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Bifurcation diagrams of a semipositone problem with concave-convex nonlinearity

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1. Introduction

We investigate the exact multiplicity of positive solutions and bifurcation diagrams of the semipositone problem

$$\begin{cases} u''(x)+\lambda f(u)=0, & -1<x<1, \\ u(-1)=u(1)=0, \end{cases} \tag{1.1}$$

where $\lambda>0$ is a bifurcation parameter. We say that $f(u)$ is semipositone if $f(0)<0$.

Semipositone problems arise in many different areas of applied mathematics and physics, such as the buckling of mechanical systems, the design of suspension bridges, chemical reactions and population models with harvesting effort; see e.g. [3, 16, 23, 28, 29]. Note that it is possible that (1.1) has non-negative solutions with interior zeros (see [10]). In this paper we will only consider the positive solutions of (1.1).

The study of semipositone problems was formally introduced by Castro and Shivaji in [10]. In general, studying positive solutions for *semipositone* problems is more difficult than that for *positone* problems. The difficulty is due to the fact that in the semipositone case, solutions have to live in regions where the nonlinear term is negative as well as positive. Due to its importance, one dimensional semipositone problems have been widely studied by many authors; see e.g. [1, 5, 8, 10, 15, 24, 31, 34]. For high dimensional results of semipositone problems; see e.g. [2, 4, 6, 7, 9, 14].

In this paper, we assume that nonlinearity $f\in C[0,\infty)\cap C^2(0,\infty)$ satisfies hypotheses (H1)–(H3) as follows:

(H1) $f(0)<0$ (semipositone).

(H2) f is concave-convex on $(0,\infty)$; that is, f has a unique positive inflection point γ such that

$$f''(u)\begin{cases} <0 & \text{on } (0,\gamma), \\ =0 & \text{when } u=\gamma, \\ >0 & \text{on } (\gamma,\infty). \end{cases} \tag{1.2}$$

(H3) f is asymptotic superlinear; that is, $\lim_{u\to\infty}(f(u)/u)=\infty$.

In addition, we assume that f satisfies either one of the following two hypotheses:

(H4a) f has exactly one positive zero a.

(H4b) f has three distinct positive zeros $a < b < c$.

Possible graphs of f satisfying (H1)–(H3) and (H4a) (resp. (H1)–(H3) and (H4b)) are illustrated in Fig. 1 (resp. Fig. 2.) For the degenerate case that f has exactly two distinct positive zeros $a < b$, the analysis for the exact multiplicity of positive solutions and bifurcation diagrams of (1.1) is the same as that f has exactly three distinct positive zeros, and hence we omit it.

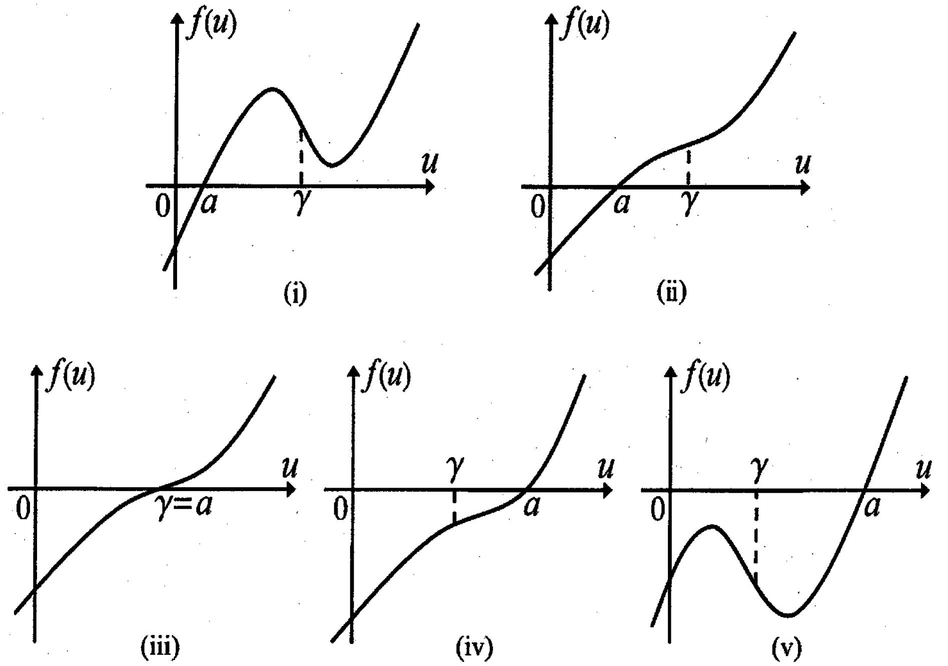

Fig. 1. Five possible graphs of f satisfying (H1)–(H3) and (H4a).
(i) $f(\gamma) > 0$, $f'(\gamma) < 0$. (ii) $f(\gamma) > 0$, $f'(\gamma) \geq 0$. (iii) $f(\gamma) = 0$. (iv) $f(\gamma) < 0$, $f'(\gamma) \geq 0$. (v) $f(\gamma) < 0$, $f'(\gamma) < 0$.

