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Global solution of the coupled KPZ equations

Masato Hoshino

Abstract

This article concerns the multi-component coupled Kardar-Parisi-Zhang (KPZ) equation and its two types of approximations. By applying the paracontrolled calculus introduced by Gubinelli et al. [7, 8], we show that two approximations have the common limit under the properly adjusted choice of renormalization factors. In particular, if the coupling constants of the nonlinear term of the coupled KPZ equation satisfy the so-called "trilinear" condition, then we show that the solution of the limit equation exists globally in time when the initial value is sampled from the stationary measure. This article is a short version of Funaki and Hoshino [5].

1 Introduction and main results

We consider the following $\mathbb{R}^d$-valued coupled KPZ equation for $h(t,x) = (h^\alpha(t,x))_{\alpha=1}^d$ defined on the one dimensional torus $\mathbb{T} \equiv \mathbb{R}/\mathbb{Z} = [0,1)$:

$$\partial_t h^\alpha = \tfrac{1}{2}\partial_x^2 h^\alpha + \tfrac{1}{2}\Gamma^\alpha_{\beta\gamma}\partial_x h^\beta \partial_x h^\gamma + \xi^\alpha, \quad x \in \mathbb{T}, \tag{1.1}$$

for $1 \le \alpha \le d$. Here summation symbols $\sum$ over β and γ are omitted by Einstein's convention. $(\Gamma^\alpha_{\beta\gamma})_{1\le\alpha,\beta,\gamma\le d}$ are given constants, and $\xi(t,x) = (\xi^\alpha(t,x))_{\alpha=1}^d$ is an $\mathbb{R}^d$-valued space-time Gaussian white noise with the covariance structure

$$E[\xi^\alpha(t,x)\xi^\beta(s,y)] = \delta^{\alpha\beta}\delta(x-y)\delta(t-s),$$

where $\delta^{\alpha\beta}$ denotes Kronecker's δ. We always assume that the coupling constants $\Gamma^\alpha_{\beta\gamma}$ satisfy the bilinear condition: $\Gamma^\alpha_{\beta\gamma} = \Gamma^\alpha_{\gamma\beta}$ for all α, β, γ.

One of the motivations to study the coupled KPZ equation (1.1) comes from the nonlinear fluctuating hydrodynamics recently discussed by Spohn and others [3, 12, 13], whose origin goes back to Landau. At least heuristically, from a microscopic system with a random evolution involves a weak asymmetry, then we can expect to obtain the coupled KPZ equation in a proper space-time scaling limit by expanding the equation to the second order.

The equation (1.1) itself is ill-posed, so that we need to introduce its approximations. Let $\eta \in C^\infty(\mathbb{R})$ be an even function satisfying $\mathrm{supp}(\eta) \subset (-\frac{1}{2}, \frac{1}{2})$ and $\int_{\mathbb{R}} \eta(x)dx = 1$. We set $\eta^\varepsilon(x) = \varepsilon^{-1}\eta(\varepsilon^{-1}x)$ for $\varepsilon > 0$ and consider the approximating equation with a proper renormalization:

$$\partial_t h^{\varepsilon,\alpha} = \tfrac{1}{2}\partial_x^2 h^{\varepsilon,\alpha} + \tfrac{1}{2}\Gamma^\alpha_{\beta\gamma}(\partial_x h^{\varepsilon,\beta}\partial_x h^{\varepsilon,\gamma} - c^\varepsilon\delta^{\beta\gamma} - B^{\varepsilon,\beta\gamma}) + \xi^\alpha * \eta^\varepsilon, \tag{1.2}$$

for $1 \le \alpha \le d$, where $c^\varepsilon = \frac{1}{\varepsilon}\|\eta\|^2_{L^2(\mathbb{R})}$ and $B^{\varepsilon,\beta\gamma}$ is a renormalization factor, which diverges as $O(-\log\varepsilon)$ as $\varepsilon \downarrow 0$ in general. For the precise value of $B^{\varepsilon,\beta\gamma}$, see [5].

Another approximation of (1.1) suitable for studying invariant measures is introduced as follows. Let $\eta_2^\varepsilon = \eta^\varepsilon * \eta^\varepsilon$ and consider the equation with a proper renormalization:

$$\partial_t \tilde{h}^{\varepsilon,\alpha} = \tfrac{1}{2}\partial_x^2 \tilde{h}^{\varepsilon,\alpha} + \tfrac{1}{2}\Gamma^\alpha_{\beta\gamma}(\partial_x \tilde{h}^{\varepsilon,\beta}\partial_x \tilde{h}^{\varepsilon,\gamma} - c^\varepsilon \delta^{\beta\gamma} - \tilde{B}^{\varepsilon,\beta\gamma}) * \eta_2^\varepsilon + \xi^\alpha * \eta^\varepsilon, \tag{1.3}$$

