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Remarks on two critical exponents for Hénon type equation on the hyperbolic space

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1 Introduction

We devote this paper to an announcement about results in [9]. More precisely, we shall introduce critical exponents with respect to the sign of radial solutions to Hénon type equation on the hyperbolic space:

$$\text{(H)} \qquad -\Delta_g u = (\sinh r)^{\alpha} |u|^{p-1} u \quad \text{in} \quad \mathbb{H}^N,$$

where $N \geq 2$, $p > 1$, and $\alpha > -2$. We denote by $\mathbb{H}^N$ the N-dimensional hyperbolic space, i.e., let $\mathbb{H}^N$ be a manifold admitting a pole o and whose metric g is defined, in the polar coordinates around o, by

$$ds^2 = dr^2 + (\sinh r)^2 d\Theta^2, \quad r > 0, \quad \Theta \in \mathbb{S}^{N-1},$$

where $d\Theta^2$ denotes the canonical metric on the unit sphere $\mathbb{S}^{N-1}$, r is the geodesic distance between o and a point (r, Θ). Moreover, Δ_g denotes the Laplace-Beltrami operator on $(\mathbb{H}^N, g)$ given by

$$\begin{aligned}\Delta_g f(r, \theta_1, ..., \theta_{N-1}) =& (\sinh r)^{-(N-1)} \partial_r \left\{ (\sinh r)^{N-1} \partial_r f(r, \theta_1, ..., \theta_{N-1}) \right\} \\ &+ (\sinh r)^{-2} \Delta_{\mathbb{S}^{N-1}} f(r, \theta_1, ..., \theta_{N-1}),\end{aligned}$$

where $f : \mathbb{H}^N \to \mathbb{R}$ is a scalar function and $\Delta_{\mathbb{S}^{N-1}}$ is the Laplace-Beltrami operator on the unit ball $\mathbb{S}^{N-1}$. In order to state known results on the sign of radial solutions to (L), we define several notations. We denote the Sobolev exponent by $p_s(N, \alpha)$, i.e.,

$$p_s(N, \alpha) = \frac{N + 2 + 2\alpha}{N - 2}.$$

Furthermore, we define several classes of radial solutions to (H) as follows:

Definition 1.1. *For each $\beta > 0$, let $u_\beta = u_\beta(r)$ be the radial solution of* (H) *satisfying $u_\beta(0) = \beta$.*

(i) *We say that u_β is Type O, if u_β has infinitely many zeros in $(0, \infty)$.*

(ii) *We say that u_β is Type R, if u_β has finitely many zeros in $(0, \infty)$ and satisfies*

$$\lim_{r \to \infty} (\sinh r)^{N-1} |u_\beta(r)| = \gamma \quad \textit{for some} \quad \gamma > 0.$$

(iii) *We say that u_β is Type S, if u_β has finitely many zeros in $(0,\infty)$ and satisfies*

$$\lim_{r\to\infty}(\sinh r)^{N-1}|u_\beta(r)| = \infty.$$

We note that Definition 1.1 is inspired by [16]. First we introduce known results on (H) for the case of $\alpha = 0$.

From 2000's, the following Lane-Emden equation on the hyperbolic space is well-investigated ([1, 3, 10, 11, 12, 14, 15]):

$$\text{(L)} \qquad -\Delta_g u = |u|^{p-1}u \quad \text{in} \quad \mathbb{H}^N,$$

where $N \geq 3$ and $p > 1$. Here, for each $\beta > 0$, we denote by $u_\beta^L = u_\beta^L(r)$ the radial solution of (L) satisfying $u_\beta^L(0) = \beta$. Then the following result is obtained:

Proposition 1.1 ([4, 12]). *Let $N \geq 3$, and $p > 1$.*

(i) *Let $p < p_s(N,0)$. Then there exists $\beta_L = \beta_L(N,p) > 0$ such that the following hold:*
- *If $\beta < \beta_L$, then u_β^L is positive in $[0,\infty)$ and is Type S;*
- *If $\beta = \beta_L$, then u_β^L is positive in $[0,\infty)$ and is Type R;*
- *If $\beta > \beta_L$, then u_β^L is sign-changing and has finitely many zeros in $[0,\infty)$.*

(ii) *Let $p \geq p_s(N,0)$. Then for any $\beta > 0$, u_β^L is positive in $[0,\infty)$ and is Type S.*

Remark that Proposition 1.1 implies that $p_s(N,0)$ is critical on the existence of sign-changing radial solutions to (L). However, $p_s(N,0)$ is not critical with respect to the existence of positive solutions of (L).

