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Comparison Theory for Cyclic Systems of Differential Equations

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Abstract.

New identities of the Picone type for a class of cyclic systems of ordinary differential equations are established and the Sturm-Picone comparison theory for such systems is developed with the help of these formulas.

Key words and phrases. Cyclic differential systems, Picone's identity, Sturmian comparison

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1 Introduction

The purpose of this paper is to provide an overview of recent comparison results of the present authors concerning the existence and the distribution of zeros of components of solutions for differential systems of the form

$$x' - p(t)\varphi_{1/\alpha}(y) = 0, \quad y' + q(t)\varphi_\alpha(x) = 0, \tag{1}$$

where α is a positive constant, p and q are continuous functions on an interval J and $\varphi_\gamma(u)$ denotes the odd function in $u \in \mathbb{R}$ defined by $\varphi_\gamma(u) = |u|^\gamma \mathrm{sgn}\, u,\ \gamma > 0$.

In establishing our results we employed new identities of the Picone type and the so called duality principle which is based on an elementary but very useful observation that if (x, y) is a solution of (1), then $(x, -y)$ and $(-x, y)$ solve the differential system

$$x' + p(t)\varphi_{1/\alpha}(y) = 0, \quad y' - q(t)\varphi_\alpha(x) = 0, \tag{2}$$

which is of the same form as (1) with the only difference that the roles of $\{x, y\}, \{p, q\}$ and $\{\alpha, 1/\alpha\}$ are interchanged.

We classify zeros of components of solutions as follows:

Let (x, y) be a solution of system (1) which satisfies $x(a) = 0$ and $y(a) \neq 0$ for some $a \in J$. A value $t = b > a$ from J is called a *conjugate* (resp. *pseudoconjugate*) point to $t = a$ if $x(b) = 0$ (resp. $y(b) = 0$).

If (x, y) is a solution of (1) which satisfies $y(a) = 0$ and $x(a) \neq 0$ for some $a \in J$, then a value $t = b > a$ from J is called a *focal* (resp. *deconjugate*) point to $t = a$ if $x(b) = 0$ (resp. $y(b) = 0$) (see [16]).

Along with (1) consider another differential system of the same form

$$X' - P(t)\varphi_{1/\alpha}(Y) = 0, \quad Y' + Q(t)\varphi_\alpha(X) = 0, \tag{3}$$

where P and Q are continuous functions on J.

It is known (see Elbert [3] and Mirzov [13]) that if

$$0 \leq p(t) \leq P(t) \text{ and } q(t) \leq Q(t) \quad (\text{or } 0 \geq p(t) \geq P(t) \text{ and } q(t) \geq Q(t))$$

for all $t \in J$ and there exists a solution (x, y) of (1) such that

$$x(a) = x(b) = 0 \quad \text{and} \quad x(t) \neq 0 \quad \text{for } t \in (a, b)$$

for some $a, b \in J, a < b$, then for any solution (X, Y) of system (3) the first component $X(t)$ must have at least one zero in $[a, b]$. Similarly, if

$$0 \leq q(t) \leq Q(t) \text{ and } p(t) \leq P(t) \text{ (or } 0 \geq q(t) \geq Q(t) \text{ and } p(t) \geq P(t))$$

for all $t \in J$ and system (1) has a solution (x, y) such that

$$y(a) = y(b) = 0 \quad \text{and} \quad y(t) \neq 0 \quad \text{for } t \in (a, b)$$

for some $a, b \in J, a < b$, then for any solution (X, Y) of system (3) the second component $Y(t)$ must vanish at some point $t = c$ in $[a, b]$.

We generalize and extend Mirzov's result in several directions. First, we show that a version of Picone's formula can be established for the pair of systems (1) and (3) which makes the proof of the Sturm-Picone comparison theorem straightforward and easy. Secondly, zeros of the component $X(t)$ (resp. $Y(t)$) are guaranteed to exist in the open interval (a, b) (stronger Sturmian conclusion) rather than in $[a, b]$ (weaker Sturmian conclusion). Finally, we establish another kind of Picone's identity for (1) and (3) which enables to generalize the point-wise comparison criterion to an integral comparison theorem of the Leighton type.