for $1 \le \alpha \le d$, where $\tilde{B}^{\varepsilon,\beta\gamma}$ is a renormalization factor, which diverges as $O(-\log \varepsilon)$ as $\varepsilon \downarrow 0$ in general. For the precise value of $\tilde{B}^{\varepsilon,\beta\gamma}$, see [5]. In [4], under the *trilinear condition* on Γ:

$$\Gamma^\alpha_{\beta\gamma} = \Gamma^\alpha_{\gamma\beta} = \Gamma^\beta_{\gamma\alpha}, \tag{1.4}$$

for all α, β, γ, the infinitesimal invariance of the smeared Wiener measure for the tilt process $\tilde{u}^\varepsilon = \partial_x \tilde{h}^\varepsilon$ of the solution $\tilde{h}^\varepsilon$ of (1.3) is shown (actually on $\mathbb{R}$ instead of $\mathbb{T}$).

When $d = 1$ and $\Gamma^\alpha_{\beta\gamma} = 1$, the solution of the equation (1.2) with $B^{\varepsilon,\beta\gamma} = 0$ converges as $\varepsilon \downarrow 0$ to the Cole-Hopf solution $h_{\mathrm{CH}}(t,x)$ of the KPZ equation [9], while the solution of the equation (1.3) with $\tilde{B}^{\varepsilon,\beta\gamma} = 0$ converges to $h_{\mathrm{CH}}(t,x) + \frac{1}{24}t$ under the equilibrium setting [6] and the non-equilibrium setting [11].

Our first goal is to study the limits of the solutions of two types of approximating equations (1.2) and (1.3) based on the paracontrolled calculus. For $\kappa \in \mathbb{R}$ and $r \in \mathbb{N}$, $(\mathcal{C}^\kappa)^r := \mathcal{B}^\kappa_{\infty,\infty}(\mathbb{T};\mathbb{R}^r)$ denotes the $\mathbb{R}^r$-valued Besov space on $\mathbb{T}$.

Theorem 1.1. (1) *If $h_0 \in (\mathcal{C}^{1/2-\delta})^d$ for some $\delta > 0$, then a unique solution h^ε of the approximating equation* (1.2) *with initial value h_0 exists up to the survival time $T^\varepsilon_{\mathrm{sur}} \in (0,\infty]$ (i.e. $T^\varepsilon_{\mathrm{sur}} = \infty$ or $\lim_{t \uparrow T^\varepsilon_{\mathrm{sur}}} \|h^\varepsilon\|_{C([0,t],(\mathcal{C}^{1/2-\delta})^d)} = \infty$). There exists $0 < T_{\mathrm{sur}} \le \liminf_{\varepsilon\downarrow 0} T^\varepsilon_{\mathrm{sur}}$ and h^ε converges to some h in $C([0,T],(\mathcal{C}^{1/2-\delta})^d)$ for every $0 < T < T_{\mathrm{sur}}$ in probability as $\varepsilon \downarrow 0$. This T_{sur} can be chosen maximal similarly to $T^\varepsilon_{\mathrm{sur}}$.*
(2) *A similar result holds for the solution $\tilde{h}^\varepsilon$ of the equation* (1.3) *with some limit $\tilde{h}$. Moreover, under a well-adjusted choice of the renormalization factors $B^{\varepsilon,\beta\gamma}$ and $\tilde{B}^{\varepsilon,\beta\gamma}$, one can make $h = \tilde{h}$.*

Our second goal is to show the global existence of the limit process h under the condition (1.4). Let μ be the distribution of $(\partial_x B^\alpha(x))_{1\le\alpha\le d, x\in\mathbb{T}}$ on the space $(\mathcal{C}_0^{-1/2-\delta})^d := \{u \in (\mathcal{C}^{-1/2-\delta})^d; \int_{\mathbb{T}} u = 0\}$ for $\delta > 0$, where $(B^\alpha)_\alpha$ are independent pinned Brownian motions such that $B^\alpha(0) = B^\alpha(1) = 0$.

Theorem 1.2. *We assume the trilinear condition* (1.4). *Then there exists a μ-full subset $H \subset (\mathcal{C}_0^{-1/2-\delta})^d$ such that, if $\partial_x h(0) \in H$, then the limit process h exists on whole $[0,\infty)$ almost surely. Precisely, both h^ϵ and $\tilde{h}^\epsilon$ exist on whole $[0,\infty)$ and converge to h in probability as $\epsilon \downarrow 0$ in $C([0,T],(\mathcal{C}^{1/2-\delta})^d)$ for every $T > 0$.*

Remark 1.1. *Proposition 5.4 of Hairer and Mattingly [10] combined with Theorem 1.2 shows that the limit process h exists on $[0,\infty)$ almost surely for* all *initial values $h(0) \in (\mathcal{C}^{1/2-\delta})^d$, since the measure μ has a dense support in $(\mathcal{C}_0^{-1/2-\delta})^d$.*

2 Solution theory of the coupled KPZ equation

In this section, we explain the local well-posedness theory of the coupled KPZ equation (1.1) by applying the paracontrolled calculus [7, 8]. For details, see Section 2 of [5].

