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Stability of stationary solutions for semilinear parabolic equations

Ryuji Kajikiya
Department of Mathematics, Faculty of Science and Engineering,
Saga University, Saga, 840-8502, Japan
E-mail: kajikiya@ms.saga-u.ac.jp

We study the stability of stationary solutions for the semilinear parabolic equation,

$$\begin{aligned} u_t - \Delta u &= f(x,u) \quad &&\text{in } \Omega \times (0,\infty), \\ u &= 0 &&\text{on } \partial\Omega \times (0,\infty), \\ u(x,0) &= u_0(x) &&\text{in } \Omega, \end{aligned} \tag{1}$$

where Ω is a bounded smooth domain in $\mathbb{R}^N$.

Definition 1. We call $u(x,t)$ a *solution* of (1) if it belongs to the following space and satisfies (1):

$$C([0,\infty); L^2(\Omega)) \cap C^1((0,\infty); L^2(\Omega)) \cap C((0,\infty); H^2(\Omega) \cap H_0^1(\Omega)).$$

We suppose the following assumption.

Assumption 2. $f(x,u)$ is a Hölder continuous function on $\overline{\Omega} \times \mathbb{R}$ which is odd with respect to u and satisfies $|f(x,u)| \le C(|u|^p + 1)$ for $u \in \mathbb{R}$ and $x \in \overline{\Omega}$, with some $C > 0$, where $1 < p < \infty$ when $N = 1, 2$ and $1 < p < N/(N-2)$ when $N \ge 3$. For each $u \ne 0$, the second partial derivative $f_{uu}(x,u)$ exists and continuous on $\overline{\Omega} \times (\mathbb{R} \setminus \{0\})$ and there exists $L, u_0 > 0$, $\theta_0 \in (0,1)$ such that $|f_{uu}(x,v)| \le L|f_u(x,u)|/u + L/u$ for $0 < u < u_0$ and $v \in [(1-\theta_0)u, (1+\theta_0)u]$. Moreover we assume

$$\frac{\partial}{\partial u}\left(\frac{f(x,u)}{u}\right) < 0 \quad \text{for } u > 0.$$

Assumption 3. Let λ_1 be the first eigenvalue of the Laplacian. We assume that

$$\limsup_{|u|\to\infty}(\max_{x\in\overline{\Omega}} f(x,u)/u) < \lambda_1, \quad \lim_{u\to 0}\left(\min_{x\in\overline{\Omega}} f_u(x,u)\right) = \infty.$$

We define

$$E(u) := \int_\Omega \left(\frac{1}{2}|\nabla u|^2 - F(x,u)\right) dx, \quad F(x,u) := \int_0^u f(x,s)ds.$$

Then $E(u)$ becomes a Lyapunov functional of (1). The stationary problem is as follows:

$$-\Delta v = f(x,v) \quad (x\in\Omega), \qquad v = 0 \quad (x \in \partial\Omega). \tag{2}$$

Proposition 4. *The following results are known. See [2, 3, 4].*

(i) *There exists a unique positive solution ϕ of (2); moreover ϕ is a minimizer of E in $H_0^1(\Omega)$ and all minimizers of E consist only of $\pm\phi$.*

(ii) *There exists a sequence v_n of non-trivial solutions for (2) such that v_n converges to zero in $C^2(\overline{\Omega})$ as $n \to \infty$.*

Definition 5. In the following, $u(t)$ means a solution of (1).

(i) A stationary solution v is called *stable* if for any $\varepsilon > 0$, there exists a $\delta > 0$ such that $\|u(0) - v\|_{H_0^1(\Omega)} < \delta$ implies $\|u(t) - v\|_{H_0^1(\Omega)} < \varepsilon$ for $t \geq 0$.

(ii) A stationary solution v is called *asymptotically stable* if v is stable and there exists a $\delta_0 > 0$ such that if $\|u(0) - v\|_{H_0^1(\Omega)} < \delta_0$ then $\lim_{t\to\infty} \|u(t) - v\|_{H_0^1(\Omega)} = 0$.