For related results concerning the existence of zeros of the components of solutions of system (1) see [2], [4] and [14]. The special case with $\alpha = 1$ was studied in [1], [10] and [11]. Comparison results for scalar half-linear ordinary differential equations of the second order can be found in [5], [6] and [12].

2 Main results

To formulate our results we use $\Phi_\gamma(U, V)$ to denote the form defined for $U, V \in \mathbb{R}$ and $\gamma > 0$ by

$$\Phi_\gamma(U, V) = |U|^{\gamma+1} + \gamma|V|^{\gamma+1} - (\gamma + 1)U\varphi_\gamma(V).$$

From the Young inequality it follows that $\Phi_\gamma(U,V) \geq 0$ for all $U, V \in \mathbb{R}$ and the equality holds if and only if $U = V$.

Our first result is the pointwise comparison criterion of the Sturm-Picone type. Its proof makes use of the following two lemmas. The first one contains identities which play a crucial role in our considerations. Formulas can be verified easily by a direct computation.

Lemma 2.1 (Picone's identity of the first kind) *Let (x,y) and (X,Y) be solutions on J of systems* (1) *and* (3), *respectively.*

(i) *If $X(t) \neq 0$ in J, then*

$$\tag{4} \begin{aligned} &\frac{d}{dt}\left\{\frac{x}{\varphi_\alpha(X)}\left[\varphi_\alpha(X)y - \varphi_\alpha(x)Y\right]\right\} = \left[Q(t) - q(t)\right]|x|^{\alpha+1} \\ &+\alpha\left[P(t) - p(t)\right]\frac{|x|^{\alpha+1}}{|X|^{\alpha+1}}|Y|^{\frac{1}{\alpha}+1} + p(t)\Phi_\alpha\big(\varphi_{1/\alpha}(y), x\varphi_{1/\alpha}(Y)/X\big). \end{aligned}$$

(ii) *If $Y(t) \neq 0$ in J, then*

$$\tag{5} \begin{aligned} &\frac{d}{dt}\left\{\frac{y}{\varphi_{1/\alpha}(Y)}\left[\varphi_{1/\alpha}(y)X - \varphi_{1/\alpha}(Y)x\right]\right\} = \left[P(t) - p(t)\right]|y|^{\frac{1}{\alpha}+1} \\ &+\frac{1}{\alpha}\left[Q(t) - q(t)\right]\frac{|y|^{\frac{1}{\alpha}+1}}{|Y|^{\frac{1}{\alpha}+1}}|X|^{\alpha+1} + q(t)\Phi_{1/\alpha}\big(\varphi_\alpha(x), y\varphi_\alpha(X)/Y\big). \end{aligned}$$

The next result shows that if certain Wronskian-like function is identically zero for a pair of vector solutions of the two-dimensional system of the form (1), then one of these solutions is a constant multiple of another in the sense specified below.

Lemma 2.2 *Let (x,y) and (X,Y) be solutions on J of the same system* (1).

(i) *If $p(t) > 0$, $x(t) \neq 0$ in J and $x(t)\varphi_{1/\alpha}\big(Y(t)\big) - X(t)\varphi_{1/\alpha}\big(y(t)\big) \equiv 0$ in J, then there exists a constant c such that $\big(X(t), Y(t)\big) = \big(cx(t), \varphi_\alpha(c)y(t)\big)$ for all $t \in J$.*

(ii) *If $q(t) > 0$, $y(t) \neq 0$ in J and $y(t)\varphi_\alpha\big(X(t)\big) - Y(t)\varphi_\alpha\big(x(t)\big) \equiv 0$ in J, then there exists a constant c such that $\big(X(t), Y(t)\big) = \big(cx(t), \varphi_{1/\alpha}(c)y(t)\big)$ for all $t \in J$.*

The first of our main results now follows. For its proof see [8].

