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Stochastic complex Ginzburg-Landau equation with space-time white noise

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Abstract

We study the stochastic complex Ginzburg-Landau equation with complex-valued space-time white noise on the three dimensional torus. This nonlinear equation is so singular that it can only be understood in a renormalized sense. We prove local well-posedness of it in the framework of paracontrolled distribution theory. This article is an announcement of the authors' full paper with the same title.

1 Introduction

In this article, we report local well-posedness of the stochastic complex Ginzburg-Landau equation (CGL) with complex-valued space-time white noise ξ in the three-dimensional torus $\mathbf{T}^3 = (\mathbf{R}/\mathbf{Z})^3$

$$
\text{(P)} \qquad \begin{cases} \partial_t u = (\mathrm{i} + \mu)\triangle u + \nu(1 - |u|^2)u + \xi & \text{on } (0,\infty) \times \mathbf{T}^3, \\ u(0,\cdot) = u_0(\cdot). \end{cases}
$$

Here, $\mathrm{i} = \sqrt{-1}$, μ is a positive constant and ν is a complex constant. There are a lot of preceding results on CGL; for example, [Hai02], [BS04b], [BS04a], [KS04], [Yan04], [Oda06], [PG11].

First of all, we explain difficulty of this problem. We rewrite (P) as $\mathcal{L}^1 u = \nu(1-|u|^2)u+u+\xi$ and consider a stationary solution to the linear equation $\mathcal{L}^1 Z = \xi$ on $(0,\infty)\times\mathbf{T}^3$, where $\mathcal{L}^1 = \partial_t - \{(\mathrm{i}+\mu)\triangle - 1\}$. Then, by setting $P_t^1 = e^{t\{(\mathrm{i}+\mu)\triangle-1\}}$ and $I(u)_t = \int_{-\infty}^t P_{t-s}^1 u_s\, ds$ for distribution-valued functions u on $[0,\infty)$, we see that the solution is given by $Z_t = I(\xi)_t$ formally and it is not a function but a distribution with respect to the space variable in dimension three. More precisely, Z_t belongs to $\mathcal{C}^{-\frac{1}{2}-\kappa}$ for any $\kappa > 0$, where $\mathcal{C}^\alpha$ is the Hölder-Besov space with the Hölder exponent $\alpha \in \mathbf{R}$; see Section 2 for definition. Hence the products Z_t^2, $Z_t\overline{Z_t}$, $Z_t^2\overline{Z_t}$ and so on are not defined a priori. Since the irregularity of the solution to (P) comes from the white noise, it is natural to guess that the space regularity of u_t is not better than that of Z_t and that the product $|u_t|^2 u_t = u_t^2 \overline{u_t}$ is not defined a priori. Hence, in order to define a notion of solution to (P), it is necessary to define the product in some way.

Hairer [Hai14] and Gubinelli-Imkeller-Perkowski [GIP15] developed great results in order to overcome such difficulty, respectively. Their works are breakthrough in the theory of singular stochastic partial differential equation and a lot of results are shown after the works; for example, [BK16], [FH14], [Hos16a], [ZZ15], [CC13], [MW16], [Hos16b], [GP17], [BB16].

We also use them to obtain local well-posedness of CGL. In the authors' full paper [IHN17], they use the both theories and establish the well-posedness; however, in this article, we only state the result obtained by the theory of paracontrolled distributions developed in [GIP15].

2 Notation

Before starting our discussion, we introduce notations. We denote by $\mathcal{D}$ the space of all smooth functions on $\mathbf{T}^3$ and by $\mathcal{D}'$ its dual. For every $\alpha \in \mathbf{R}$, we denote by $\mathcal{C}^\alpha$ the Hölder-Besov space, which is defined by the completion of the space of smooth functions on $\mathbf{T}^3$ under the Hölder-Besov norm $\|\cdot\|_{\mathcal{C}^\alpha}$. To define the norm, we use the Littlewood-Paley block $\{\triangle_m = \mathcal{F}^{-1}\rho_m\mathcal{F}\}_{m=-1}^{\infty}$, where $\mathcal{F}$ and $\mathcal{F}^{-1}$ are the Fourier transformation and its inverse, respectively, and $\{\rho_m\}_{m=-1}^{\infty}$ is the dyadic partition of unity. The norm is defined by

$$\|f\|_{\mathcal{C}^\alpha} = \sup_{m \geq -1} 2^{m\alpha} \|\triangle_m f\|_{L^\infty}.$$

We denote by $C_T\mathcal{C}^\alpha$ the space of all $\mathcal{C}^\alpha$-valued continuous functions on $[0,T]$ for every $T > 0$. We define $C_T^\delta \mathcal{C}^\alpha$ by the space of all δ-Hölder continuous functions from $[0,T]$ to $\mathcal{C}^\alpha$ and set $\mathcal{L}_T^{\alpha,\delta} = C_T\mathcal{C}^\alpha \cap C_T^\delta \mathcal{C}^{\alpha-2\delta}$.