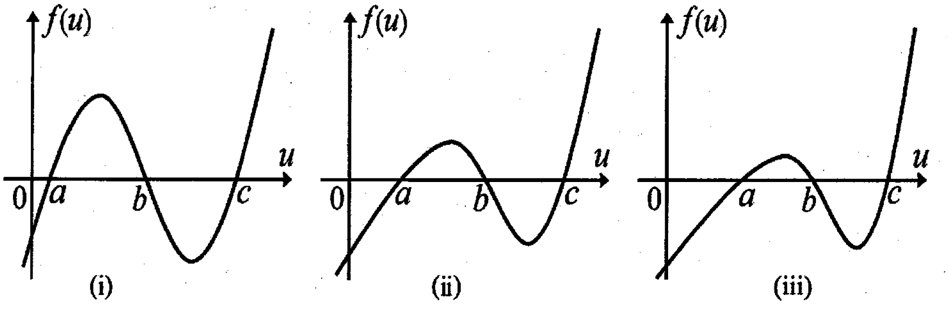

Fig. 2. Three possible graphs of f satisfying (H1)–(H3) and (H4b).
(i) $F(b) > 0$. (ii) $F(b) = 0$. (iii) $F(b) < 0$.

Let $F(u) = \int_0^u f(t)dt$. We notice that:

(I) If f satisfying (H1)–(H3) and (H4a) (resp. (H1)–(H3), (H4b) and $F(b) \leq 0$), there exists a unique $\mu \in (a, \infty)$ (resp. $\mu \in (c, \infty)$) such that

$$F(u) \begin{cases} \leq (\not\equiv)\ 0 & \text{on } (0, \mu), \\ = 0 & \text{when } u = \mu, \\ > 0 & \text{on } (\mu, \infty). \end{cases} \tag{1.3}$$

(II) If f satisfying (H1)–(H3), (H4b) and $F(b) > 0$, there exist two numbers $\bar{b} \in (a, b)$ and $\bar{c} \in (c, \infty)$ such that

$$F(u) \begin{cases} < 0 & \text{on } (0, \bar{b}), \\ = 0 & \text{when } u = \bar{b}, \\ > 0 & \text{on } (\bar{b}, b], \\ < F(b) & \text{on } (b, \bar{c}), \\ = F(b) & \text{when } u = \bar{c}, \\ > F(b) & \text{on } (\bar{c}, \infty). \end{cases} \tag{1.4}$$

We define the bifurcation diagram of (1.1)

$$\Sigma = \left\{(\lambda, \|u_\lambda\|_\infty) : \lambda > 0 \text{ and } u_\lambda \text{ is a positive solution of (1.1)}\right\}.$$

We say that, on the $(\lambda, ||u||_\infty)$-plane, the bifurcation diagram Σ is reversed S-shaped (see Fig. 3) if Σ is a continuous curve and there exist $0 < \lambda_* < \lambda^* < \infty$ such that Σ has *exactly two* turning points at some points $(\lambda_*, \|u_{\lambda_*}\|_\infty)$ and $(\lambda^*, \|u_{\lambda^*}\|_\infty)$, and

(i) $\lambda_* < \lambda^*$ and $\|u_{\lambda_*}\|_\infty < \|u_{\lambda^*}\|_\infty$,

(ii) at $(\lambda_*, \|u_{\lambda_*}\|_\infty)$ the curve turns to the *right*,

(iii) at $(\lambda^*, \|u_{\lambda^*}\|_\infty)$ the curve turns to the *left*.

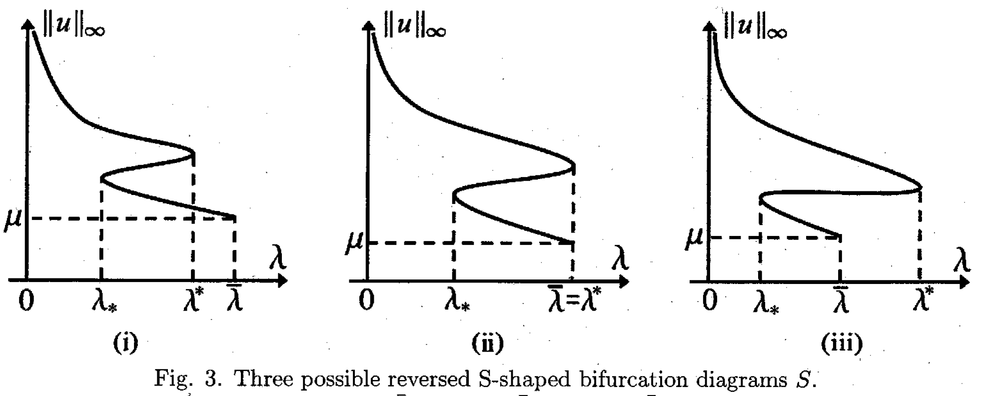

Fig. 3. Three possible reversed S-shaped bifurcation diagrams S. (i) $\lambda^* < \bar{\lambda}$. (ii) $\lambda^* = \bar{\lambda}$. (iii) $\lambda^* > \bar{\lambda}$.