(iii) A stationary solution v is called *exponentially stable* if v is stable and there exist constants $C, \lambda, \delta_0 > 0$ such that $\|u(0) - v\|_{H_0^1(\Omega)} < \delta_0$ implies $\|u(t) - v\|_{H_0^1(\Omega)} \leq Ce^{-\lambda t}$ for all $t \geq 0$.

We state the main results.

Theorem 6. *For any $u_0 \in H_0^1(\Omega)$, (1) has a bounded global solution $u(t)$ in $H_0^1(\Omega)$. The orbit of solution $u(t)$ is relatively compact. The ω limit set is a non-empty subset of the set of stationary solutions.*

Theorem 7. *There exists an $\varepsilon_0 > 0$ such that if v is a stationary solution satisfying $\|v\|_\infty < \varepsilon_0$, then it is not asymptotically stable. Furthermore, if v is isolated from other stationary solutions, it is unstable. The zero solution is unstable.*

Theorem 8. *The unique positive stationary solution ϕ is exponentially stable. Moreover the exponent is the the first eigenvalue of the linearlized operator $-\Delta - f_u(x,\phi)$. Denote it by $\mu > 0$. Then there exists a $\delta > 0$ such that if $u(t)$ is a solution of (1) satisfying $\|u(0) - \phi\|_{H_0^1} < \delta$, then $\|u(t) - \phi\|_{H_0^1} \le Ce^{-\mu t}$ for $t \ge 0$ with some $C > 0$.*

The exponent μ is optimal. Indeed, we have the theorem below.

Theorem 9. *Let $u_0 \in H_0^1(\Omega)$ satisfy either*

$$u_0(x) \ge (1+\delta_0)\phi(x) \quad \text{or} \quad 0 < u_0(x) \le (1-\delta_0)\phi(x).$$

with some $\delta_0 \in (0,1)$. Then there exists a $c > 0$ such that a solution $u(t)$ with the initial data $u(0) = u_0$ satisfies

$$\|u(t) - \phi\|_{H_0^1} \ge \|u(t) - \phi\|_2 \ge ce^{-\mu t} \quad \text{for } t \ge 0.$$

Let $N = 1$, $\Omega = (0,1)$ and $f(x,u) \equiv f(u)$. Then the stationary problem is rewritten as

$$-v'' = f(v) \quad (x \in (0,1)), \qquad v(0) = v(1) = 0. \tag{3}$$

If a solution $v(x)$ of (3) has exactly k zeros in the interval $(0,1)$, we call it a *k-nodal solution.* The next result is known (see [6] and [7]).

Proposition 10. *Let $N = 1$, $\Omega = (0,1)$ and $f(x,u) \equiv f(u)$. Then for each $k \ge 1$, there exists a unique $(k-1)$-nodal solution v_k of (3) satisfying $v'(0) > 0$. The set of all solutions for (3) consists of $\pm v_k$ with $k \in \mathbb{N}$ and the zero solution.*

Let v_k be a stationary solution as above. Then we have the next result.

Theorem 11. *The positive stationary solution v_1 and the negative stationary solution $-v_1$ are exponentially stable with the exact exponent μ and all the other stationary solutions are unstable.*

When $f(x,u) = |u|^{p-1}u$, the results above were obtained in a joint work with Professor Akagi [1]. Theorems in this paper are extensions of those results to more general functions $f(x,u)$. From now on, we put $f(x,u) = |u|^{p-1}u$ with $0 < p < 1$ for simplicity. We prove the stability only of positive stationary solution.

Lemma 12. *$E(u)$ is a Lyapunov functional.*

Proof. For a solution $u(t)$ of (1), a direct computation shows

$$\begin{aligned}\frac{d}{dt}E(u(t)) &= \int_\Omega \left(\nabla u \nabla u_t - |u|^{p-1}uu_t\right) dx \\ &= \int_\Omega \left((-\Delta u - |u|^{p-1}u)u_t\right) dx = -\int_\Omega |u_t|^2 dx \le 0.\end{aligned}$$

Therefore E is a Lyapunov functional. □

Lemma 13. *The unique positive stationary solution ϕ is isolated from other stationary solutions.*

Proof. Suppose on the contrary that there exists a sequence $\{u_n\}$ of stationary solutions which converges to ϕ in $H_0^1(\Omega)$. Then the elliptic regularity theorem shows that this convergence is valid in the strong topology in $C^2(\overline{\Omega})$. Since the outward normal derivative $\partial\phi/\partial\nu$ is negative on $\partial\Omega$, it holds that $\partial u_n/\partial\nu < 0$ also for n large. Therefore $u_n > 0$ in Ω for n large. This contradicts the uniqueness of the positive stationary solution. □

We shall show the asymptotic stability of the unique positive stationary solution ϕ.