Here, the assertion for the case of $p < p_s(N,0)$ and $\beta = \beta_L$ in Proposition 1.1 was proved in [12]. Indeed, making use of the variational methods, they proved the existence of the positive radial solution of Type S. The rest of the assertions of Proposition 1.1 were obtained in [4]. Furthermore, in [4] they obtained the precise result on the asymptotic behavior of radial solutions. Indeed, they showed that the decay rate of radial solutions of Type S is given by

$$\text{(1.1)} \qquad \lim_{r\to\infty} r^{\frac{1}{p-1}}|u_\beta^L(r)| = \left(\frac{N-1}{p-1}\right)^{\frac{1}{p-1}}.$$

Next, we state known results on the stability of solutions to (L). In [2], it was proved that there exist stable, positive, and radial solutions of (L) for any $p > 1$. This result implies that critical exponents with respect to the stability of solutions do not exist for (L).

On the other hand, we observe from the case of the Euclidean space that critical exponents depend on a weight of equation. Indeed, we introduce the results on the existence of critical exponents for the following Hénon equation in $\mathbb{R}^N$:

$$\text{(E)} \qquad -\Delta u = |x|^\alpha |u|^{p-1}u \quad \text{in} \quad \mathbb{R}^N,$$

where $N \geq 3$, $p > 1$, and $\alpha > -2$. Then all regular radial solutions to (E) are positive for $p \geq p_s(N,\alpha)$ and have infinitely many zeros for $p < p_s(N,\alpha)$ (e.g., see [16]). Hence, $p_s(N,\alpha)$ is critical not only on the positivity of radial solutions but also on the existence of sign-changing radial solutions of (E). Moreover, there exists a critical exponent $p_{JL}(N,\alpha) > p_s(N,\alpha)$ on the stability of solutions to (E). Indeed, if $p < p_{JL}(N,\alpha)$, then there exist no non-trivial stable solutions, and if $p \geq p_{JL}(N,\alpha)$, then there exist stable, radial, and positive solutions ([5, 6]). Here, the exponent $p_{JL}(N,0)$ is called the Joseph-Lundgren exponent.

The critical exponents of (E) depend on the weight. Thus, it is natural to investigate the existence of critical exponents for (L) with a weight. Indeed, we considered the Hénon type equation (H) on $\mathbb{H}^N$ in [7]-[8]. Here the weight of (H) denotes the power of the volume element of $\mathbb{H}^N$.

We have already obtained the result on the existence of a critical exponent on the stability of solutions to (H) in [7]-[8]. Before we state our results for (H), we prepare some notations. For each $\beta > 0$, we denote by $u_\beta = u_\beta(r)$ the family of radial regular solutions of (H) with $u_\beta(0) = \beta$, i.e., u_β is the solution of the following initial value problem:

$$\text{(Hr)} \qquad \begin{cases} u''(r) + \dfrac{N-1}{\tanh r}u'(r) + (\sinh r)^\alpha |u(r)|^{p-1}u(r) = 0 \quad \text{in} \quad (0,+\infty), \\ u(0) = \beta. \end{cases}$$

Moreover, we define the exponent $p_c(N,\alpha)$ by

$$p_c(N,\alpha) := \begin{cases} +\infty & \text{if} \quad N \leq 1 + 4\alpha, \\ \dfrac{(N-1)^2 - 2\alpha(N-1) - 2\alpha^2 + 2\alpha\sqrt{2\alpha(N-1)+\alpha^2}}{(N-1)(N-4\alpha-1)} & \text{if} \quad N > 1 + 4\alpha. \end{cases}$$

Then we obtained the following result:

Proposition 1.2 ([7, 8]). *Let $N \geq 2$ and $\alpha > 0$. Then the following hold:*

(i) *If $1 < p < p_c(N,\alpha)$, then* (H) *has no non-trivial stable solutions;*

(ii) *If $p > p_c(N,\alpha)$, then there exists $\bar{\beta} = \bar{\beta}(N,p,\alpha) > 0$ such that u_β is stable for any $\beta \in (0,\bar{\beta}]$.*

Here, we mention the case where $\alpha \geq (N-1)/4$. The condition $\alpha \geq (N-1)/4$ yields $p_c(N,\alpha) = +\infty$ from the definition of $p_c(N,\alpha)$. Therefore, the assertion (i) in Proposition 1.2 implies that (H) has no non-trivial stable solutions. Namely, if $\alpha \geq (N-1)/4$, then the case of the assertion (ii) does not occur.