Next we introduce the notion of paradifferential calculus. For every $f \in \mathcal{C}^\alpha$ and $g \in \mathcal{C}^\beta$, we define the resonance $f \odot g$ and the paraproduct $f \otimes g$. They give the decomposition $fg = f \otimes g + f \odot g + f \ominus g$. The paraproduct $f \otimes g$ can be defined for any $\alpha, \beta \in \mathbf{R}$, but the resonance $f \odot g$ can be defined for $\alpha + \beta > 0$. Hence, in order define products fg, it is necessary that $\alpha + \beta > 0$ holds.

For more information about the Hölder-Besov spaces and the paradifferential calculus, we consult [BCD11].

3 Main result

In this section, we state our main result and give a sketch of the proof.

We define a solution to (P) as a limit of solutions to renormalized equations. To introduce the renormalized equations, we explain how to mollify the white noise. Roughly speaking, we define smeared noise ξ^ϵ for a parameter $0 < \epsilon < 1$ by cutting off high frequency part of the Fourier transform of ξ. Let χ be a smooth function defined on $\mathbf{R}^3$ such that (1) $\operatorname{supp}\chi \subset B(0,1)$, where $B(x,r)$ denotes the open ball of radius $r > 0$ and center $x \in \mathbf{R}^3$, (2) $\chi(0) = 1$. We set $\chi^\epsilon(k) = \chi(\epsilon k)$ for every $k \in \mathbf{Z}^3$. Define $\mathbf{e}_k(x) = e^{-2\pi i k\cdot x}$ for every $k \in \mathbf{Z}^3$ and $x \in \mathbf{T}^3$. Here, the dot $\cdot$ denotes the usual inner product. We define ξ^ϵ by

$$\xi^\epsilon = \sum_{k\in\mathbf{Z}^3} \chi^\epsilon(k)\hat{\xi}(k)\mathbf{e}_k.$$

Here, $\{\hat{\xi}(k)\}_{k\in\mathbf{Z}^3}$ denotes the Fourier transform of ξ and it has the same law with independent copies of complex-valued white noise on $\mathbf{R}$. We see that $\xi^\epsilon \to \xi$ in an appropriate topology. For such smeared noise ξ^ϵ, we consider the renormalized equation

$$\text{(P')} \quad \begin{cases} \partial_t u^\epsilon = (\mathrm{i} + \mu)\triangle u^\epsilon + \nu(1 - |u^\epsilon|^2)u^\epsilon + \nu\mathfrak{c}^\epsilon u^\epsilon + \xi^\epsilon, & \text{on } (0,\infty)\times\mathbf{T}^3, \\ u(0,\cdot) = u_0(\cdot). \end{cases}$$

Here $\mathfrak{c}^\epsilon$ is a complex constant defined by $\mathfrak{c}^\epsilon = 2(\mathfrak{c}_1^\epsilon - \overline{\nu\mathfrak{c}_{2,1}^\epsilon} - 2\nu\mathfrak{c}_{2,2}^\epsilon)$, where $\mathfrak{c}_1^\epsilon$, $\overline{\mathfrak{c}_{2,1}^\epsilon}$ and $\mathfrak{c}_{2,2}^\epsilon$ are complex constants specified later. We note that $|\mathfrak{c}^\epsilon| \to \infty$ as $\epsilon \downarrow 0$. We can make sense of a solution to (P) as the limit of solutions to (P'). The next is our main result:

Theorem 1. *Let $u_0 \in \mathcal{C}^{-\frac{2}{3}+\kappa'}$ for $0 < \kappa' \ll 1$. Consider* (P'). *There exist a unique process u^ϵ and a random time T_*^ϵ such that*

- *u^ϵ solves* (P') *on $[0,T_*^\epsilon) \times \mathbf{T}^3$,*
- *T_*^ϵ converges to some a.s. positive random time T_* in probability,*
- *u^ϵ converges to some process u defined on $[0,T_*) \times \mathbf{T}^3$ in the sense that $\sup_{0\le s\le T_*/2} \|u_s^\epsilon - u_s\|_{\mathcal{C}^{-\frac{2}{3}+\kappa'}} \to 0$ as $\epsilon \to 0$ in probability. Furthermore, u is independent of the choice of ξ^ϵ.*

The proof of this theorem consists of a deterministic part and a probabilistic part. In the next subsections, we explain them and show the theorem.