Proof of asymptotic stability. Let ϕ be the unique positive stationary solution. Choose $\varepsilon_0 > 0$ so small that there are no stationary solutions in $B(\phi, \varepsilon_0)$ except for ϕ, where

$$B(\phi, \varepsilon_0) := \{v \in H_0^1(\Omega) : \|v - \phi\|_{H_0^1} < \varepsilon_0\}.$$

Define $d := \inf_{H_0^1} E(u)$. Then $E(u) = d$ if and only if $u = \pm\phi$. Give $\varepsilon \in (0, \varepsilon_0)$ arbitrarily. We shall show

$$d_\varepsilon := \inf\{E(v) : v \in H_0^1(\Omega),\ \|v - \phi\|_{H_0^1} = \varepsilon\} > d.$$

Suppose that this claim is false, i.e., $d_\varepsilon = d$. Then there exists a sequence $v_n \in H_0^1(\Omega)$ such that $\|v_n - \phi\|_{H_0^1} = \varepsilon$ and $E(v_n) \to d_\varepsilon = d$. Since v_n is bounded in $H_0^1(\Omega)$, it has a convergent subsequence (denoted by v_n again) to a weak limit $v \in H_0^1(\Omega)$. This convergence is valid in the strong topology in $L^{p+1}(\Omega)$. Accordingly, we have

$$\begin{aligned}&\frac{1}{2}\|\nabla v_n\|_2^2 = E(v_n) + \frac{1}{p+1}\|v_n\|_{p+1}^{p+1} \\ &\qquad \to d + \frac{1}{p+1}\|v\|_{p+1}^{p+1} \le E(v) + \frac{1}{p+1}\|v\|_{p+1}^{p+1} = \frac{1}{2}\|\nabla v\|_2^2.\end{aligned}$$

Hence, $\limsup_{n\to\infty}\|\nabla v_n\|_2 \le \|\nabla v\|_2$. The weak convergence shows that $\liminf_{n\to\infty}\|\nabla v_n\|_2 \ge \|\nabla v\|_2$. Therefore $\|\nabla v_n\|_2$ converges to $\|\nabla v\|_2$, and hence v_n strongly converges to v. Thus $\|v-\phi\|_{H_0^1} = \varepsilon$ and $E(v) = d$. This is a contradiction. Consequently, $d_\varepsilon > d$.

Since $d < d_\varepsilon$, we can choose $\delta \in (0,\varepsilon)$ so small that $E(u_0) < d_\varepsilon$ for $u_0 \in B(\phi,\delta)$. Let $u_0 \in B(\phi,\delta)$ and let $u(t)$ be a solution of (1) satisfying $u(0) = u_0$. We shall show that

$$u(t) \in B(\phi,\varepsilon) \quad \text{for all } t > 0. \tag{4}$$

If this would be proved, then ϕ is stable. Suppose that the assertion above is false. Then there exists a $t_0 > 0$ such that $u(t_0) \in \partial B(\phi,\varepsilon)$. Then $E(u(t_0)) \ge d_\varepsilon$. Since E is a Lyapunov functional,

$$d_\varepsilon \le E(u(t_0)) \le E(u_0) < d_\varepsilon.$$

A contradiction occurs. Hence (4) is true and ϕ is stable.

Since the orbit is relatively compact, $u(t)$ converges to a stationary solution along a subsequence. Since ϕ is the unique stationary solution in $B(\phi,\varepsilon)$, $u(t)$ itself (without a subsequence) converges to ϕ. Therefore ϕ is asymptotically stable. □

Since $0 < p < 1$, $\phi(x)^{p-1}$ has a singularity on $\partial\Omega$. However we have the next result (see [5]).