We observe from Proposition 1.2 that the exponent $p_c(N,\alpha)$ is critical on the existence of non-trivial stable solutions of (H) and the result is completely different from that of (L). Hence, we can expect that there is also a critical exponent with respect to the existence of sign-changing radial solutions of (H).

Following the motivation, we shall investigate the sign of solutions to (H). In order to state our main results, we set the exponent $p_b(N,\alpha)$ as

$$p_b(N,\alpha)=\frac{N-1+2\alpha}{N-1}.$$

Then, focusing on radial solutions to (H), we obtain the following result on the existence of critical exponents with respect to the sign of radial solutions:

Theorem 1.1 ([9]). *Let $N\geq 3$, $p>1$, and $\alpha>0$.*

(i) *Let $p<p_b(N,\alpha)$. Then for any $\beta>0$, u_β is Type O.*

(ii) *Let $p_b(N,\alpha)<p<p_s(N,\alpha)$. Then there exists $\beta_H=\beta_H(N,p,\alpha)>0$ such that the following hold:*
If $\beta<\beta_H$, then u_β is positive in $[0,\infty)$ and is Type S;
If $\beta=\beta_H$, then u_β is positive in $[0,\infty)$ and is Type R;
If $\beta>\beta_H$, then u_β is sign-changing and has finitely many zeros in $[0,\infty)$.

(iii) *Let $p>p_s(N,\alpha)$, then for any $\beta>0$, u_β is positive in $[0,\infty)$ and is Type S.*

Remark that Theorem 1.1 also holds true for $\alpha=0$ by Proposition 1.1. Here, there exist two critical exponents on the sign of radial solutions to (H). Indeed, $p_b(N,\alpha)$ is critical on the positivity of radial solutions, while $p_s(N,\alpha)$ is also critical on the existence of sign-changing radial solutions.

We obtain further results on radial solutions of (H). Positive solutions of Type S satisfy the following asymptotic behavior:

Theorem 1.2 ([9]). *Let $N\geq 3$, $\alpha>0$, and the pair (p,β) satisfy*

$$p_b(N,\alpha)<p<p_s(N,\alpha),\ \ 0<\beta<\beta_H,\quad or\quad p>p_s(N,\alpha),\ \ \beta>0.$$

Then it holds that

$$\lim_{r\to+\infty}u_\beta(r)\,(\sinh r)^{\frac{\alpha}{p-1}}=\left\{\frac{\alpha}{p-1}\left(N-1-\frac{\alpha}{p-1}\right)\right\}^{\frac{1}{p-1}}.$$

We observe from (1.1) and Theorem 1.2 that the decay rate of radial positive solutions of Type S for the case of $\alpha>0$ is different from that for the case of $\alpha=0$.

For the equation (L), the existence of sign-changing solutions of Type S has been already proved by Theorem 2.4 in [4] for $1<p<p_s(N,0)$. On the other hand, from the result in the Euclidean space (e.g., see [16]), we expect that there exists sign-changing solutions of type R for the equation (H). Indeed, the following result is obtained:

Theorem 1.3 ([9]). *Let $N\geq 3$, $\alpha>-2$, and $\max\{1,p_b(N,\alpha)\}<p<p_s(N,\alpha)$. Then there exist strictly increasing positive divergent sequences $\{\beta_i\}_{i\in\mathbb{N}}$ and $\{\gamma_i\}_{i\in\mathbb{N}}$ such that u_{β_i} has just i zeros on $[0,+\infty)$ and satisfies $(\sinh r)^{N-1}u_{\beta_i}(r)\to(-1)^i\gamma_i$ as $r\to\infty$.*

Remark that $\beta_0 = \beta_L$ if $\alpha = 0$, and $\beta_0 = \beta_H$ if $\alpha > 0$, where β_L and β_H are defined in Proposition 1.1 and Theorem 1.1 respectively. Theorem 1.3 implies that there exist radial solutions of Type R for the case of $p_b(N,\alpha) < p < p_s(N,\alpha)$ and $\beta > \beta_H$ in Theorem 1.1. Furthermore, for the equation (L), i.e., for the case of $\alpha = 0$, Theorem 1.3 also clarifies the existence of radial solutions of Type R when $1 < p < p_s(N,0)$ and $\beta > \beta_L$ in Proposition 1.1.