Theorem 2.1 (Pointwise comparison) (i) *Suppose that (x,y) and (X,Y) are solutions of* (1) *and* (3), *respectively, satisfying $x(b) = 0$, $x(t) \neq 0$ for $t \in (a,b)$ and either $x(a) = 0$ or $x(a) \neq 0$, $X(a) \neq 0$ and*

$$\frac{y(a)}{\varphi_\alpha(x(a))} \geq \frac{Y(a)}{\varphi_\alpha(X(a))}.$$

Let

$$\tag{6} 0 < p(t) \leq P(t), \quad q(t) \leq Q(t), \quad t \in [a,b].$$

If, moreover, $X(t)^2 + Y(t)^2 > 0$ in $[a,b]$ and either the strict inequality holds in at least one of inequalities (6) *throughout some subinterval of (a,b) or*

$$\tag{7} x(t)\varphi_{1/\alpha}\big(Y(t)\big) - X(t)\varphi_{1/\alpha}\big(y(t)\big) \not\equiv 0 \quad \text{in } (a,b),$$

then $X(t)$ has at least one zero in the open interval (a,b).

(ii) *Suppose that (x,y) and (X,Y) are solutions of (1) and (3), respectively, satisfying $y(b)=0$, $y(t)\neq 0$ for $t\in(a,b)$ and either $y(a)=0$ or $y(a)\neq 0$, $Y(a)\neq 0$ and*

$$\frac{x(a)}{\varphi_{1/\alpha}(y(a))}\geq\frac{X(a)}{\varphi_{1/\alpha}(Y(a))}.$$

Let

$$p(t)\leq P(t),\quad 0<q(t)\leq Q(t),\quad t\in J. \tag{8}$$

If, moreover, $X(t)^2+Y(t)^2>0$ in $[a,b]$ and either the strict inequality holds in at least one of inequalities (8) throughout some subinterval of (a,b) or

$$y(t)\varphi_\alpha\big(X(t)\big)-Y(t)\varphi_\alpha\big(x(t)\big)\not\equiv 0\quad \textit{in } (a,b), \tag{9}$$

then $Y(t)$ has at least one zero in the open interval (a,b).

Example 2.1. Consider the systems

$$x'-k^{\alpha+1}\varphi_{1/\alpha}(y)=0,\quad y'+m^{\alpha+1}\varphi_\alpha(x)=0, \tag{10}$$

and

$$X'-K^{\alpha+1}\varphi_{1/\alpha}(Y)=0,\quad Y'+M^{\alpha+1}\varphi_\alpha(X)=0, \tag{11}$$

where $0<k<K$ and $0<m<M$ are constants. Let $\sin_\alpha$ (resp. $\cos_\alpha$) denote the first (resp. the second) component of the solution of the system

$$u'-\varphi_{1/\alpha}(v)=0,\quad v'+\varphi_\alpha(u)=0, \tag{12}$$

satisfying the initial condition

$$u(0)=0,\quad v(0)=\left(\frac{2}{\alpha+1}\right)^{\frac{\alpha}{\alpha+1}}. \tag{13}$$

It is known that $\sin_\alpha t$ and $\cos_\alpha t$ are periodic oscillatory functions with the period

$$\pi_\alpha:=\frac{2\alpha^{\frac{1}{\alpha+1}}\pi}{(\alpha+1)\sin\frac{\pi}{\alpha+1}}$$

(see [7]). Notice that $\pi_\alpha=\pi_{1/\alpha}$.

Systems (10) and (11) have the particular oscillatory solutions

$$(x_1,y_1)=\big(k\sin_\alpha(k^\alpha mt),m^\alpha\cos_\alpha(k^\alpha mt)\big),$$

$$(x_2,y_2)=\big(k^\alpha\cos_{1/\alpha}(km^{1/\alpha}t),m\sin_{1/\alpha}(km^{1/\alpha}t)\big),$$

and

$$(X_1,Y_1)=\big(K\sin_\alpha(K^\alpha Mt),M^\alpha\cos_\alpha(K^\alpha Mt)\big),$$

$$(X_2, Y_2) = \left(K^\alpha \cos_{1/\alpha}(KM^{1/\alpha}t), M \sin_{1/\alpha}(KM^{1/\alpha}t)\right),$$

respectively.

Theorem 2.1 guarantees that for any solution (X, Y) of system (11) the first component $X(t)$ has at least one zero between each pair of consecutive zeros of $x_1(t)$ which are spaced regularly at the distance $\pi_\alpha/(k^\alpha m)$. If, in particular, (X, Y) is such that $X(0) = 0$, then $X(t)$ vanishes again before $x_1(t)$ does, that is, it must have a zero in the open interval $(0, \pi_\alpha/(k^\alpha m))$. The function (X_1, Y_1) is an example of such solution of (11).