Moreover, we say that, on the $(\lambda, ||u||_\infty)$-plane, the bifurcation diagram Σ is broken reversed S-shaped (see Fig. 4) if Σ has two connected branches such that

(i) the lower branch of Σ has exactly one turning point $(\lambda_*, \|u_{\lambda_*}\|_\infty)$ where the curve turns to the *right*,

(ii) the upper branch of Σ is a monotone decreasing curve.

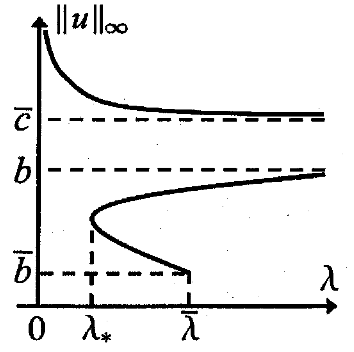

Fig. 4. Broken reversed S-shaped bifurcation diagram S.

If the nonlinearity $f \in C^2$ is *convex* on $[0, \infty)$, it is easy to check that the bifurcation diagram Σ of semipositone problem (1.1) is a monotone decreasing curve on the $(\lambda, ||u||_\infty)$-plane; see Castro and Shivaji [10] for the details. In [5, 8, 10, 34], semipositone problems with *concave* nonlinearities have been extensively studied. More precisely, the bifurcation diagram Σ of semipositone problem (1.1) is either a $\subset$-shaped curve, a monotone decreasing curve, or an empty set on the $(\lambda, ||u||_\infty)$-plane. If the nonlinearity $f \in C^2$ is *convex-concave* on $[0, \infty)$, the exact multiplicity result of positive solutions and bifurcation diagram Σ of semipositone problem (1.1) remain the same as those for concave nonlinearities $f \in C^2$ on $[0, \infty)$; see Ouyang and Shi [30].

It is well known in the literature that the study of positive solutions to semipositone problems with *concave-convex* nonlinearities is mathematically challenging. If $f \in C^2[0, \infty)$ satisfies (H1)–(H3), (H4a) and some suitable conditions, Castro and Shivaji [10, Theorem 1.1(C)] used the quadrature method (time map method) to prove that (1.1) has at least three solutions for positive λ in certain range. In the following Theorem 1.1, Shi and Shivaji [31, Theorem 3.1] and Gadam and Iaia [15, Theorem 1] proved that the bifurcation diagram Σ of (1.1) is reversed S-shaped on the $(\lambda, ||u||_\infty)$-plane, respectively. If $f \in C^2[0, \infty)$ satisfies (H1)–(H3) and (H4b), in the following Theorem 1.2, Shi and Shivaji [31, Theorem 4.1] proved that the bifurcation diagram Σ of (1.1) is broken reversed S-shaped on the $(\lambda, ||u||_\infty)$-plane. The approach in Shi and Shivaji [31, Theorems 3.1 and 4.1] mainly used bifurcation theory of Crandall and Rabinowitz [12]. While the approach in Gadam and Iaia [15, Theorem 1] used some variational techniques with respect to parameters and some time map techniques. For more researches about the exact multiplicity of concave-convex nonlinearity problem, see [11, 18, 19, 20, 21, 33, 35].

Define

$$\theta(u) \equiv 2F(u) - uf(u) \quad \text{for } u \geq 0. \tag{1.5}$$

Theorem 1.1 (See Fig. 1(i)–(ii) and Fig. 3). *Consider* (1.1). *Assume that* $f \in C^2[0, \infty)$ *satisfies (H1)–(H3), (H4a),*

$$F(\gamma) > 0, \tag{1.6}$$

and

$$\theta(\gamma) \geq 0. \tag{1.7}$$

Then the bifurcation diagram Σ *of* (1.1) *is reversed S-shaped on the* $(\lambda, ||u||_\infty)$*-plane.*

Theorem 1.2 (See Fig. 2(i) and Fig. 4). *Consider* (1.1). *Assume that* $f \in C^2[0,\infty)$ *satisfies (H1)–(H3) and (H4b),* (1.6), (1.7) *and*

$$J(\bar{c}) \leq 0, \tag{1.8}$$

where $J(u) \equiv f^2(u) - F(u)f'(u)$ *and* $\bar{c}$ *is defined in* (1.4). *Then the bifurcation diagram* Σ *of* (1.1) *is broken reversed S-shaped on the* $(\lambda, ||u||_\infty)$*-plane.*