2.1 Preliminary consideration due to formal expansion

In the equation (1.1), we think of the noise as the leading term and the nonlinear term as its perturbation. Although we eventually take $a = 1$, we put $a > 0$ in front of the nonlinear term:

$$\mathcal{L}h^\alpha = \frac{a}{2}\Gamma^\alpha_{\beta\gamma}\partial_x h^\beta \partial_x h^\gamma + \xi^\alpha, \tag{2.1}$$

where $\mathcal{L} = \partial_t - \frac{1}{2}\partial_x^2$. Then, at least formally, we can expand the solution h as $h^\alpha = \sum_{k=0}^\infty a^k h_k^\alpha$. By comparing the terms of order a^0, a^1, a^2, a^3 in both sides of (2.1), we obtain the following identities:

$$\begin{aligned}
\mathcal{L}h_0^\alpha &= \xi^\alpha,\\
\mathcal{L}h_1^\alpha &= \tfrac{1}{2}\Gamma^\alpha_{\beta\gamma}\partial_x h_0^\beta \partial_x h_0^\gamma,\\
\mathcal{L}h_2^\alpha &= \Gamma^\alpha_{\beta\gamma}\partial_x h_1^\beta \partial_x h_0^\gamma,\\
\mathcal{L}h_3^\alpha &= \tfrac{1}{2}\Gamma^\alpha_{\beta\gamma}\partial_x h_1^\beta \partial_x h_1^\gamma + \Gamma^\alpha_{\beta\gamma}\partial_x h_2^\beta \partial_x h_0^\gamma.
\end{aligned} \tag{2.2}$$

The first equation determines h_0^α. Even though the products in the right hand side are ill-defined because $h_0^\alpha \in \mathcal{C}^{1/2-} := \cap_{\delta>0}\mathcal{C}^{1/2-\delta}$ in x, we just *assume* that $h_1^\alpha \in \mathcal{C}^{1-}$ and $h_2^\alpha \in \mathcal{C}^{3/2-}$, at this moment. When ξ^α is replaced by the smeared noise $\xi^{\varepsilon,\alpha} := \xi^\alpha * \eta^\varepsilon$, these products make sense after the renormalization (2.6). We denote $h_0^\alpha, h_1^\alpha, h_2^\alpha$ with stationary initial values by $H_{\mathsf{I}}^\alpha, H_{\mathsf{Y}}^\alpha, H_{\mathsf{Y}'}^\alpha$, respectively. Then the equation (2.1) (with $a = 1$) for $h^\alpha = H_{\mathsf{I}}^\alpha + H_{\mathsf{Y}}^\alpha + H_{\mathsf{Y}'}^\alpha + h_{\geq 3}^\alpha$ can be rewritten into an equation for the remainder $h_{\geq 3}$:

$$\mathcal{L}h_{\geq 3}^\alpha = \Phi^\alpha + \mathcal{L}h_3^\alpha, \tag{2.3}$$

where $\Phi^\alpha = \Phi^\alpha(H_{\mathsf{I}}, H_{\mathsf{Y}}, H_{\mathsf{Y}'}, h_{\geq 3})$ is given by

$$\Phi^\alpha = \Gamma^\alpha_{\beta\gamma}\partial_x h_{\geq 3}^\beta \partial_x H_{\mathsf{I}}^\gamma + \Gamma^\alpha_{\beta\gamma}(\partial_x H_{\mathsf{Y}'}^\beta + \partial_x h_{\geq 3}^\beta)\partial_x H_{\mathsf{Y}}^\gamma + \tfrac{1}{2}\Gamma^\alpha_{\beta\gamma}(\partial_x H_{\mathsf{Y}'}^\beta + \partial_x h_{\geq 3}^\beta)(\partial_x H_{\mathsf{Y}'}^\gamma + \partial_x h_{\geq 3}^\gamma).$$

To solve (2.3), we need to introduce four more objects as driving terms:

$$\begin{aligned}
&H_{\mathsf{YY}}^{\beta\gamma} = \tfrac{1}{2}\partial_x H_{\mathsf{Y}}^\beta \partial_x H_{\mathsf{Y}}^\gamma, && H_{\mathsf{Y}'\mathsf{I}}^{\beta\gamma} = \partial_x H_{\mathsf{Y}'}^\beta \odot \partial_x H_{\mathsf{I}}^\gamma,\\
&H_{<}^\alpha = \text{“stationary solution of } \mathcal{L}H_{<}^\alpha = \partial_x H_{\mathsf{I}}^\alpha\text{”}, && H_{<\mathsf{I}}^{\beta\gamma} = \partial_x H_{<}^\beta \odot \partial_x H_{\mathsf{I}}^\gamma.
\end{aligned}$$