Similarly, the second component $Y(t)$ of any solution (X, Y) of (11) must vanish at least once between any consecutive zeros of $y_2(t)$. The function (X_2, Y_2) provides an example of the solution of (11) with $Y_2(0) = 0$ for which the first positive zero $\pi_{1/\alpha}/(KM^{1/\alpha})$ of the component $Y_2(t)$ precedes the first positive zero of $y_2(t)$.

Example 2.2. Compare (1) with

$$X' - p^*\varphi_{1/\alpha}(Y) = 0, \quad Y' + q^*\varphi_\alpha(X) = 0, \tag{B}$$

where $p^* = \max_{t\in[a,b]} p(t)$ and $q^* = \max_{t\in[a,b]} q(t)$. From Theorem 2.1 it follows that the first (resp. the second) component x of the solution (x, y) of system (1) does not have more zeros in (a, b) than the first (resp. the second) component of the solutions of (B).

System (B) has the oscillatory solution

$$\left(p^{*\frac{1}{\alpha+1}} \sin_\alpha \left(p^{*\frac{\alpha}{\alpha+1}} q^{*\frac{1}{\alpha+1}} t\right), q^{*\frac{\alpha}{\alpha+1}} \cos_\alpha \left(p^{*\frac{\alpha}{\alpha+1}} q^{*\frac{1}{\alpha+1}} t\right)\right), \tag{14}$$

where $\sin_\alpha$ (resp. $\cos_\alpha$) denotes the generalized sine function (resp. generalized cosine function) defined in Example 2.1.

The interval between consecutive zeros of the first (second) component of solution (14) of (B) is

$$\frac{\pi_\alpha}{p^{*\frac{\alpha}{\alpha+1}} q^{*\frac{1}{\alpha+1}}}.$$

If, therefore,

$$p^{*\frac{\alpha}{\alpha+1}} q^{*\frac{1}{\alpha+1}} < \frac{\pi_\alpha}{b-a}, \tag{15}$$

then for no solution of the given system (1) the first component can have more than one zero in the interval (a, b).

Now, as a second comparison system consider

$$X' - p_*\varphi_{1/\alpha}(Y) = 0, \quad Y' + q_*\varphi_\alpha(X) = 0, \tag{C}$$

where $p_* = \min_{t\in[a,b]} p(t)$ and $q_* = \min_{t\in[a,b]} q(t)$.

The first components of the solutions of (1) oscillate at least as rapidly as those of (C). System (C) has the oscillatory solution

$$\left(p_*^{\frac{1}{\alpha+1}} \sin_\alpha \left(p_*^{\frac{\alpha}{\alpha+1}} q_*^{\frac{1}{\alpha+1}} t\right), q_*^{\frac{\alpha}{\alpha+1}} \cos_\alpha \left(p_*^{\frac{\alpha}{\alpha+1}} q_*^{\frac{1}{\alpha+1}} t\right)\right), \tag{16}$$

so that the interval between consecutive zeros of the first (second) component of (16) is

$$\frac{\pi_\alpha}{p_*^{\frac{\alpha}{\alpha+1}} q_*^{\frac{1}{\alpha+1}}}.$$

It follows that a sufficient condition that the first (second) components of the solutions of the given system (1) should have at least m zeros in (a, b) is that

$$p_*^{\frac{\alpha}{\alpha+1}} q_*^{\frac{1}{\alpha+1}} \geq \frac{m\pi_\alpha}{b-a}. \tag{17}$$

In particular, a sufficient condition that system (1) should possess a solution the first (second) component of which has a zero in (a, b) is that

$$p_*^{\frac{\alpha}{\alpha+1}} q_*^{\frac{1}{\alpha+1}} \geq \frac{\pi_\alpha}{b-a}. \tag{18}$$

Example 2.3. Compare (1) with the Euler-type system

$$x' - p(t)\varphi_{1/\alpha}(y) = 0, \quad y' + \left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1} \frac{p(t)}{P(t)^{\alpha+1}} \varphi_\alpha(x) = 0, \tag{19}$$

where $\int_0^\infty p(t)dt = \infty$ and $P(t) = \int_0^t p(s)ds$. System (19) has the nonoscillatory solution

$$x(t) = P(t)^{\frac{\alpha}{\alpha+1}}, \quad y(t) = \left(\frac{\alpha}{\alpha+1}\right)^\alpha P(t)^{-\frac{\alpha}{\alpha+1}}, \quad t > 0.$$

Thus, all nontrivial solutions of (1) are *nonoscillatory* if

$$q(t) \leq \left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1} \frac{p(t)}{P(t)^{\alpha+1}} \tag{20}$$

for all sufficiently large t.