In Theorem 2.1(i) stated below, we improve the results in Theorem 1.1; that is, we give weaker conditions $f(\gamma) > 0$, (C1), (C2) for (1.6), (1.7); see Theorem 2.1(i) and Remark 1 for the details. For Theorem 1.2, Shi and Shivaji [31, p. 570, lines 1–2] conjectured that (1.8) is not necessary. In Theorem 2.2(i) stated below, we prove this conjecture. Moreover, we give weaker conditions $F(b) > 0$, (C1$'$), (C2$'$) for (1.6), (1.7); see Theorem 2.2(i) and Remark 2 for the details.

In recent years, singular semipositone problems have been studied extensively in the literature (see [17, 26, 27, 32, 36, 37] and the references therein). Our approaches used in this paper also can be applied to this study. If $f \in C^2(0,\infty)$, $\lim_{u\to 0^+} f(u) = -\infty$, $F(u_0) > 0$ for some $u_0 > 0$, and f satisfies hypotheses (H2) and (H3), under the same additional conditions in Theorems 2.1 and 2.2, we can prove that all the results still hold; see Remark 3 for the details.

In Section 3, we give two examples for Theorems 2.1 and 2.2. The first one is the semipositone problem with cubic nonlinearity of the form

$$\begin{cases} u''(x) + \lambda(u^3 - Au^2 + Bu - C) = 0, & -1 < x < 1, \\ u(-1) = u(1) = 0, \quad A, B, C > 0. \end{cases} \tag{1.9}$$

For (1.9), Shi and Shivaji [31, Section 5] proved Theorem 1.1 holds if

$$C \leq \frac{A^3}{54}, \tag{1.10}$$

$$C < \frac{6AB - A^3}{36}, \tag{1.11}$$

and either

$$A^2 \leq 3B, \tag{1.12}$$

or

$$A^2 > 3B, \quad (9AB - 2A^3 - 27C)^2 - 4(A^2 - 3B)^3 > 0. \tag{1.13}$$

In Theorem 3.1 stated below, we prove the same results for (1.9) under weaker condition

$$C \leq \frac{16}{729}A^3$$

for (1.10). Moreover,

(i) If $A^2 \leq 3B$, we give weaker condition $C < \frac{9AB-2A^3}{27}$ for (1.11).

(ii) If $A^2 > 3B$, we give weaker condition $C < \frac{9AB-2A^3}{27} - \frac{2(A^2-3B)^{3/2}}{27}$ for (1.11) and (1.13).

See Theorem 3.1 and Remark 5 stated below for the details.

Another example is the semipositone problem with cubic nonlinearity of the form

$$\begin{cases} u''(x) + \lambda(u-a)(u-b)(u-c) = 0, & -1 < x < 1, \\ u(-1) = u(1) = 0, & 0 < a < b < c. \end{cases} \tag{1.14}$$

For problem (1.14), a subcase of problem (1.9), we are able to determine completely the exact multiplicity of positive solutions and bifurcation diagrams of (1.14). In Theorem 3.2 stated below, we prove that the bifurcation diagram Σ of (1.14) is broken reversed S-shaped on the $(\lambda, \|u\|_\infty)$-plane (see Fig. 4) if $2b(a+c) > b^2 + 6ac$; while it is a monotone decreasing curve on the $(\lambda, \|u\|_\infty)$-plane (see Fig. 5) if $2b(a+c) \leq b^2 + 6ac$. See Theorem 3.2 stated below depicted below for the details.

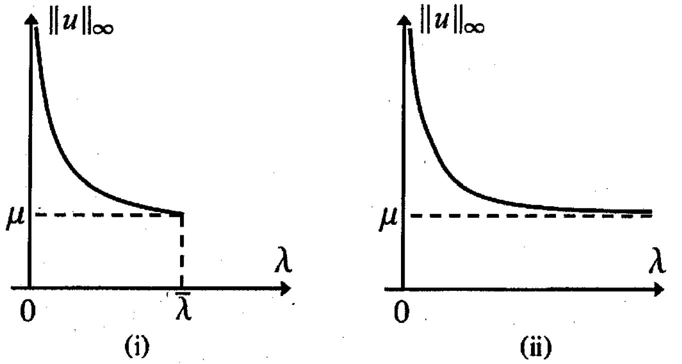

Fig. 5. Two possible monotone decreasing bifurcation diagrams S.