Here $\circledcirc$ and $\odot$ are Bony's paraproducts; see [7, 8] for details. Now we divide $h_{\geq 3}^\alpha$ into the sum of two parts f^α and g^α: $h_{\geq 3}^\alpha = f^\alpha + g^\alpha$, which solve

$$\begin{aligned}
\mathcal{L}f^\alpha &= \Gamma^\alpha_{\beta\gamma}(\partial_x H_{\mathsf{Y}'}^\beta + \partial_x f^\beta + \partial_x g^\beta) \circledcirc \partial_x H_{\mathsf{I}}^\gamma,\\
\mathcal{L}g^\alpha &= \Gamma^\alpha_{\beta\gamma}(\partial_x H_{\mathsf{Y}'}^\beta + \partial_x f^\beta + \partial_x g^\beta)(\odot + \circledcirc)\partial_x H_{\mathsf{I}}^\gamma + \text{other terms},
\end{aligned} \tag{2.4}$$

respectively. Here, the implicit term contain sufficiently regular functions, so that they are well-defined if $H_{\mathsf{YY}}^{\beta\gamma}, H_{\mathsf{Y}'\mathsf{I}}^{\beta\gamma} \in \mathcal{C}^{0-}$ are given. By the commutator estimate (Lemma 2.4 of [7]), the term $\partial_x f^\beta \odot \partial_x H_{\mathsf{I}}^\gamma$ is defined if $H_{<\mathsf{I}}^{\beta\gamma} = \partial_x H_{<}^\beta \odot \partial_x H_{\mathsf{I}}^\gamma \in \mathcal{C}^{0-}$ is given.

2.2 Deterministic solution theory

Fix $\kappa \in (\frac{1}{3}, \frac{1}{2})$. The driver of the coupled KPZ equation is the element $\mathbb{H}$ of the form

$$
\begin{aligned}
\mathbb{H} := &((H^{\alpha}_{\mathrm{I}}), (H^{\alpha}_{\mathrm{Y}}), (H^{\alpha}_{\mathrm{Y}'}), (H^{\beta\gamma}_{\mathrm{W}}), (H^{\beta\gamma}_{\mathrm{Y}''}), (H^{\alpha}_{<}), (H^{\beta\gamma}_{<'}))\\
&\in C([0,T],(\mathcal{C}^{\kappa})^d) \times C([0,T],(\mathcal{C}^{2\kappa})^d) \times \{C([0,T],(\mathcal{C}^{\kappa+1})^d) \cap C^{1/4}([0,T],(\mathcal{C}^{\kappa+1/2})^d)\}\\
&\times C([0,T],(\mathcal{C}^{2\kappa-1})^{d^2}) \times C([0,T],(\mathcal{C}^{2\kappa-1})^{d^2}) \times C([0,T],(\mathcal{C}^{\kappa+1})^d) \times C([0,T],(\mathcal{C}^{2\kappa-1})^{d^2}),
\end{aligned}
$$

which satisfies $\mathcal{L}H_{<} = \partial_x H_{\mathrm{I}}$. We denote by $|\!|\!|\mathbb{H}|\!|\!|_T$ the product norm of $\mathbb{H}$ on the above space.

Fix $\lambda \in (\frac{1}{3}, \kappa)$ and $\mu \in (-\lambda, \lambda]$. For an $\mathcal{D}'(\mathbb{T}, \mathbb{R}^d)$-valued functions $f = (f^{\alpha})_{\alpha=1}^d$ and $g = (g^{\alpha})_{\alpha=1}^d$ on $[0,T]$, we write $(f,g) \in \mathcal{D}^{\lambda,\mu}_{\mathrm{KPZ}}([0,T])$ if

$$
\begin{aligned}
&\|(f,g)\|_{\mathcal{D}^{\lambda,\mu}_{\mathrm{KPZ}}([0,T])} :=\\
&\sup_{t\in[0,T]} t^{\frac{\lambda-\mu}{2}} \|f(t)\|_{(\mathcal{C}^{\lambda+1})^d} + \sup_{t\in[0,1]} \|f(t)\|_{(\mathcal{C}^{\mu+1})^d} + \sup_{s<t\in[0,T]} s^{\frac{\lambda-\mu}{2}} \frac{\|f(t)-f(s)\|_{(\mathcal{C}^{\lambda+1/2})^d}}{|t-s|^{1/4}}\\
&+ \sup_{t\in[0,T]} t^{\lambda-\mu} \|g(t)\|_{(\mathcal{C}^{2\lambda+1})^d} + \sup_{t\in[0,1]} \|g(t)\|_{(\mathcal{C}^{2\mu+1})^d} + \sup_{s<t\in[0,T]} s^{\lambda-\mu} \frac{\|g(t)-g(s)\|_{(\mathcal{C}^{2\lambda+1/2})^d}}{|t-s|^{1/4}}
\end{aligned}
$$

is finite.