In order to prove Theorems 1.1–1.3, we need to verify the existence and the uniqueness of a solution of (Hr). In particular, for the proof of Theorem 1.3, we shall study a solution of the following initial value problem:

$$
(1.2) \qquad \begin{cases} u''(r) + \dfrac{N-1}{\tanh r}u'(r) + \lambda u(r) + (\sinh r)^{\alpha}|u(r)|^{p-1}u(r) = 0 \quad \text{in} \quad (0,\infty), \\ u(0) = \beta. \end{cases}
$$

where $N \geq 2$, $p > 1$, $\alpha > -2$, and $\lambda \geq 0$. Remark that we impose only the value of $u(0)$ in (1.2) and do not impose the value of $u'(0)$. The existence of the solution to (1.2) is proved as follows:

Theorem 1.4. *Let $N \geq 2$, $p > 1$, $\alpha > -2$, and $\lambda \geq 0$. For each $\beta > 0$, then there exists a unique solution $u_\beta \in C([0,\infty)) \cap C^2((0,\infty))$ of the initial value problem* (1.2).

Remark 1.1. *In particular, the solution $u_\beta \in C([0,\infty)) \cap C^2((0,\infty))$, stated in Theorem* 1.4, *satisfies the following*:

$$
\begin{aligned} &u_\beta \in C^2([0,\infty)) \quad \text{if} \quad \alpha \geq 0; \\ &u_\beta \in C^1([0,\infty)) \quad \text{if} \quad -1 \leq \alpha < 0. \end{aligned}
$$

For the proof of Theorems 1.1–1.3, see [9]. We devote the rest of this paper to proving Theorem 1.4 and Remark 1.1. More precisely, in Lemmas 2.1-2.2, making use of the successive approximation and the fixed point theorem, we shall show the existence and the uniqueness of a solution to an integral equation in a local interval. Moreover, we observe from Lemmas 2.3-2.4 that a solution of the integral equation is a "local-in-time" solution of (1.2). Lemmas 2.3-2.4 are obtained by direct calculations and the asymptotic behavior of the derivative of the solution at the origin. Then using the fact that the solution is bounded in C^1, we prove that the solution can be extended globally in Lemma 2.5. Hence we complete the proof of Theorem 1.4. For the proof of Remark 1.1, using the integral equation, we study the asymptotic behaviors of the derivative and the second derivative of the solution to (1.2) at the origin in Lemmas 2.6-2.7. The proofs of Lemmas 2.6-2.7 are the modification of Proposition 4.4 in [13].

2 Proof of Theorem 1.4 and Remark 1.1

In the following, for each $\beta > 0$, we consider the initial value problem (1.2). To begin with, we shall prove the existence and uniqueness of a solutions to (1.2).

We start with the existence and the uniqueness of "local-in-time solution" to (1.2). Now we study the solutions of the following integral equations:

$$u(r) = \beta - \int_0^r \frac{1}{(\sinh s)^{N-1}} \int_0^s (\sinh t)^{N-1}\{\lambda u(t) + (\sinh t)^\alpha |u(t)|^{p-1}u(t)\}dtds. \tag{2.1}$$

Lemma 2.1. *Let $N \geq 2$, $p > 1$, $\alpha > -2$, $\beta > 0$, and $\lambda \geq 0$. Then there exists a constant $\delta > 0$ such that the integral equation* (2.1) *has a unique solution $u \in C([0,\delta])$ with $u(0) = \beta$.*

Proof. To begin with, we shall show the existence of solution to (2.1) by the successive approximation. For this purpose, we define notations. Set the function space X by

$$X := \{u \in C([0,\delta]) : |u| \leq M \quad \text{in} \quad [0,\delta]\},$$

where $\delta > 0$ is sufficiently small and $M > 0$ is the constant satisfying

$$2\beta < M. \tag{2.2}$$

Moreover, we define the mapping $\Phi : C([0,\delta]) \to C([0,\delta])$ by

$$\Phi(u)(r) = \beta - \int_0^r \frac{1}{(\sinh s)^{N-1}} \int_0^s (\sinh t)^{N-1}\{\lambda u(t) + (\sinh t)^\alpha |u(t)|^{p-1}u(t)\}dtds.$$

Furthermore, we define inductively the sequence $\{u_i\}_{i\in\mathbb{N}}$ as

$$u_0 = \beta, \quad u_{i+1} = \Phi(u_i) \quad \text{in} \quad [0,\delta].$$

Then we claim that

$$|u_i(r)| \leq M \quad \text{for any} \quad i \in \mathbb{N} \quad \text{and} \quad r \in [0,\delta]. \tag{2.3}$$

