \tilde{\phi} &= \left(\sum_{i=1}^{n-1} \partial_1 S^i \partial_{s^i}, \dots, \sum_{i=1}^{n-1} \partial_n S^i \partial_{s^i} \right) \tilde{\phi},
\end{aligned}$$

$$\Delta^\Gamma \tilde{\phi} = \left(\sum_{i=1}^{n-1} \Delta S^i \partial_{s^i} + \sum_{i,j=1}^{n-1} \nabla S^i \cdot \nabla S^j \partial^2_{s^i s^j} \right) \tilde{\phi},$$

and $V(t,s)$ is the inward normal velocity of the interface γ_t at $X_0(t,s)$, namely,

$$V(t,s) = \partial_t d(t, X_0(t,s)). \tag{2.5}$$

We denote by $\kappa_1, \cdots, \kappa_{n-1}, 0$ the eigenvalues of the Hessian $D_x^2 d(t,x)$ with corresponding normalized eigenvectors $\tau_1, \cdots, \tau_{n-1}$, ∇d. Set

$$\kappa(t,s) := (n-1)\bar{\kappa}_{\gamma_t} = \sum_{i=1}^{n-1} \kappa_i = \Delta d(t, X_0(t,s)), \tag{2.6}$$

where $\bar{\kappa}_{\gamma_t}$ is the mean curvature of γ_t at $x = X_0(t,s)$. Set

$$b(t,s) := -\nabla d \cdot \nabla \Delta d(t,x)|_{x=X_0(t,s)} = \sum_{i=1}^{n-1} \kappa_i^2. \tag{2.7}$$

3 Formal expansion of the solution u^ε

The equation (2.1) is expressed as

$$0 = f(u^\varepsilon(t,x)) + \varepsilon^2(-\partial_t u^\varepsilon(t,x) + \Delta u^\varepsilon(t,x) + v(t)) - \varepsilon \lambda_\varepsilon(t), \tag{3.1}$$

where

$$\lambda_\varepsilon(t) := \varepsilon^{-1} \fint_D f(u^\varepsilon(t,\cdot)). \tag{3.2}$$

We define $h_\varepsilon(t,s)$ by

$$\tilde{\gamma}_t^\varepsilon \equiv \{x \in D \,|\, u^\varepsilon(t,x) = 0\} = \{X(t,r,s) \,|\, r = \varepsilon h_\varepsilon(t,s), s \in \mathcal{S}\}, \tag{3.3}$$

and $\rho = \rho^\varepsilon(t,x)$ by

$$\rho^\varepsilon(t,x) = \frac{d(t,x) - \varepsilon h_\varepsilon(t, \mathbf{S}(t,x))}{\varepsilon}.$$

We denote by $\tilde{u}^\varepsilon = \tilde{u}^\varepsilon(t,\rho,s)$ the function $u^\varepsilon = u^\varepsilon(t,x)$ viewed under the coordinate (t,ρ,s) related by $x = X_0(t,s) + \varepsilon(\rho + h_\varepsilon(t,s))\mathbf{n}(t,s)$. In the following, we will write $\tilde{u}^\varepsilon$ for u. Then we have

$$\begin{aligned} 0 = &\left[\partial_\rho^2 u + f(u)\right] + \varepsilon\left[(-V(t,s) + \Delta d)\partial_\rho u - \lambda_\varepsilon(t)\right] \\ &+ \varepsilon^2\left[(\Delta^\Gamma u - \partial_t^\Gamma u) + (\partial_t^\Gamma h_\varepsilon - \Delta^\Gamma h_\varepsilon)\partial_\rho u\right] \\ &+ \varepsilon^2\left[|\nabla^\Gamma h_\varepsilon|^2 \partial_\rho^2 u - 2\nabla^\Gamma h_\varepsilon \cdot \nabla^\Gamma \partial_\rho u\right] + \varepsilon^2 v(t). \end{aligned} \tag{3.4}$$

Suppose that u and h_ε have the *inner* asymptotic expansions:

$$\begin{aligned} &u(t,\rho,s) = m(\rho) + \varepsilon u_0(t,\rho,s) + \varepsilon^2 u_1(t,\rho,s) + \varepsilon^3 u_2(t,\rho,s) + \cdots, \\ &\varepsilon h_\varepsilon(t,s) = \varepsilon h_1(t,s) + \varepsilon^2 h_2(t,s) + \varepsilon^3 h_3(t,s) + \cdots, \quad (t,\rho,s) \in [0,T^v] \times \mathbb{R} \times \mathcal{S}, \end{aligned} \tag{3.5}$$

where m is the standing wave solution determined by $m'' + f(m) = 0$ on $\mathbb{R}$, $m(\pm\infty) = \pm 1$, $m(0) = 0$. On the other hand, assume that λ_ε and $u^\pm$ have the *outer* asymptotic expansions:

$$
\begin{aligned}
&\lambda_\varepsilon(t) = \lambda_0(t) + \varepsilon\lambda_1(t) + \varepsilon^2\lambda_2(t) + \varepsilon^3\lambda_3(t) + \cdots, \\
&u^\pm(t) = \pm 1 + \varepsilon u_0^\pm(t) + \varepsilon^2 u_1^\pm(t) + \varepsilon^3 u_2^\pm(t) + \cdots, \quad t \in [0, T^v].
\end{aligned} \tag{3.6}
$$