The following theorem is due to the paracontrolled calculus and fixed point theorem.

Theorem 2.1 (Theorem 2.1 of [5]). (1) *Let $T > 0$ and $\mathbb{H} \in \mathcal{H}^{\kappa}_{\mathrm{KPZ}}$. Then for every initial value $(f(0), g(0)) \in (\mathcal{C}^{\mu+1})^d \times (\mathcal{C}^{2\mu+1})^d$ the system (2.4) admits a unique solution in $\mathcal{D}^{\lambda,\mu}_{\mathrm{KPZ}}([0,T_*])$ up to the time*

$$
T_* = C(1 + \|f(0)\|_{(\mathcal{C}^{\mu+1})^d} + \|g(0)\|_{(\mathcal{C}^{2\mu+1})^d} + |\!|\!|\mathbb{H}|\!|\!|_T^3)^{-\frac{2}{\kappa-\lambda}} \wedge T,
$$

where C is a universal constant depending only on κ, λ, μ and T. The solution satisfies

$$
\|(f,g)\|_{\mathcal{D}^{\lambda,\mu}_{\mathrm{KPZ}}([0,T_*])} \leq C'(1 + \|f(0)\|_{(\mathcal{C}^{\mu+1})^d} + \|g(0)\|_{(\mathcal{C}^{2\mu+1})^d} + |\!|\!|\mathbb{H}|\!|\!|_T^3),
$$

with a universal constant C'.
(2) *Let $T_{\mathrm{sur}} \leq T$ be the maximal time such that the unique solution of the system (2.4) exists on $[0, T_{\mathrm{sur}})$. The map $(f(0), g(0), \mathbb{H}) \mapsto T_{\mathrm{sur}}$ is lower semi-continuous. If $T_{\mathrm{sur}} < T$, then*

$$
\lim_{t\uparrow T_{\mathrm{sur}}} \|h\|_{C([0,t],(\mathcal{C}^{\kappa\wedge(\mu+1)\wedge(2\mu+1)})^d)} = \infty,
$$

where $h = S_{\mathrm{KPZ}}(f(0), g(0), \mathbb{H}) := H_{\mathrm{I}} + H_{\mathrm{Y}} + H_{\mathrm{Y}'} + f + g$. The map S_{KPZ} is continuous.

We do similar arguments for the equation with $*\eta_2^{\varepsilon}$ for the nonlinear term:

$$
\partial_t \tilde{h}^{\alpha} = \tfrac{1}{2}\partial_x^2 \tilde{h}^{\alpha} + \tfrac{1}{2}\Gamma^{\alpha}_{\beta\gamma}(\partial_x \tilde{h}^{\beta} \partial_x \tilde{h}^{\gamma}) * \eta_2^{\varepsilon} + \xi^{\alpha} \tag{2.5}
$$

and construct a solution map $h = S^{\varepsilon}_{\mathrm{KPZ}}(f(0), g(0), \mathbb{H})$ corresponding to the equation (2.5), though the driver $\mathbb{H}$ satisfies $\mathcal{L}H_{<} = \partial_x H_{\mathrm{I}} * \eta_2^{\varepsilon}$. Furthermore, we have the following convergence result.

Theorem 2.2 (Theorem 2.2 of [5]). *If $(f^{\varepsilon}(0), g^{\varepsilon}(0)) \to (f(0), g(0))$ in $(\mathcal{C}^{\mu+1})^d \times (\mathcal{C}^{2\mu+1})^d$ and $\mathbb{H}^{\varepsilon} \to \mathbb{H}$ in $\mathcal{H}^{\kappa}_{\mathrm{KPZ}}$, then we have $S^{\varepsilon}_{\mathrm{KPZ}}(f^{\varepsilon}(0), g^{\varepsilon}(0), \mathbb{H}^{\varepsilon}) \to S_{\mathrm{KPZ}}(f(0), g(0), \mathbb{H})$.*