When $i = 0$, we observe from (2.2) that

$$u_0(r) = \beta < M \quad \text{in} \quad [0,\delta].$$

Next, we assume that (2.3) holds for the case of i. Recalling $\alpha > -2$, we find

$$\begin{aligned}
&|u_{i+1}(r)| \\
&\leq \beta + \int_0^r \frac{1}{(\sinh s)^{N-1}} \int_0^s \{\lambda M(\sinh t)^{N-1} + M^p(\sinh t)^{N-1+\alpha}\}dtds \\
&\leq \beta + \int_0^r \frac{1}{(\sinh s)^{N-1}} \int_0^s \{\lambda M(\sinh t)^{N} + M^p(\sinh t)^{N+\alpha}\}\frac{dt}{\tanh t}ds \\
&= \beta + \int_0^r \frac{1}{(\sinh s)^{N-1}} \left\{\frac{\lambda M}{N}(\sinh s)^N + \frac{M^p}{N+\alpha}(\sinh s)^{N+\alpha}\right\} ds \\
&= \beta + \int_0^r \left\{\frac{\lambda M}{N}(\sinh s) + \frac{M^p}{N+\alpha}(\sinh s)^{\alpha+1}\right\} ds \\
&\leq \beta + \int_0^r \left\{\frac{\lambda M}{N}(\sinh s)^2 + \frac{M^p}{N+\alpha}(\sinh s)^{\alpha+2}\right\} \frac{ds}{\tanh s} \\
&= \beta + \frac{\lambda M}{2N}(\sinh \delta)^2 + \frac{M^p}{(N+\alpha)(\alpha+2)}(\sinh \delta)^{\alpha+2}.
\end{aligned} \tag{2.4}$$

Since we may take $\delta > 0$ satisfying

$$\frac{\lambda}{2N}(\sinh\delta)^2 + \frac{M^{p-1}}{(N+\alpha)(\alpha+2)}(\sinh\delta)^{\alpha+2} < \frac{1}{2},$$

we deduce from (2.2) and (2.4) that

$$|u_{i+1}(r)| < M. \tag{2.5}$$

Therefore (2.3) holds for any $i \in \mathbb{N}$ and $r \in [0, \delta]$, i.e., $u_i \in X$ for any $i \in \mathbb{N}$. Next, we shall show that the mapping $\Phi : X \to X$ is the contraction mapping. Making use of the mean value theorem, we observe that for $u, \tilde{u} \in X$ and $r \in [0, \delta]$,

$$\begin{aligned}
&|\Phi(u)(r) - \Phi(\tilde{u})(r)| \\
&\le \int_0^r \frac{1}{(\sinh s)^{N-1}} \int_0^s \{\lambda|u(t) - \tilde{u}(t)|(\sinh t)^{N-1} + |u^p(t) - \tilde{u}^p(t)|(\sinh t)^{N-1+\alpha}\} dtds \\
&\le \int_0^r \frac{1}{(\sinh s)^{N-1}} \int_0^s \{\lambda(\sinh t)^{N-1} + pM^{p-1}(\sinh t)^{N-1+\alpha}\} dtds \cdot \|u - \tilde{u}\|_{C([0,\delta])} \\
&\le \int_0^r \frac{1}{(\sinh s)^{N-1}} \int_0^s \{\lambda(\sinh t)^{N} + pM^{p-1}(\sinh t)^{N+\alpha}\} \frac{dt}{\tanh t} ds \cdot \|u - \tilde{u}\|_{C([0,\delta])} \\
&= \int_0^r \left\{\frac{\lambda}{N}\sinh s + \frac{pM^{p-1}}{N+\alpha}(\sinh s)^{\alpha+1}\right\} ds \cdot \|u - \tilde{u}\|_{C([0,\delta])} \\
&\le \int_0^r \left\{\frac{\lambda}{N}(\sinh s)^2 + \frac{pM^{p-1}}{N+\alpha}(\sinh s)^{\alpha+2}\right\} \frac{ds}{\tanh s} \cdot \|u - \tilde{u}\|_{C([0,\delta])} \\
&= \left\{\frac{\lambda}{2N}(\sinh\delta)^2 + \frac{pM^{p-1}}{(N+\alpha)(\alpha+2)}(\sinh\delta)^{\alpha+2}\right\} \|u - \tilde{u}\|_{C([0,\delta])}.
\end{aligned}$$