2. Main results

The main results are next Theorems 2.1 and 2.2. We combine several different techniques developed in the latest decade [13, 15, 22, 25] to prove Theorems 2.1 and 2.2. In Theorem 2.1(i), we improve the results in Theorem 1.1; that is, we weaken conditions $f(\gamma) > 0$, (C1), (C2) for (1.6), (1.7). In Theorem 2.2(i), we improve the results in Theorem 1.2; that is, we weaken conditions $F(b) > 0$, (C1′), (C2′) for (1.6), (1.7).

We first recall the function $\theta(u) = 2F(u) - uf(u)$ defined in (1.5) and the positive numbers $\mu, \bar{b}, \bar{c}$ defined in (1.3) and (1.4).

Theorem 2.1. *Consider* (1.1). *Assume that* $f \in C[0,\infty) \cap C^2(0,\infty)$ *satisfies (H1)–(H3) and (H4a). Then the following assertions (i) and (ii) hold:*

(i) (See Fig. 1(i)–(ii) and Fig. 3.) If $f(\gamma) > 0$ *and*

(C1) There exists $\bar{\mu} \in (\mu, \infty)$ *such that* $\theta(\bar{\mu}), \theta'(\bar{\mu}) \geq 0$.

Also assume that one of the following conditions (C2)–(C4) holds:

(C2) $\frac{u\theta'(u)}{\theta(u)}$ *is nonincreasing on* $(\mu, \bar{\mu})$.

(C3) $\frac{uf'(u)}{f(u)}$ *is nonincreasing on* $(\mu, \bar{\mu})$.

(C4) $f''(u) < 0$ *on* $(\mu, \bar{\mu})$.

Then the bifurcation diagram Σ *of* (1.1) *is reversed S-shaped on the* $(\lambda, ||u||_\infty)$*-plane. Moreover, there exist* $\lambda^* > \lambda_* > 0$ *and* $\bar{\lambda} > \lambda_*$ *such that:*

Case 1: *(See Fig. 3(i).) If* $\lambda^* < \bar{\lambda}$, *then* (1.1) *has exactly three positive solutions* u_λ, v_λ, w_λ *with* $u_\lambda < v_\lambda < w_\lambda$ *for* $\lambda_* < \lambda < \lambda^*$, *exactly two positive solutions* (v_λ, w_λ *with* $v_\lambda < w_\lambda$) *for* $\lambda = \lambda_*$ *and* (u_λ, v_λ *with* $u_\lambda < v_\lambda$) *for* $\lambda = \lambda^*$, *exactly one positive solution* (w_λ) *for* $0 < \lambda < \lambda_*$ *and* (u_λ) *for* $\lambda^* < \lambda \le \bar{\lambda}$, *and no positive solution for* $\lambda > \bar{\lambda}$.

Case 2: *(See Fig. 3(ii).) If* $\lambda^* = \bar{\lambda}$, *then* (1.1) *has exactly three positive solutions* u_λ, v_λ, w_λ *with* $u_\lambda < v_\lambda < w_\lambda$ *for* $\lambda_* < \lambda < \lambda^*$, *exactly two positive solutions* (v_λ, w_λ *with* $v_\lambda < w_\lambda$) *for* $\lambda = \lambda_*$ *and* (u_λ, v_λ *with* $u_\lambda < v_\lambda$) *for* $\lambda = \lambda^*$, *exactly one positive solution* w_λ *for* $0 < \lambda < \lambda_*$, *and no positive solution for* $\lambda > \lambda^*$.

Case 3: *(See Fig. 3(iii).) If* $\lambda^* > \bar{\lambda}$, *then* (1.1) *has exactly three positive solutions* u_λ, v_λ, w_λ *with* $u_\lambda < v_\lambda < w_\lambda$ *for* $\lambda_* < \lambda \le \bar{\lambda}$, *exactly two positive solutions* v_λ, w_λ *with* $v_\lambda < w_\lambda$ *for* $\lambda = \lambda_*$ *and* $\bar{\lambda} < \lambda < \lambda^*$, *exactly one positive solution* w_λ *for* $0 < \lambda < \lambda_*$ *and* $\lambda = \lambda^*$, *and no positive solution for* $\lambda > \lambda^*$.

More precisely,

$$\lim_{\lambda \to 0^+} \|w_\lambda\|_\infty = \infty \text{ and } \lim_{\lambda \to \bar{\lambda}^-} \|u_\lambda\|_\infty = \|u_{\bar{\lambda}}\|_\infty = \mu.$$

(ii) *(See Fig. 1(iii)–(v) and Fig. 5(i).) If* $f(\gamma) \le 0$, *then the bifurcation diagram* Σ *of* (1.1) *is a monotone decreasing curve on the* $(\lambda, ||u||_\infty)$*-plane. Moreover, there exists* $\bar{\lambda} > 0$ *such that* (1.1) *has exactly one positive solution* u_λ *for* $0 < \lambda \le \bar{\lambda}$, *and no positive solution for* $\lambda > \bar{\lambda}$. *More precisely,*

$$\lim_{\lambda \to 0^+} \|u_\lambda\|_\infty = \infty \text{ and } \lim_{\lambda \to \bar{\lambda}^-} \|u_\lambda\|_\infty = \|u_{\bar{\lambda}}\|_\infty = \mu.$$