2.3 Renormalization

From now we consider the $\mathbb{R}^d$-valued space-time white noise ξ. By replacing ξ^α by $\xi^{\varepsilon,\alpha} = \xi^\alpha * \eta^\varepsilon$ in (1.1) and introducing the renormalization factors $-c^\varepsilon$, $C^{\varepsilon,\beta\gamma}$ and $D^{\varepsilon,\beta\gamma}$, we obtain the renormalized driver $\mathbb{H}^\varepsilon$ corresponding to ξ^ε, which is defined by the solutions of

$$\begin{aligned} &\mathcal{L}H^{\varepsilon,\alpha}_{\mathsf{I}} = \xi^{\varepsilon,\alpha}, && \mathcal{L}H^{\varepsilon,\alpha}_{\mathsf{Y}} = \tfrac{1}{2}\Gamma^\alpha_{\beta\gamma}(\partial_x H^{\varepsilon,\beta}_{\mathsf{I}}\partial_x H^{\varepsilon,\gamma}_{\mathsf{I}} - c^\varepsilon\delta^{\beta\gamma}), \\ &\mathcal{L}H^{\varepsilon,\alpha}_{\mathsf{Y}\mathsf{Y}} = \Gamma^\alpha_{\beta\gamma}\partial_x H^{\varepsilon,\beta}_{\mathsf{Y}}\partial_x H^{\varepsilon,\gamma}_{\mathsf{I}}, && \mathcal{L}H^{\varepsilon,\alpha}_{<} = \partial_x H^{\varepsilon,\alpha}_{\mathsf{I}} \end{aligned} \tag{2.6}$$

with stationary initial values, and products

$$\begin{aligned} &H^{\varepsilon,\beta\gamma}_{\mathsf{V}\mathsf{Y}\mathsf{Y}} = \tfrac{1}{2}(\partial_x H^{\varepsilon,\beta}_{\mathsf{Y}}\partial_x H^{\varepsilon,\gamma}_{\mathsf{Y}} - C^{\varepsilon,\beta\gamma}), \qquad H^{\varepsilon,\beta\gamma}_{\mathsf{Y}\mathsf{Y}\mathsf{I}} = \partial_x H^{\varepsilon,\beta}_{\mathsf{Y}\mathsf{Y}} \odot \partial_x H^{\varepsilon,\gamma}_{\mathsf{I}} - D^{\varepsilon,\beta\gamma}, \\ &H^{\varepsilon,\beta\gamma}_{<\mathsf{I}} = \partial_x H^{\varepsilon,\beta}_{<} \odot \partial_x H^{\varepsilon,\gamma}_{\mathsf{I}}. \end{aligned}$$

We see that $h^\varepsilon := S_{\mathrm{KPZ}}(f(0), g(0), \mathbb{H}^\varepsilon)$ solves (1.2) with $B^{\varepsilon,\beta\gamma} = C^{\varepsilon,\beta\gamma} + 2D^{\varepsilon,\beta\gamma}$.

By replacing ξ^α by $\xi^{\varepsilon,\alpha}$ in (2.5) and introducing the renormalization factors $\tilde{C}^{\varepsilon,\beta\gamma}$, $\tilde{D}^{\varepsilon,\beta\gamma}$, we again obtain the renormalized driver $\tilde{\mathbb{H}}^\varepsilon$ corresponding to the approximating equation (1.3), which is defined by the similar way to $\mathbb{H}^\varepsilon$ with C^ε and D^ε replaced by $\tilde{C}^\varepsilon$ and $\tilde{D}^\varepsilon$, respectively. We see that $\tilde{h}^\varepsilon := S^\varepsilon_{\mathrm{KPZ}}(f(0), g(0), \tilde{\mathbb{H}}^\varepsilon)$ solves (1.3) with $\tilde{B}^{\varepsilon,\beta\gamma} = \tilde{C}^{\varepsilon,\beta\gamma} + 2\tilde{D}^{\varepsilon,\beta\gamma}$.

Theorems 2.1 and 2.2 combined with the following result prove Theorem 1.1.

Theorem 2.3 (Theorem 3.2 of [5]). *There exists an $\mathcal{H}^\kappa_{\mathrm{KPZ}}$-valued random variable $\mathbb{H}$ such that, for every $T > 0$ and $p \geq 1$,*

$$E\|\mathbb{H}\|^p_T < \infty, \quad \lim_{\varepsilon\downarrow 0} E\|\mathbb{H}^\varepsilon - \mathbb{H}\|^p_T = \lim_{\varepsilon\downarrow 0} E\|\tilde{\mathbb{H}}^\varepsilon - \mathbb{H}\|^p_T = 0.$$

In particular, both $h^\varepsilon = S_{\mathrm{KPZ}}(f(0), g(0), \mathbb{H}^\varepsilon)$ and $\tilde{h}^\varepsilon = S^\varepsilon_{\mathrm{KPZ}}(f(0), g(0), \tilde{\mathbb{H}}^\varepsilon)$ converge to $h = S_{\mathrm{KPZ}}(f(0), g(0), \mathbb{H})$ in probability as $\varepsilon \downarrow 0$.