4 Inductive scheme to determine coefficients

Set

$$\nu_k = (u_k, h_k, \lambda_k, u_k^\pm), \quad k = 0, 1, \ldots, K. \tag{4.1}$$

Then, ν_k will be inductively determined in such a manner that all k-th order terms (those of order $O(\varepsilon^k)$) vanish when we substitute these expansions in (3.4), Indeed, we find

$$u_0(t, \rho, s) = -\lambda_0(t)\theta_1(\rho), \quad u_0^\pm(t) := \frac{\lambda_0(t)}{f'(\pm 1)}, \tag{4.2}$$

where $\theta_1 = \theta_1(\rho)$ is a smooth function. Furthermore, u_k and $u_k^\pm$ are determined by a function $A^{k-1} = A^{k-1}(\lambda_0, u_i, h_i, 0 \le i \le k-1)$, h_k and λ_k, (see [2] for details). Set

$$u_{k,\varepsilon}^{\mathrm{in}}(t, x) = m(\rho) + \sum_{i=0}^{k} \varepsilon^{i+1} u_i(t, \rho, \mathbf{S}(t, x)), \tag{4.3}$$

$$u_{k,\varepsilon,\pm}^{\mathrm{out}}(t) = \pm 1 + \sum_{i=0}^{k} \varepsilon^{i+1} u_i^\pm(t), \tag{4.4}$$

and define $u_k^\varepsilon(t, x)$ by connecting (4.3) and (4.4) smoothly (see [2]).

5 Bounds for derivatives of X_0, $\partial^{\mathbf{m}} X_0$ and $\mathbf{S}$

Definition 5.1. *For $k \in \mathbb{Z}_+$, $T > 0$ and $g \in C^\infty(\mathbb{R}_+)$, we define $|g|_k \equiv |g|_{k,T}$ as*

$$|g|_{k,T} = \sum_{i=0}^{k} \sup_{t \in [0,T]} \left| \frac{d^i g}{dt^i}(t) \right|. \tag{5.1}$$

We take a class $\mathcal{V}$ of functions $v \in C^\infty(\mathbb{R}_+)$ and $T > 0$ satisfying that

$$C_{\mathcal{V},T} = \max(C_{\mathcal{V},T}^{(1)}, C_{\mathcal{V},T}^{(2)}) < \infty, \tag{5.2}$$

where

$$
\begin{aligned}
C_{\mathcal{V},T}^{(1)} := \sup_{\substack{v \in \mathcal{V}, s \in \mathcal{S}, \\ r \in (-3\delta, 3\delta)}} \Big\{ &|\partial^{\mathbf{m}} d(\cdot, X_0(\cdot, s))|_{0,T}, |\partial^{\mathbf{m}} S^l(\cdot, X(\cdot, r, s))|_{0,T}, \\
&|\partial^{\mathbf{m}} X_0(\cdot, s)|_{0,T}; 1 \le l \le n-1, |\mathbf{m}| \le M \Big\} < \infty,
\end{aligned} \tag{5.3}
$$

and

$$C^{(2)}_{\mathcal{V},T} := \sup_{v\in\mathcal{V},s\in\mathcal{S},t\in[0,T]} \Big\{ \big(\alpha_-(t,s)\big)^{-1}, \, |\gamma_t^v|^{-1} \Big\} < \infty. \tag{5.4}$$

Here, in $C^{(1)}_{\mathcal{V},T}$, $M = M(K) \in \mathbb{N}$ denotes the maximal number of the degrees of spatial derivatives taken over the terms appearing in A^K, h_k and λ_k, $\partial^{\mathbf{m}} = \partial^{m_1}_{x^1} \cdots \partial^{m_n}_{x^n}$, $|\mathbf{m}| = \sum_{i=1}^n m_i$ for $m = (m_1, \cdots, m_n) \in (\mathbb{Z}_+)^n$ and $\delta > 0$ is chosen as in Section 2. Moreover, in $C^{(2)}_{\mathcal{V},T}$,

$$\alpha_-(t,s) \equiv \alpha_-^v(t,s) := \inf_{\xi\in\mathbb{R}^{n-1}:|\xi|=1} \big(\alpha(t,s)\xi, \xi\big),$$

where $\alpha(t,s) = (\alpha_{ij}(t,s))_{1\le i,j\le n-1}$ is the matrix defined by $\alpha_{ij} = \nabla S^i \cdot \nabla S^j$, and $(\cdot,\cdot)$ and $|\cdot|$ denote the inner product and the norm of $\mathbb{R}^{n-1}$, respectively.