3.1 Deterministic part

In the deterministic part, we construct the solution map of (P) from the space $\mathcal{X}_{T_*}^\kappa$ of driving vectors to the space $\mathcal{D}_{T_*}^{\kappa,\kappa'}$ of solutions, where T_* is a life time of a solution and κ, κ' are positive small parameters, and show that the solution map is continuous. In this part, we rely on a method introduced in [MW16]. To be precise, for every $0 < \kappa < \kappa' < 1/18$ and $T > 0$, we call a vector of space-time distributions

$$X = (X^{\mathsf{I}}, X^{\mathsf{V}}, X^{\bar{\mathsf{V}}}, X^{\mathsf{Y}}, X^{\mathsf{Y}^{\bullet}}, X^{\bar{\mathsf{Y}}^{\bullet}}, X^{\tau_1}, X^{\tau_2}, X^{\tau_3}, X^{\tau_4}, X^{\tau_5}, X^{\tau_6}, X^{\tau_7}, X^{\tau_8})$$
$$\in C_T\mathcal{C}^{-\frac{1}{2}-\kappa} \times (C_T\mathcal{C}^{-1-\kappa})^2 \times (C_T\mathcal{C}^{1-\kappa})^2 \times \mathcal{L}_T^{\frac{1}{2}-\kappa, \frac{1}{4}-\frac{1}{2}\kappa} \times (C_T\mathcal{C}^{-\kappa})^6 \times (C_T\mathcal{C}^{-\frac{1}{2}-\kappa})^2$$

which satisfies $\mathcal{L}^1 X^{\mathsf{Y}} = X^{\mathsf{V}}$ and $\mathcal{L}^1 X^{\mathsf{Y}^{\bullet}} = X^{\bar{\mathsf{V}}}$ a *driving vector* of (P). We denote by $\mathcal{X}_T^{\kappa}$ the set of all driving vectors. The definition of $\mathcal{D}_T^{\kappa,\kappa'}$ is a little complicated. Because we transform (P) to a system of two equations for (v, w) so that $u = X^{\mathsf{I}} - \nu X^{\bar{\mathsf{Y}}^{\bullet}} + v + w$ solves (P). The space $\mathcal{D}_T^{\kappa,\kappa'}$ is where (v, w) lives.

We explain the meanings of the graphical symbols I, V, $\bar{\mathsf{V}}$, Y,.... They are just coordinates mathematically; however, the dot and the line are icons for the white noise and the operation I, respectively. Hence, I represents $I(\xi) = Z$. Moreover, $\bar{\mathsf{I}}$ and $\bar{\mathsf{V}}$ are icons for the complex conjugate of Z and the product $Z\overline{Z}$, respectively. So $\bar{\mathsf{Y}}^{\bullet}$ means $I(Z^2\overline{Z})$. Finally, $\bullet$ at the bottom denotes the resonance term; τ_1 represents $I(Z^2\overline{Z}) \odot Z$.

3.2 Probabilistic part

In the probabilistic part, we construct a driving vector X^ϵ from the smeared noise ξ^ϵ with a parameter $0 < \epsilon < 1$ and show convergence of X^ϵ as $\epsilon \downarrow 0$. More precisely, we set $X^{\epsilon,\mathsf{I}} = Z^\epsilon = I(\xi^\epsilon)$, $X^{\epsilon,\bar{\mathsf{I}}} = \overline{Z^\epsilon}$ and $X^{\epsilon,\mathsf{V}} = (Z^\epsilon)^2$; however, since $\mathfrak{c}_1^\epsilon = \boldsymbol{E}[Z_t^\epsilon \overline{Z_t^\epsilon}]$ diverges as $\epsilon \downarrow 0$, we need to consider renormalization and set $X^{\epsilon,\bar{\mathsf{V}}} = Z^\epsilon \overline{Z^\epsilon} - \mathfrak{c}_1^\epsilon$. In order to define $X^{\epsilon,\tau}$ for $\bar{\mathsf{Y}}^{\bullet}$, τ_3, τ_4, τ_1 and τ_8, it is necessary to consider renormalization. The other renormalization constants are $\mathfrak{c}_{2,1}^\epsilon = \frac{1}{2}\boldsymbol{E}[X_{(t,x)}^{\epsilon,\mathsf{Y}} \odot X_{(t,x)}^{\epsilon,\bar{\bar{\mathsf{V}}}}]$ and $\mathfrak{c}_{2,2}^\epsilon = \boldsymbol{E}[X_{(t,x)}^{\epsilon,\mathsf{Y}^{\bullet}} \odot X_{(t,x)}^{\epsilon,\bar{\mathsf{V}}}]$. Note that the constants $\mathfrak{c}_1^\epsilon$, $\mathfrak{c}_{2,1}^\epsilon$ and $\mathfrak{c}_{2,2}^\epsilon$ look dependent on (t,x) but they are not. To show convergence of X^ϵ, we express $\Delta_m X^\tau$ by complex Itô-Wiener integrals and estimate their kernels. This method is established in [GP17]. For definition and properties of complex Itô-Wiener integrals, see [Itô52].

3.3 Comments on our main result

From the deterministic part and the probabilistic part, we can show our main theorem. In fact, we see that u^ϵ is given by substitution X^ϵ into the solution map. From the continuity of the solution map and convergence of $\{X^\epsilon\}_{0<\epsilon<1}$, we see that $\{u^\epsilon\}_{0<\epsilon<1}$ convergence to some process u, where u is given by substitution X into the solution map.

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