Taking $\delta > 0$ sufficiently small, we obtain

$$\|\Phi(u) - \Phi(\tilde{u})\|_{C([0,\delta])} \le \frac{1}{2}\|u - \tilde{u}\|_{C([0,\delta])}. \tag{2.6}$$

Hence $\Phi : X \to X$ is the contraction mapping. Therefore, $\{u_i\}_{i\in\mathbb{N}}$ is the Cauchy sequence in $C([0, \delta])$ and there exists $u \in X$ satisfying (2.1). Here, we shall verify that $u(0) = \beta$. From $u \in C([0, \delta])$, there exists $C > 0$ such that $|u| \le C$ in $[0, \delta]$. Then, for $r \in (0, \delta)$, the following estimate holds:

$$\begin{aligned}
|u(r) - \beta| &\le \int_0^r \frac{1}{(\sinh s)^{N-1}} \int_0^s \{\lambda C(\sinh t)^{N-1} + C^p(\sinh t)^{N-1+\alpha}\} dtds \\
&\le \int_0^r \frac{1}{(\sinh s)^{N-1}} \left\{\frac{\lambda C}{N}(\sinh s)^N + \frac{C^p}{N+\alpha}(\sinh s)^{N+\alpha}\right\} ds \\
&\le \int_0^r \left\{\frac{\lambda C}{N}(\sinh s)^2 + \frac{C^p}{N+\alpha}(\sinh s)^{\alpha+2}\right\} \frac{ds}{\tanh s} \\
&\le \frac{\lambda C}{2N}(\sinh r)^2 + \frac{C^p}{(N+\alpha)(\alpha+2)}(\sinh r)^{\alpha+2}.
\end{aligned} \tag{2.7}$$

Therefore we obtain $u(r) \to \beta$ as $r \to 0$, i.e., $u(0) = \beta$. Finally, we shall show the uniqueness of the solution to (2.1) in $[0, \delta]$. In particular, we claim that there exists $\tilde{\delta} \in (0, \delta]$ such that the solution of (2.1) is unique in $C([0, \tilde{\delta}])$. Suppose not, for any $\varepsilon \in (0, \delta]$, there exist $u, \tilde{u} \in C([0, \varepsilon])$ satisfying (2.1) and $u \not\equiv \tilde{u}$ in $[0, \varepsilon]$. Here, by the same calculation as in (2.7), we can verify that $u(0) = \tilde{u}(0) = \beta$. Thus we observe from (2.2) that there exists $\tilde{\delta} \in (0, \delta]$ such that

$$|u| \leq M \quad \text{and} \quad |\tilde{u}| \leq M \quad \text{in} \quad [0, \tilde{\delta}].$$

Then, we deduce from (2.6) that

$$\|u - \tilde{u}\|_{C([0,\tilde{\delta}])} = \|\Phi(u) - \Phi(\tilde{u})\|_{C([0,\tilde{\delta}])} \leq \frac{1}{2}\|u - \tilde{u}\|_{C([0,\tilde{\delta}])}.$$

This is a contradiction. Hence, there exists $\tilde{\delta} \in (0, \delta]$ such that the solution of (2.1) is unique in $C([0, \tilde{\delta}])$ and we complete the proof. □

Lemma 2.2. *Let $N \geq 2$, $p > 1$, $R > 0$, $\alpha > -2$, $\beta_1, \beta_2 \in \mathbb{R}$, and $\lambda \geq 0$. Then there exists a constant $\delta > 0$ such that the integral equation*

$$\begin{aligned} u(r) = \beta_1 &+ \int_R^r \frac{\beta_2(\sinh R)^{N-1}}{(\sinh s)^{N-1}} ds \\ &- \int_R^r \frac{1}{(\sinh s)^{N-1}} \int_R^s (\sinh t)^{N-1}\{\lambda u(t) + (\sinh t)^\alpha |u(t)|^{p-1}u(t)\} dt ds. \end{aligned} \tag{2.8}$$

has a unique solution $u \in C([R, R+\delta])$.