Remark 1. *Assume that* $f \in C[0,\infty) \cap C^2(0,\infty)$ *satisfies (H1)–(H3) and (H4a). If* (1.6) *holds, then* $f(\gamma) > 0$ *and* $\gamma > \mu$ *by* (1.3)*; see Fig. 1(i)–(ii). Let* $\bar{\mu} = \gamma$, *then (C1) and (C4) hold by (H2) and* (1.7). *We also note that (C3) and (C4) are both sufficient conditions of (C2) by the proof of Hung and Wang [22, Theorem 2.1]; see also Lemma 3.2 stated below. So, conditions* $f(\gamma) > 0$, *(C1) and (C2) are necessary conditions of* (1.6) *and* (1.7). *This implies that Theorem 2.1(i) is more general than Theorem 1.1.*

Theorem 2.2. *Consider* (1.1). *Assume that* $f \in C[0,\infty) \cap C^2(0,\infty)$ *satisfies (H1)–(H3) and (H4b). Then the following assertions (i)–(iii) hold:*

(i) *(See Fig. 2(i) and Fig. 4.) If* $F(b) > 0$ *and*

(C1′) *There exists* $\hat{b} \in (\bar{b}, b]$ *such that* $\theta(\hat{b}), \theta'(\hat{b}) \ge 0$.

Also assume one of the following conditions (C2′)–(C4′) holds:

(C2′) $\frac{u\theta'(u)}{\theta(u)}$ *is nonincreasing on* $(\bar{b}, \hat{b})$.

(C3′) $\frac{uf'(u)}{f(u)}$ *is nonincreasing on* $(\bar{b}, \hat{b})$.

(C4′) $f''(u) < 0$ *on* $(\bar{b}, \hat{b})$.

Then the bifurcation diagram Σ *of* (1.1) *is broken reversed S-shaped on the* $(\lambda, ||u||_\infty)$-*plane. Moreover, there exist* $\bar{\lambda} > \lambda_* > 0$ *such that* (1.1) *has exactly three positive solutions* u_λ, v_λ, w_λ *with* $u_\lambda < v_\lambda < w_\lambda$ *for all* $\lambda_* < \lambda \leq \bar{\lambda}$, *exactly two positive solutions* v_λ, w_λ *with* $v_\lambda < w_\lambda$ *for* $\lambda = \lambda_*$ *and* $\lambda > \bar{\lambda}$, *and exactly one positive solution* w_λ *for* $0 < \lambda < \lambda_*$. *More precisely,*

$$\lim_{\lambda \to \bar{\lambda}^-} \|u_\lambda\|_\infty = \|u_{\bar{\lambda}}\|_\infty = \bar{b}, \lim_{\lambda \to \infty} \|v_\lambda\|_\infty = b, \lim_{\lambda \to 0^+} \|w_\lambda\|_\infty = \infty, \lim_{\lambda \to \infty} \|w_\lambda\|_\infty = \bar{c}.$$

(ii) *(See Fig. 2(ii) and Fig. 5(ii).) If* $F(b) = 0$, *then the bifurcation diagram* Σ *of* (1.1) *is a monotone decreasing curve on the* $(\lambda, ||u||_\infty)$-*plane. Moreover,* (1.1) *has exactly one positive solution* u_λ *for all* $\lambda > 0$. *More precisely,*

$$\lim_{\lambda \to 0^+} \|u_\lambda\|_\infty = \infty \text{ and } \lim_{\lambda \to \infty} \|u_\lambda\|_\infty = \mu.$$

(iii) *(See Fig. 2(iii) and Fig. 5(i).) If* $F(b) < 0$, *then the bifurcation diagram* Σ *of* (1.1) *is a monotone decreasing curve on the* $(\lambda, ||u||_\infty)$-*plane. Moreover, there exists* $\bar{\lambda} > 0$ *such that* (1.1) *has exactly one positive solution* u_λ *for* $0 < \lambda \leq \bar{\lambda}$, *and no positive solution for* $\lambda > \bar{\lambda}$. *More precisely,*

$$\lim_{\lambda \to 0^+} \|u_\lambda\|_\infty = \infty \text{ and } \lim_{\lambda \to \bar{\lambda}^-} \|u_\lambda\|_\infty = \|u_{\bar{\lambda}}\|_\infty = \mu.$$