Lemma 14. *The linearlized operator $-\Delta - p\phi^{p-1}$ is self-adjoint and has a compact resolvent in $L^2(\Omega)$.*

By the lemma above, $-\Delta - p\phi^{p-1}$ has discrete eigenvalues in $\mathbb{R}$. Since ϕ is a positive solution of (2) with $f(x,u) \equiv |u|^{p-1}u$, it satisfies $(-\Delta-\phi^{p-1})\phi = 0$. Therefore the first eigenvalue of $-\Delta - \phi^{p-1}$ is zero. Since $-\phi^{p-1} < -p\phi^{p-1}$, we have the result below.

Lemma 15. *The first eigenvalue of $-\Delta - p\phi^{p-1}$ is positive.*

Let μ and $\psi(x)$ be the first eigenvalue and the eigenfunction of $-\Delta - p\phi^{p-1}$, that is,

$$(-\Delta - p\phi^{p-1})\psi = \mu\psi, \quad \psi > 0 \quad \text{in } \Omega, \qquad \psi = 0 \quad \text{on } \partial\Omega.$$

Moreover, we assume that $\|\nabla\psi\|_2 = 1$. Since $\phi,\psi > 0$ $(x \in \Omega)$, $\phi,\psi \in C^2(\overline{\Omega})$, $\partial\phi/\partial\nu, \partial\psi/\partial\nu < 0$ $(x \in \partial\Omega)$, there exists a $c_0 > 0$ such that $c_0 \le \phi(x)/\psi(x)$ for $x \in \Omega$. For $c \in \mathbb{R}$, we define

$$U(x,t;c) := \phi(x) + ce^{-\mu t}\psi(x). \tag{5}$$

The next three lemmas are proved in our paper [5].

Lemma 16. *For $-c_0 < c < \infty$, $U(x,t;c)$ is a positive supersolution of* (1).

Let λ_1 be the first eigenvalue of $-\Delta$ and let ϕ_1 be the corresponding eigenfunction, i.e.,

$$-\Delta\phi_1 = \lambda_1\phi_1, \quad \phi_1 > 0 \quad (x \in \Omega), \qquad \phi_1 = 0 \quad (x \in \partial\Omega).$$

Define $\xi(t) := \mu(e^{\mu t}+1)^{-1}$. For $\varepsilon > 0$ small, we define

$$V(x,t;\varepsilon) := \phi(x) - \varepsilon^2\xi(t)\psi(x) + \varepsilon^3 e^{-2\mu t}\phi_1(x). \tag{6}$$

Lemma 17. *For $\varepsilon > 0$ small, $V(x,t;\varepsilon)$ is a positive subsolution of* (1).

Using the supersolution $U(x,t;c)$ defined by (5) and the subsolution $V(x,t;\varepsilon)$ given by (6), we can obtain the next lemma.

Lemma 18. *Let $u(x,t)$ be a solution of* (1) *with its initial data $u(0)$ close to ϕ. Let $t_0 > 0$. Then there exists a constant $C > 0$ such that*

$$\left\| \frac{u(\cdot,t)}{\phi} - 1 \right\|_{L^\infty(\Omega)} \le Ce^{-\mu t} \quad \textit{for } t \ge t_0.$$

For $1 < q < \infty$, we define $Au := -\Delta u$ with its domain $D(A)$,

$$D(A) := W^{2,q}(\Omega) \cap W_0^{1,q}(\Omega).$$

Then the fractional power A^α with $\alpha > 0$ is well-defined. Denote its definition domain by $X(\alpha,q)$, i.e.,

$$X(\alpha,q) := \{u \in L^q(\Omega) : A^\alpha u \in L^q(\Omega)\},$$

This is a Banach space equipped with the norm,

$$\|u\|_{X(\alpha,q)} := \|A^\alpha u\|_q \quad \text{for } u \in X(\alpha,q).$$

We shall prove Theorem 9 only and we refer to our paper [5] for proofs of other theorems.