Assumption 5.1. *There exist some $N = N(K) \in \mathbb{N}$, $T = T(\mathcal{V}) > 0$ and $C_1 = C_1(C_{\mathcal{V},T}, K, T) > 0$ such that*

$$\sup_{1\le i\le n} \sup_{s\in\mathcal{S}} |\partial_t^k \partial^{\mathbf{m}} X_0^i(\cdot,s)|_{0,T} \le C_1(1+|v|_{N,T})^N, \tag{5.5}$$

$$\sup_{1\le i\le n-1} \sup_{r\in(-3\delta,3\delta),\, s\in\mathcal{S}} |\partial_t^k \partial^{\mathbf{m}} S^i(\cdot, X(\cdot,r,s))|_{0,T} \le C_1(1+|v|_{N,T})^N, \tag{5.6}$$

for $k = 0, 1, \cdots, K$, $|\mathbf{m}| \le M$ and $v \in \mathcal{V}$.

Under the choice $\mathcal{V} \equiv \mathcal{V}(\omega) = \{\alpha\dot{w}^\varepsilon; 0 < \varepsilon \le \varepsilon_0^*\}$ for sufficiently small $\varepsilon_0^* > 0$, Assumption 5.1 determines $\tau(\omega) := T(\mathcal{V}(\omega))$, up to which two bounds (5.5) and (5.6) hold. Indeed, Assumption 5.1 is true for some $T = \tau(\omega) > 0$ under a two-dimensional setting as long as the limit curve γ_t is convex (see [2]).

6 Estimates for u_k and $u_k^\pm$

Under these settings, one can obtain estimates for u_k and $u_k^\pm$.

Proposition 6.1. *For every $k = 0, 1, \ldots, K$,*

$$\sup_{(t,\rho,s)\in[0,T]\times\mathbb{R}\times\mathcal{S}} \big\{ |u_k(t,\rho,s)|, |u_k^\pm(t)| \big\} \le (C_2K_2)^{C_2K_2}, \tag{6.1}$$

holds for some $C_2 = C_2(C_{\mathcal{V},T}, T) > 0$ and

$$K_2 \equiv K_2(v) = e^{n_1(K)(1+|v|_{n_1(K)})^{n_1(K)}(T\vee 1)},$$

with some $n_1 = n_1(K) \in \mathbb{N}$.

Corollary 6.2. *We assume Assumptions 1.1, 1.2 and 5.1, and define $G_\varepsilon \ge e^e, 0 < \varepsilon \le 1$, from H_ε appearing in Assumption 1.1 by the relation*

$$\log\log G_\varepsilon = H_\varepsilon^{2n_1(K)}, \tag{6.2}$$

where $n_1(K) \in \mathbb{N}$ is the number determined by Proposition 6.1. Then, we have

$$\lim_{\varepsilon\downarrow 0} G_\varepsilon = \infty, \quad \lim_{\varepsilon\downarrow 0} \frac{G_\varepsilon}{|\log\varepsilon|} = 0. \tag{6.3}$$

Furthermore, u_k and $u_k^\pm$ determined from $v(t) = \alpha\dot{w}^\varepsilon(t)$ as above satisfy

$$\sup_{(t,\rho,s)\in[0,T(\omega)]\times\mathbb{R}\times\mathcal{S}} \{|u_k(t,\rho,s)|, |u_k^\pm(t)|\} \le G_\varepsilon, \quad 0 \le k \le K, \tag{6.4}$$

for every sufficiently small $\varepsilon > 0$ and every $\omega \in \Omega$, where $T = T(\omega) := \inf_{0<\varepsilon\le\varepsilon_0^*} \sigma^\varepsilon > 0$.

7 Estimate for the difference between v_K^ε and u^ε

Set

$$\psi(\varepsilon) = \big(\log\log\log|\log\varepsilon|\big)^{\tilde\beta}, \tag{7.1}$$

with $\tilde\beta > 0$ and let $W_\varepsilon(t), \varepsilon > 0$ be the stopped process of w, that is, $W_\varepsilon(t) = w(t \wedge \tau(\varepsilon))$, where $\tau(\varepsilon)$ is the first exit time of $w(t)$ from the interval $I_\varepsilon = (-\psi(\varepsilon), \psi(\varepsilon))$. We define $w^\varepsilon(t)$ by

$$w^\varepsilon(t) = \int_0^\infty \eta_{\psi(\varepsilon)}(t-s)W_\varepsilon(s)ds, \quad \eta_{\psi(\varepsilon)}(s) = \psi(\varepsilon)\eta\big(\psi(\varepsilon)s\big), \tag{7.2}$$

and η is a non-negative C^∞-function on $\mathbb{R}$, whose support is contained in $(0,1)$, satisfying $\int_{\mathbb{R}}\eta(u)du = 1$. We can show that the diverging speed of the noise is sufficiently slow in such a way that Assumption 1.1 holds.