Remark 2. *Assume that* $f \in C[0, \infty) \cap C^2(0, \infty)$ *satisfies (H1)–(H3) and (H4b). If* (1.6) *holds, then* $F(b) > 0$ *and* $\gamma > \bar{b}$ *by* (1.4); *see Fig. 2(i). Let* $\hat{b} = \gamma$, *then (C1′) and (C4′) hold by (H2) and* (1.7). *We also note that conditions (C3′) and (C4′) are both sufficient conditions of (C2′) by the proof of Hung and Wang [22, Theorem 2.1]; see also Lemma 3.2 stated below. So conditions* $F(b) > 0$, *(C1′) and (C2′) are necessary conditions of* (1.6) *and* (1.7). *This implies that Theorem 2.2(i) is more general than Theorem 1.2. In addition, we prove that condition* (1.8) *is not necessary in Theorem 2.2(i), and thus we prove the conjecture in Shi and Shivaji [31, p. 570, lines 1–2].*

Remark 3. *If* $f \in C^2(0, \infty)$ *satisfies all hypotheses Theorem 2.1 (resp. Theorem 2.2), with (H1) replaced by (H1′):*

(H1′) $\lim_{u \to 0^+} f(u) = -\infty$ *(singular semipositone) and* $F(u_0) > 0$ *for some* $u_0 > 0$.

Then the same arguments in the proof of Theorem 2.1 (resp. Theorem 2.2) can apply to prove the same results in Theorem 2.1 (resp. Theorem 2.2).

Remark 4. *If* $f \in C[0, \infty) \cap C^2(0, \infty)$ *satisfies all hypotheses in Theorem 2.1 (resp. Theorem 2.2), with (H3) replaced by (H3′):*

(H3′) *There exists a number* $p_0 > 0$ *such that* $f(u)/u$ *is strictly increasing on* (p_0, ∞) *and* $\lim_{u \to \infty}(f(u)/u) = k \in (0, \infty)$.

Then the same arguments in the proof of Theorem 2.1 (resp. Theorem 2.2) can apply to determine the same shapes of bifurcation diagrams Σ *of* (1.1) *and prove that* Σ *approaches the line* $\lambda = \frac{4\pi^2}{k}$ *as* $\|u_\lambda\|_\infty \to \infty$. *Thus we are able to obtain the similar exact multiplicity of positive solutions.*

3. Two examples

In this section, we give two examples for Theorems 2.1 and 2.2. In particular, we are able to determine completely the exact multiplicity of positive solutions and bifurcation diagrams of (1.14) in Theorem 3.2.

Theorem 3.1 (See Fig. 3). *Consider problem* (1.9)

$$\begin{cases} u''(x) + \lambda(u^3 - Au^2 + Bu - C) = 0, & -1 < x < 1, \\ u(-1) = u(1) = 0, \ A, B, C > 0. \end{cases}$$

If

$$C \leq \frac{16}{729} A^3, \tag{3.1}$$

and either

$$A^2 \leq 3B, \quad C < \frac{9AB - 2A^3}{27}, \tag{3.2}$$

or

$$3B < A^2 < 4B, \quad C < \frac{9AB - 2A^3}{27} - \frac{2(A^2 - 3B)^{3/2}}{27}. \tag{3.3}$$

Then the bifurcation diagram Σ of (1.9) *is reversed S-shaped on the $(\lambda, ||u||_\infty)$-plane. Moreover, there exist $\lambda^* > \lambda_* > 0$ and $\bar{\lambda} > \lambda_*$ such that:*

Case 1: *(See Fig. 3(i).) If $\lambda^* < \bar{\lambda}$, then* (1.9) *has exactly three positive solutions u_λ, v_λ, w_λ with $u_\lambda < v_\lambda < w_\lambda$ for $\lambda_* < \lambda < \lambda^*$, exactly two positive solutions (v_λ, w_λ with $v_\lambda < w_\lambda$) for $\lambda = \lambda_*$ and (u_λ, v_λ with $u_\lambda < v_\lambda$) for $\lambda = \lambda^*$, exactly one positive solution (w_λ) for $0 < \lambda < \lambda_*$ and (u_λ) for $\lambda^* < \lambda \leq \bar{\lambda}$, and no positive solution for $\lambda > \bar{\lambda}$.*

Case 2: *(See Fig. 3(ii).) If $\lambda^* = \bar{\lambda}$, then* (1.9) *has exactly three positive solutions u_λ, v_λ, w_λ with $u_\lambda < v_\lambda < w_\lambda$ for $\lambda_* < \lambda < \lambda^*$, exactly two positive solutions (v_λ, w_λ with $v_\lambda < w_\lambda$) for $\lambda = \lambda_*$ and (u_λ, v_λ with $u_\lambda < v_\lambda$) for $\lambda = \lambda^*$, exactly one positive solution w_λ for $0 < \lambda < \lambda_*$, and no positive solution for $\lambda > \lambda^*$.*