Criterion (20) is sharp in the sense that if for some $\varepsilon > 0$

$$q(t) \geq (1+\varepsilon)\left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1} p(t)P(t)^{-\alpha-1} \tag{21}$$

for all large t, then all nontrivial solutions of (1) are *oscillatory* (see Mirzov [13]).

We now apply Theorem 2.1 to get information about the arrangement of zeros of components of oscillatory solutions of system (1).

Theorem 2.2. *Assume that the functions $p(t)$ and $q(t)$ are increasing (or decreasing) on $[0, \infty)$. Let (x, y) be an oscillatory solution of system (1) and let $\{\sigma_k\}_{k=1}^\infty$ and $\{\tau\}_{k=1}^\infty$ denote the respective sequences of zeros of $x(t)$ and $y(t)$. Then, the sequences $\{\sigma_{k+1} - \sigma_k\}$ and $\{\tau_{k+1} - \tau_k\}$ are decreasing (or increasing).*

The proof of the above theorem can be found in [8].

Remark 2.1 (Critique of Picone's identities of the first kind) Formulas (4) and (5) are relatively simple and their use makes the proof of the pointwise comparison theorem straightforward and easy, but because of the presence of the components X and Y of the solution of system (3) in the second terms on the right-hand sides of (4) and (5), they are not suitable for establishing integral comparison theorems of the Leighton-type. This observation motivated our attempts to establish another Picone-type identity that would not have this handicap. The following lemma contains formulas of this kind.

Lemma 2.3 (Picone's identity of the second kind) *Let (x,y) and (X,Y) be solutions on J of systems* (1) *and* (3), *respectively.*

(i) *If $X(t) \neq 0$, $p(t) \geq 0$ and $P(t) > 0$ in J, then*

$$\frac{d}{dt}\left\{\frac{x}{\varphi_\alpha(X)}\left[\varphi_\alpha(X)y - \varphi_\alpha(x)Y\right]\right\} \tag{22}$$

$$= \left[Q(t)-q(t)\right]|x|^{\alpha+1}+p(t)\left[1-\left(p(t)/P(t)\right)^{\alpha}\right]|y|^{\frac{1}{\alpha}+1}+P(t)^{-\alpha}\Phi_\alpha\left(p(t)\varphi_{1/\alpha}(y), P(t)\frac{x}{X}\varphi_{1/\alpha}(Y)\right).$$

(ii) *If $Y(t) \neq 0$, $q(t) \geq 0$ and $Q(t) > 0$ in J, then*

$$\frac{d}{dt}\left\{\frac{y}{\varphi_{1/\alpha}(Y)}\left[\varphi_{1/\alpha}(y)X - \varphi_{1/\alpha}(Y)x\right]\right\} \tag{23}$$

$$= \left[P(t)-p(t)\right]|y|^{\frac{1}{\alpha}+1}+q(t)\left[1-\left(q(t)/Q(t)\right)^{1/\alpha}\right]|x|^{\alpha+1}+Q(t)^{-1/\alpha}\Phi_{1/\alpha}\left(q(t)\varphi_\alpha(x), Q(t)\frac{y}{Y}\varphi_\alpha(X)\right).$$

Now, assuming that system (3) has a solution (X,Y) such that its component $X(t)$ (resp. $Y(t)$) does not vanish in (a,b) where a and b are consecutive zeros of the first (resp. the second) component of solution (x,y) of comparison system (1) satisfying conditions (24) (resp. (25)) below, and integrating (22) (resp. (23)) from a to b, we are led to a contradiction which establishes the truth of following integral comparison theorem. (For details of the proof see [9].)