3 Global existence

When $d = 1$, the global existence of the solution of the KPZ equation was obtained by Gubinelli and Perkowski [8], using the Cole-Hopf transform. In the multi-component case, however, such transform does not work in general, so that the global existence is nontrivial. In this section, by similar arguments to Da Prato and Debussche [1], we show the global existence for initial values sampled from the invariant measure of (1.1), under the trilinear condition (1.4).

3.1 Solution theory of the coupled Burgers equation

Precisely, the process which has the invariant measure is the derivative $u = \partial_x h$, which solves the coupled stochastic Burgers equation

$$\partial_t u^\alpha = \tfrac{1}{2}\partial_x^2 u^\alpha + \tfrac{1}{2}\Gamma^\alpha_{\beta\gamma}\partial_x(u^\beta u^\gamma) + \partial_x \xi^\alpha. \tag{3.1}$$

We can apply the paracontrolled calculus to (3.1) and construct a well-posed solution map

$$S_{\mathrm{CSB}} : (\mathcal{C}^\mu_0)^d \times (\mathcal{C}^{2\mu}_0)^d \times \mathcal{U}^\kappa_{\mathrm{CSB}} \ni (v(0), w(0), \mathbb{U}) \mapsto u \in C([0, T_{\mathrm{sur}}), (\mathcal{C}_0^{(\kappa-1)\wedge\mu\wedge 2\mu})^d)$$

similarly to the coupled KPZ equation. From now we set $\mu = \frac{\kappa-1}{2}$, so that $(\kappa-1)\wedge\mu\wedge 2\mu = \kappa - 1$. Indeed, these two solution maps S_{KPZ} and S_{CSB} are equivalent. If h solves (1.1), then $u = \partial_x h$ solves (3.1). Conversely, the solution $\hat{h}$ of

$$\partial_t \hat{h}^\alpha = \tfrac{1}{2}\partial_x^2 \hat{h}^\alpha + \tfrac{1}{2}\Gamma^\alpha_{\beta\gamma} u^\beta u^\gamma + \sigma^\alpha_\beta \xi^\beta$$

coincides with the original h. Hence the global existence of u is equivalent to that of h.

3.2 Invariant measure of the coupled Burgers equation

We can constructed a $\mathcal{U}^\kappa_{\mathrm{CSB}}$-valued random variable $\mathbb{U}$ such that $u = S_{\mathrm{CSB}}(v(0), w(0), \mathbb{U})$ solves the equation (3.1) with space-time white noise ξ. Note that renormalization factors vanish because we take the derivative ∂_x.

Let μ be the distribution of $(\partial_x B^\alpha(x))_{1\le\alpha\le d, x\in\mathbb{T}}$, where $(B^\alpha)_\alpha$ are independent pinned Brownian motions such that $B^\alpha(0) = B^\alpha(1) = 0$. μ is an invariant measure of the Ornstein-Uhlenbeck process u determined by

$$\mathcal{L}u^\alpha = \partial_x \xi^\alpha. \tag{3.2}$$

Under the trilinear condition (1.4), μ is invariant under the equation (3.1). To prove this fact, we consider the approximation

$$\partial_t u^{N,\alpha} = \tfrac{1}{2}\partial_x^2 u^{N,\alpha} + F^\alpha_N(u^N) + \partial_x \xi^\alpha, \tag{3.3}$$

for $N \in \mathbb{N}$, where

$$F^\alpha_N(u^N) = \tfrac{1}{2}\Gamma^\alpha_{\beta\gamma}\partial_x P_N(P_N u^{N,\beta} P_N u^{N,\gamma}),$$

and $P_N = \psi(N^{-1}D)$ is the Fourier multiplier defined by an even cut-off function $\psi \in C_0^\infty(\mathbb{R})$ taking values in $[0,1]$ and supported in the interval $[-1,1]$. Since F_N depends on finitely many Fourier components of u^N, the equation (3.3) is well-posed.

Proposition 3.1 (Proposition 5.5 and Theorem 5.6 of [5]). (1) *If the trilinear condition* (1.4) *holds, the solution u^N of* (3.3) *exists globally in time, and admits μ as an invariant measure.*
(2) *Let u^N and u be the solution of* (3.3) *and* (3.1) *respectively, with common initial value $u_0 \in (\mathcal{C}_0^{\kappa-1})^d$. Then u_N converges to u in probability as $N \to \infty$ in $C([0, T_{\mathrm{sur}}), (\mathcal{C}_0^{\kappa-1})^d)$.*

Proof. In (1), the identity

$$\langle F^\alpha_N(u), u^\alpha\rangle_{L^2(\mathbb{T})} = -\tfrac{1}{2}\Gamma^\alpha_{\beta\gamma}\langle P_N u^\beta P_N u^\gamma, \partial_x P_N u^\alpha\rangle_{L^2(\mathbb{T})} = 0. \tag{3.4}$$

has an essential role. The invariance of μ under (u^N) follows by Echeverría's criterion [2] by using (3.4). (2) is an application of the paracontrolled calculus. □

3.3 Global existence for a.e.-initial values

We can prove the following result in a similar way to Theorem 5.1 of [1]. Our main result of this section is formulated as follows.