Case 3: *(See Fig. 3(iii).) If $\lambda^* > \bar{\lambda}$, then* (1.9) *has exactly three positive solutions u_λ, v_λ, w_λ with $u_\lambda < v_\lambda < w_\lambda$ for $\lambda_* < \lambda \leq \bar{\lambda}$, exactly two positive solutions v_λ, w_λ with $v_\lambda < w_\lambda$ for $\lambda = \lambda_*$ and $\bar{\lambda} < \lambda < \lambda^*$, exactly one positive solution w_λ for $0 < \lambda < \lambda_*$ and $\lambda = \lambda^*$, and no positive solution for $\lambda > \lambda^*$.*

More precisely,

$$\lim_{\lambda \to 0^+} \|w_\lambda\|_\infty = \infty \text{ and } \lim_{\lambda \to \bar{\lambda}^-} \|u_\lambda\|_\infty = \|u_{\bar{\lambda}}\|_\infty = \mu.$$

Remark 5. *Comparing conditions* (3.1)–(3.3) *in Theorem 5.1 with conditions* (1.10)–(1.13) *in Shi and Shivaji [31, Section 5], we obtain that:*

(i) (3.1) *is weaker than* (1.10).

(ii) *If $A^2 \leq 3B$, it is easy to check that condition $C < \frac{9AB-2A^3}{27}$ is weaker than* (1.11).

(iii) *If $3B < A^2 < 4B$, it is easy to check that condition $C < \frac{9AB-2A^3}{27} - \frac{2(A^2-3B)^{3/2}}{27}$ is weaker than* (1.11) *and* (1.13).

(iv) If $A^2 \geq 4B$, it is easy to check that (1.11) *and* (1.13) *can not hold together.*

(v) By above part (i)–(iv), we obtain that conditions ((3.1), and either (3.2) or (3.3)) in Theorem 5.1 are weaker than conditions ((1.10), (1.11), and either (1.12) or (1.13)) in Shi and Shivaji [31, Section 5].

Theorem 3.2. *Consider problem* (1.14)

$$\begin{cases} u''(x) + \lambda(u-a)(u-b)(u-c) = 0, & -1 < x < 1, \\ u(-1) = u(1) = 0, & 0 < a < b < c. \end{cases}$$

Then the following assertions (i)–(iii) hold:

(i) (See Fig. 4.) If $2b(a+c) > b^2 + 6ac$, then the bifurcation diagram Σ of (1.14) is broken reversed S-shaped on the $(\lambda, ||u||_\infty)$-plane. Moreover, there exist $\bar{\lambda} > \lambda_* > 0$ such that (1.14) has exactly three positive solutions u_λ, v_λ, w_λ with $u_\lambda < v_\lambda < w_\lambda$ for all $\lambda_* < \lambda \leq \bar{\lambda}$, exactly two positive solutions v_λ, w_λ with $v_\lambda < w_\lambda$ for $\lambda = \lambda_*$ and $\lambda > \bar{\lambda}$, and exactly one positive solution w_λ for $0 < \lambda < \lambda_*$. More precisely,

$$\lim_{\lambda \to \bar{\lambda}^-} \|u_\lambda\|_\infty = \|u_{\bar{\lambda}}\|_\infty = \bar{b},\ \lim_{\lambda \to \infty} \|v_\lambda\|_\infty = b,\ \lim_{\lambda \to 0^+} \|w_\lambda\|_\infty = \infty,\ \lim_{\lambda \to \infty} \|w_\lambda\|_\infty = \bar{c}.$$

(ii) (See Fig. 5(ii).) If $2b(a+c) = b^2 + 6ac$, then the bifurcation diagram Σ of (1.14) is a monotone decreasing curve on the $(\lambda, ||u||_\infty)$-plane. Moreover, (1.14) has exactly one positive solution u_λ for all $\lambda > 0$. More precisely,

$$\lim_{\lambda \to 0^+} \|u_\lambda\|_\infty = \infty \text{ and } \lim_{\lambda \to \infty} \|u_\lambda\|_\infty = \mu.$$

(iii) (See Fig. 5(i).) If $2b(a+c) < b^2 + 6ac$, then the bifurcation diagram Σ of (1.14) is a monotone decreasing curve on the $(\lambda, ||u||_\infty)$-plane. Moreover, there exists $\bar{\lambda} > 0$ such that (1.14) has exactly one positive solution u_λ for $0 < \lambda \leq \bar{\lambda}$, and no positive solution for $\lambda > \bar{\lambda}$. More precisely,

$$\lim_{\lambda \to 0^+} \|u_\lambda\|_\infty = \infty \text{ and } \lim_{\lambda \to \bar{\lambda}^-} \|u_\lambda\|_\infty = \|u_{\bar{\lambda}}\|_\infty = \mu.$$

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