Proof. To begin with, we shall show the existence of solution to (2.8) by the successive approximation. For this purpose, we define notations. We define X as

$$X := \{u \in C([R, R+\delta]) : |u| \leq M \quad \text{in} \quad [R, R+\delta]\},$$

where $\delta > 0$ is sufficiently small and $M > 0$ is the constant with

$$|\beta_1| + |\beta_2| \leq \frac{M}{2}. \tag{2.9}$$

Moreover we denote by $\Phi : C([R, R+\delta]) \to C([R, R+\delta])$,

$$\begin{aligned} \Phi(u)(r) = \beta_1 &+ \int_R^r \frac{\beta_2(\sinh R)^{N-1}}{(\sinh s)^{N-1}} ds \\ &- \int_R^r \frac{1}{(\sinh s)^{N-1}} \int_R^s (\sinh t)^{N-1}\{\lambda u(t) + (\sinh t)^\alpha |u(t)|^{p-1}u(t)\} dt ds. \end{aligned}$$

Set inductively the sequence $\{u_i\}_{i\in\mathrm{N}}$ as

$$u_0(r) = \beta_1 + \int_R^r \frac{\beta_2(\sinh R)^{N-1}}{(\sinh s)^{N-1}} ds, \quad u_{i+1}(r) = \Phi(u_i)(r) \quad \text{in} \quad [R, R+\delta].$$

Then we claim that

$$|u_i(r)| \le M \quad \text{for any} \quad i \in \mathbb{N} \quad \text{and} \quad r \in [R, R+\delta]. \tag{2.10}$$

For the case of $i = 0$, it follow from $\delta < 1$ and (2.9) that

$$|u_0(r)| \le |\beta_1| + \frac{|\beta_2|(\sinh R)^{N-1}}{(\sinh R)^{N-1}}(r - R) \le |\beta_1| + |\beta_2|\delta < |\beta_1| + |\beta_2| < M.$$

Next, we assume that (2.10) holds for the case of i. For the case of $i+1$, we derive

$$\begin{aligned}
(2.11)\qquad & |u_{i+1}(r)| \\
&\le |\beta_1| + |\beta_2| + \int_R^r \frac{1}{(\sinh s)^{N-1}} \int_R^s \{\lambda M (\sinh t)^{N-1} + M^p (\sinh t)^{N-1+\alpha}\} dt ds \\
&\le |\beta_1| + |\beta_2| + \int_R^r \frac{1}{(\sinh s)^{N-1}} \int_R^s \{\lambda M (\sinh t)^{N} + M^p (\sinh t)^{N+\alpha}\} \frac{dt}{\tanh t} ds \\
&\le |\beta_1| + |\beta_2| + \int_R^r \frac{1}{(\sinh s)^{N-1}} \left\{ \frac{\lambda M}{N} (\sinh s)^N + \frac{M^p}{N+\alpha} (\sinh s)^{N+\alpha} \right\} ds \\
&\le |\beta_1| + |\beta_2| + \int_R^r \left\{ \frac{\lambda M}{N} (\sinh s)^2 + \frac{M^p}{N+\alpha} (\sinh s)^{\alpha+2} \right\} \frac{ds}{\tanh s} \\
&\le |\beta_1| + |\beta_2| + \frac{\lambda M}{2N} \{(\sinh(R+\delta))^2 - (\sinh R)^2\} \\
&\quad + \frac{M^p}{(N+\alpha)(\alpha+2)} \{(\sinh(R+\delta))^{\alpha+2} - (\sinh R)^{\alpha+2}\}.
\end{aligned}$$

Here, we may take $\delta > 0$ such that

$$\begin{aligned}
\frac{M^{p-1}}{(N+\alpha)(\alpha+2)} & \{(\sinh(R+\delta))^{\alpha+2} - (\sinh R)^{\alpha+2}\} \\
& + \frac{\lambda}{2N} \{(\sinh(R+\delta))^2 - (\sinh R)^2\} \le \frac{1}{2}.
\end{aligned} \tag{2.12}$$

Using (2.9) and (2.11)-(2.12), we see that

$$|u_{i+1}(r)| < M.$$

Therefore (2.10) holds for any $i \in \mathbb{N}$ and $r \in [R, R+\delta]$, i.e., $u_i \in X$ for any $i \in \mathbb{N}$. Next, we shall show that $\Phi : X \to X$ is the contraction mapping. We observe from the mean value theorem that for $u, \tilde{u} \in X$ and $r \in [R, R+\delta]$,