Theorem 2.3 (On conjugate and deconjugate points)

(i) *Suppose that $p(t) \geq 0$, $P(t) > 0$ in J, and system* (1) *possesses a solution (x,y) such that $x(t)$ has consecutive zeros a and b, $a < b$, in J and*

$$\int_a^b \left\{\left[Q(t)-q(t)\right]|x|^{\alpha+1} + p(t)\left[1-\left(p(t)/P(t)\right)^{\alpha}\right]|y|^{\frac{1}{\alpha}+1}\right\}dt \geq 0. \tag{24}$$

Then, for any solution (X,Y) of system (3) *the component $X(t)$ has a zero in (a,b), or else $(X(t),Y(t))$ is a constant multiple of $(x(t),y(t))$ which is possible only if $p(t) \equiv P(t)$ and $q(t) \equiv Q(t)$ in (a,b).*

(ii) *Suppose that $q(t) \geq 0$ and $Q(t) > 0$ in J and there exists a solution (x,y) of* (1) *such that $y(t)$ has consecutive zeros a and b, $a < b$, in J and*

$$\int_a^b \left\{q(t)\left[1-\left(q(t)/Q(t)\right)^{\alpha}\right]|x|^{\alpha+1} + \left[P(t)-p(t)\right]|y|^{\frac{1}{\alpha}+1}\right\}dt \geq 0. \tag{25}$$

Then, for any solution (X,Y) of system (3) the component $Y(t)$ has a zero in (a,b), or else $(X(t),Y(t))$ is a constant multiple of $(x(t),y(t))$ which is possible only if $p(t)\equiv P(t)$ and $q(t)\equiv Q(t)$ in (a,b).

Remark 2.2. If the pointwise inequalities

$$0<p(t)\le P(t),\quad q(t)\le Q(t),\quad t\in J, \tag{26}$$

hold for all $t\in J$, then the integral condition (24) is clearly satisfied and the conclusion (i) of Theorem 2.3 is true. Similarly, if

$$p(t)\le P(t),\quad 0<q(t)\le Q(t) \tag{27}$$

for all $t\in J$, then (25) holds and the conclusion (ii) of Theorem 2.3 follows.

Remark 2.3. In the special case where $p(t)\equiv P(t)$ and $q(t)\equiv Q(t)$ (i.e. systems (1) and (3) coincide), from Theorem 2.3 we obtain the generalization of the classical Sturm separation theorem.

The following result generalizes and extends to systems (1) and (2) the comparison theorem for the scalar second-order half-linear differential equations given in [17]. For the proof see [9].

Theorem 2.4. (On pseudoconjugate and focal points) *Let (x,y) and (X,Y) be solutions on J of systems (1) and (3), respectively.*

(i) *Suppose that $p(t)\ge 0$, $P(t)>0$ in J, $x(a)=y(b)=0$, $a<b$, with $y(t)\ne 0$ on $[a,b)$ and*

$$V_\alpha[x,y]:=\int_a^b\{[Q(t)-q(t)]|x|^{\alpha+1}+p(t)[1-(p(t)/P(t))^\alpha]|y|^{\frac{1}{\alpha}+1}\}dt\ge 0. \tag{28}$$

Then, for any solution (X,Y), with $X\not\equiv 0$, of system (3) satisfying $X(a)=0$ there is a value $c\in(a,b]$ such that $Y(c)=0$. Moreover, $c=b$ only if (X,Y) is a constant multiple of (x,y).

(ii) *Suppose that $q(t)\ge 0$, $Q(t)>0$ in J, $y(a)=x(b)=0, a<b$, with $x(t)\ne 0$ on $[a,b)$, and*

$$\int_a^b\left\{q(t)[1-(q(t)/Q(t))^\alpha]|x|^{\alpha+1}+[P(t)-p(t)]|y|^{\frac{1}{\alpha}+1}\right\}dt\ge 0. \tag{29}$$

Then, for any solution (X,Y), with $Y\not\equiv 0$, of system (3) satisfying $Y(a)=0$ there is a value $c\in(a,b]$ such that $X(c)=0$. Moreover, $c=b$ only if (X,Y) is a constant multiple of (x,y).

Remark 2.4. If the pointwise inequalities (26) hold on $[a,b]$, then (24) is satisfied and the conclusion (i) of Theorem 2.4 follows. Similarly, the satisfaction of the inequalities (27) on $[a,b]$ implies that the assertion of Theorem 2.4 (ii) holds true.

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