Lemma 7.1. *For $w^\varepsilon(t)$ defined by (7.2), we have*

$$|\dot{w}^\varepsilon|_{k,T} \le k|\eta|_{k+2}\psi(\varepsilon)^{k+2}, \quad k \in \mathbb{Z}_+. \tag{7.3}$$

Furthermore, Assumption 1.1 holds for this $w^\varepsilon(t)$ by taking $H_\varepsilon = n_1(K)|\eta|_{n_1(K)+2}\psi(\varepsilon)^{n_1(K)+2}$.

Set

$$\Phi_k^\varepsilon(t) = \int_D \Big(\partial_t u_k^\varepsilon(t,x) - v(t)\Big)dx, \tag{7.4}$$

and let us set

$$v_k^\varepsilon(t,x) = u_k^\varepsilon(t,x) - \frac{1}{|D|}\int_0^t \Phi_k^\varepsilon(s)ds, \quad 0 \le k \le K. \tag{7.5}$$

In order to prove Theorem 1.1, we need to obtain the error estimate between v_K^ε and u^ε. We take initial data $g^\varepsilon = g^\varepsilon(x)$ of (1.1) or (2.1) satisfying

$$g^\varepsilon(x) = u_K^\varepsilon(0,x) + \phi^\varepsilon(x), \tag{7.6}$$

$$\|\phi^\varepsilon\|_{L^2(D)} \le C_3^{-\frac{1}{p}}\varepsilon^K, \tag{7.7}$$

$$\int_D \phi^\varepsilon(x)dx = 0, \tag{7.8}$$

for sufficiently small $\varepsilon > 0$, where $C_3 > 0$ is a certain constant independent of ε. Recall that $K > \max(n+2, 6)$ is assumed. Then we have

Lemma 7.2 ([1]). *For a bounded domain $D \subset \mathbb{R}^n$, let $p = \min\{\frac{4}{n}, 1\}$. Then there exists $C_n(D) > 0$ such that for every $R \in H^1(D)$ with $\int_D R(x)dx = 0$,*

$$\|R\|_{L^{2+p}(D)}^{2+p} \leq C_n(D)\|R\|_{L^2(D)}^p \|\nabla R\|_{L^2(D)}^2, \tag{7.9}$$

holds.

Theorem 7.3. *Assume* (7.6)–(7.8) *for the initial data g^ε. Then, for sufficiently small $\varepsilon > 0$*

$$\sup_{t\in[0,T]} \|v_K^\varepsilon(t) - u^\varepsilon(t)\|_{L^2(D)} \leq C_4 \varepsilon^{K-1} |\log \varepsilon|, \tag{7.10}$$

holds for some constant $C_4 > 0$ independent of ε.

References

[1] X. CHEN, D. HILHORST, E. LOGAK, *Mass conserving Allen-Cahn equation and volume preserving mean curvature flow*, Interfaces Free Bound., **12** (2010), 527–549.

[2] T. FUNAKI, S. YOKOYAMA, *Sharp interface limit for stochastically perturbed mass conserving Allen-Cahn equation*, arXiv:1610.01263.

[3] C.M. ELLIOTT AND H. GARCKE, *Existence results for diffusive surface motion laws*, Adv. Math. Sci. Appl., **7** (1997), 467–490.

[4] J. ESCHER AND G. SIMONETT, *The volume preserving mean curvature flow near spheres*, Proc. Amer. Math. Soc., **126** (1998), 2789–2796.

[5] T. FUNAKI, *The scaling limit for a stochastic PDE and the separation of phases*, Probab. Theory Relat. Fields, **102** (1995), 221–288.

[6] T. FUNAKI, *Singular limit for stochastic reaction-diffusion equation and generation of random interfaces*, Acta Math. Sin. (Engl. Ser.), **15** (1999), 407–438.

[7] G. HUISKEN, *The volume preserving mean curvature flow*, J. Reine Angew. Math., **382** (1987), 35–48.

[8] P.L. LIONS AND P.E. SOUGANIDIS, *Fully nonlinear stochastic partial differential equations: non-smooth equations and applications*, C. R. Acad. Sci. Paris Ser. I Math., **327** (1998), 735–741.

[9] H. WEBER, *On the short time asymptotic of the stochastic Allen-Cahn equation*, Ann. Inst. Henri Poincare Probab. Stat., **46** (2010), 965–975.