Proof of Theorem 9. Let $u(x,t)$ be a solution of (1) such that $\|u(0)-\phi\|_{H_0^1}$ is small enough. We have only to prove

$$\|u(t)-\phi\|_{C^1} \le Ce^{-\mu t} \quad \text{for } t \ge T,$$

with $T > 0$ large. Fix $T > 0$ so large that $u(x,T) > 0$ in Ω. Rewrite it as $u_0(x)$. Then $u_0 \in X(\alpha,q)$. We have

$$u_t - \Delta u = u^p, \quad -\Delta\phi = \phi^p.$$

We define

$$v(x,t) := u(x,t) - \phi(x), \quad v_0 := u_0 - \phi, \quad g(x,t) := u(x,t)^p - \phi(x)^p.$$

Then it follows that

$$v_t - \Delta v = g(x,t), \quad v|_{\partial\Omega} = 0, \quad v(\cdot,0) = v_0.$$

This is rewritten as

$$v(t) = e^{-tA}v_0 + \int_0^t e^{-(t-s)A}g(s)ds \quad \text{in } L^q(\Omega), \tag{7}$$

Recall that λ_1 and μ are the first eigenvalues of $-\Delta$ and $-\Delta - p\phi^{p-1}$, respectively. Hence $\lambda_1 > \mu$. Fix λ satisfying $\mu < \lambda < \lambda_1$. Then it is known that

$$\|A^\alpha e^{-tA}v\|_q \le C_{\alpha,q}t^{-\alpha}e^{-\lambda t}\|v\|_q \quad \text{for } v \in L^q(\Omega).$$

Applying A^α to both sides of (7), we obtain

$$A^\alpha v(t) = e^{-tA}A^\alpha v_0 + \int_0^t A^\alpha e^{-(t-s)A}g(s)ds \quad \text{in } L^q(\Omega).$$

Taking the L^q norm, we get

$$\|v(t)\|_{X(\alpha,q)} \le e^{-\lambda t}\|v_0\|_{X(\alpha,q)} + C_{\alpha,q}\int_0^t (t-s)^{-\alpha}e^{-\lambda(t-s)}\|g(s)\|_q ds.$$

Let us estimate $\|g(s)\|_q$. Using the inequality $0 \le (t^p - s^p)/(t-s) \le s^{p-1}$ for $t, s > 0$, we find

$$|g(x,s)| = \left|\frac{u^p - \phi^p}{u - \phi}(u-\phi)\right| \le \phi^{p-1}|u-\phi|.$$

Hence

$$\|g(s)\|_\infty \le \|\phi^p((u/\phi)-1)\|_\infty \le Ce^{-\mu s}.$$

Employing this inequality, we get

$$\|v(t)\|_{X(\alpha,q)} \le e^{-\lambda t}\|v_0\|_{X(\alpha,q)} + C_{\alpha,q}\int_0^t (t-s)^{-\alpha}e^{-\lambda(t-s)}e^{-\mu s}ds.$$

Putting $\tau = t - s$ and using $\lambda > \mu$, we obtain

$$\int_0^t (t-s)^{-\alpha}e^{-\lambda(t-s)}e^{-\mu s}ds \le Ce^{-\mu t}\int_0^\infty \tau^{-\alpha}e^{-(\lambda-\mu)\tau}d\tau.$$

Therefore

$$\|v(t)\|_{X(\alpha,q)} \le e^{-\lambda t}\|v_0\|_{X(\alpha,q)} + \tilde{C}_{\alpha,q}e^{-\mu t}.$$

Give $\theta \in (0,1)$. Choose $\alpha \in (0,1)$ close to 1 and take q large enough. Then the embedding $X(\alpha,q) \hookrightarrow C^{1,\theta}(\overline{\Omega})$ holds.

$$\|v(t)\|_{C^1} \le Ce^{-\lambda t}\|v_0\|_{X(\alpha,q)} + Ce^{-\mu t}.$$

Since $\lambda > \mu$, we have

$$\|u(t) - \phi\|_{C^1} = \|v(t)\|_{C^1} \le Ce^{-\mu t}.$$

The proof is complete. □

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