Theorem 3.2. *We assume the trilinear condition* (1.4)*. Then, for every $T > 0$ and μ-a.e. $u_0 \in (\mathcal{C}_0^{\kappa-1})^d$, there exists a unique solution u of the equation* (3.1)*. This solution satisfies for every $p \ge 1$,*

$$E\|u\|^p_{C([0,T],(\mathcal{C}_0^{\kappa-1})^d)} < \infty.$$

In particular, $T_{\mathrm{sur}} = \infty$ a.s.

Proof. We denote by $u^N(\cdot, u(0))$ the solution of (3.3) with initial value $u(0)$. With the help of local well-posedness (like Theorem 2.1) for the stochastic Burgers equation, we have the estimate

$$\int_{(\mathcal{C}_0^{\kappa-1})^d} E[\sup_{t\in[0,T]} \|u^N(t,u(0))\|^p_{(\mathcal{C}_0^{\kappa-1})^d}]\mu_A(du(0)) \lesssim_p 1.$$

The strong convergence of u^N to u combined with this estimate shows Theorem 3.2. $\square$

Remark 3.1. *Theorem 3.2 combined with Proposition 5.4 of [10] implies the global existence of the solution h of the coupled KPZ equation* (1.1)*, as mentioned in Theorem 1.2 and Remark 1.1. Global existence of the solution $\tilde{h}^\varepsilon$ of* (1.3) *can be obtained by a similar argument, since $\tilde{u}^\varepsilon = \partial_x \tilde{h}^\varepsilon$ admits μ^ε as an invariant measure, where μ^ε is the distribution of $(\partial_x B^\alpha * \eta^\varepsilon(x))_{1\le\alpha\le d, x\in\mathbb{T}}$.*

Under the trilinear condition (1.4)*, global existence of the solution h^ε of* (1.2)*, or equivalently that of the solution u^ε of*

$$\partial_t u^{\varepsilon,\alpha} = \tfrac{1}{2}\partial_x^2 u^{\varepsilon,\alpha} + \tfrac{1}{2}\Gamma^\alpha_{\beta\gamma}\partial_x(u^{\varepsilon,\beta}u^{\varepsilon,\gamma}) + \partial_x \xi^{\varepsilon,\alpha}, \tag{3.5}$$

is obtained as follows. First, we can show that if the initial value u_0^ε satisfies $E\|u_0^\varepsilon\|^2_{L^2(\mathbb{T},\mathbb{R}^d)} < \infty$ then the solution u^ε exists globally and satisfies

$$E[\|u^\varepsilon\|^2_{C([0,T],L^2(\mathbb{T},\mathbb{R}^d))}] < \infty$$

for every $T > 0$. This is obtained by applying the Itô's formula and using the identity (3.4) *again. Second, we consider the case that $u_0 \in (\mathcal{C}_0^{-1/2-\delta})^d$. We fix $T > 0$. By Theorem 2.1, for every $K > 0$ there exists (deterministic) $t = t(u_0, K) \in (0, T]$ such that*

$$u_t^{\varepsilon,K} = \begin{cases} u_t^\varepsilon, & \|\mathbb{H}^\varepsilon\|_t \le K, \\ 0, & \textit{otherwise} \end{cases}$$

satisfies $\|u_t^{\varepsilon,K}\|_{L^2(\mathbb{T},\mathbb{R}^d)} \lesssim 1 + \|u_0\|_{(\mathcal{C}_0^{-1/2-\delta})^d} + K^3$, so that $E\|u_t^{\varepsilon,K}\|^2_{L^2(\mathbb{T},\mathbb{R}^d)} < \infty$. Since the solution of (3.5) *with initial value $u_t^{\varepsilon,K}$ exists globally, we have*

$$P(u^\varepsilon \in C([0,T],(\mathcal{C}_0^{-1/2-\delta})^d)) \ge P(\|\mathbb{H}^\varepsilon\|_t \le K) \ge P(\|\mathbb{H}^\varepsilon\|_T \le K).$$

By letting $K \to \infty$, we have that u^ε exists up to the time T almost surely. Since $T > 0$ is arbitrary, we have the global existence of u^ε.

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Graduate School of Mathematical Sciences
The University of Tokyo
Komaba, Tokyo 153-8914, Japan
e-mail: hoshino@ms.u-tokyo.ac.jp