$$
\begin{aligned}
&|\Phi(u)(r) - \Phi(\tilde{u})(r)| \\
&\leq \int_R^r \frac{1}{(\sinh s)^{N-1}} \int_R^s \{\lambda |u(t) - \tilde{u}(t)| (\sinh t)^{N-1} + |u^p(t) - \tilde{u}^p(t)| (\sinh t)^{N-1+\alpha}\} dt ds \\
&\leq \int_R^r \frac{1}{(\sinh s)^{N-1}} \int_R^s \{\lambda (\sinh t)^N + pM^{p-1} (\sinh t)^{N+\alpha}\} \frac{dt}{\tanh t} ds \cdot \|u - \tilde{u}\|_{C([R,R+\delta])} \\
&\leq \int_R^r \left\{ \frac{\lambda}{N} (\sinh s)^2 + \frac{pM^{p-1}}{N+\alpha} (\sinh s)^{\alpha+2} \right\} \frac{ds}{\tanh s} \cdot \|u - \tilde{u}\|_{C([R,R+\delta])} \\
&\leq \frac{\lambda}{2N} \{(\sinh(R+\delta))^2 - (\sinh R)^2\} \|u - \tilde{u}\|_{C([R,R+\delta])} \\
&\quad + \frac{pM^{p-1}}{(N+\alpha)(\alpha+2)} \{(\sinh(R+\delta))^{\alpha+2} - (\sinh R)^{\alpha+2}\} \|u - \tilde{u}\|_{C([R,R+\delta])}.
\end{aligned}
$$

Hence, by the smallness of $\delta > 0$, the following inequality holds:

$$
\|\Phi(u) - \Phi(\tilde{u})\|_{C([R,R+\delta])} \leq \frac{1}{2} \|u - \tilde{u}\|_{C([R,R+\delta])}. \tag{2.13}
$$

Therefore $\Phi : X \to X$ is the contraction mapping. Thus, $\{u_i\}_{i \in \mathbb{N}}$ is the Cauchy sequence in $C([R, R+\delta])$ and there exists $u \in C([R, R+\delta])$ satisfying (2.8). Then $u(R) = \beta_1$. Finally, we show the uniqueness of the solution to (2.8) in $[R, R+\delta]$. Now, we claim that there exists $\tilde{\delta} \in (0, \delta]$ such that the solution of (2.8) is unique in $C([R, R+\tilde{\delta}])$. Suppose not, for any $\varepsilon \in (0, \delta]$, there exist $u, \tilde{u} \in C([R, R+\varepsilon])$ satisfying (2.8) and $u \not\equiv \tilde{u}$ in $[R, R+\varepsilon]$. Then we see that $u(R) = \tilde{u}(R) = \beta_1$. Thus we observe from (2.9) that there exists $\tilde{\delta} \in (0, \delta]$ such that

$$
|u| \leq M \quad \text{and} \quad |\tilde{u}| \leq M \quad \text{in} \quad [R, R+\tilde{\delta}].
$$

Then, we derive from (2.13)

$$
\|u - \tilde{u}\|_{C([R,R+\tilde{\delta}])} = \|\Phi(u) - \Phi(\tilde{u})\|_{C([R,R+\tilde{\delta}])} \leq \frac{1}{2} \|u - \tilde{u}\|_{C([R,R+\tilde{\delta}])},
$$

and this is a contradiction. Thus there exists $\tilde{\delta} \in (0, \delta]$ such that the solution of (2.8) is unique in $C([R, R+\tilde{\delta}])$. We complete the proof. □

Then, the integral equations in Lemmas 2.1-2.2 correspond to the following initial value problems in Lemmas 2.3-2.4 respectively.

Lemma 2.3. *Let $N \geq 2$, $p > 1$, $\alpha > -2$, $\beta > 0$, $\delta > 0$, and $\lambda \geq 0$. Then the following two statements are equivalent:*

(i) $u \in C([0, \delta]) \cap C^2((0, \delta])$ *satisfies*

$$
\begin{cases}
u''(r) + \dfrac{N-1}{\tanh r} u'(r) + \lambda u + (\sinh r)^\alpha |u(r)|^{p-1} u(r) = 0 & \text{in} \quad (0, \delta), \\
u(0) = \beta.
\end{cases} \tag{2.14}
$$

(ii) $u \in C([0,\delta])$ *satisfies* (2.1).

Moreover, in both cases, the following asymptotic behavior holds:

$$\lim_{r\to 0}(\sinh r)u'(r) = 0. \tag{2.15}$$

Lemma 2.4. *Let* $N \geq 2$, $p > 1$, $R > 0$, $\alpha > -2$, $\beta_1, \beta_2 \in \mathbb{R}$, $\delta > 0$, *and* $\lambda \geq 0$. *Then the following two statements are equivalent:*

(i) $u \in C([R, R+\delta]) \cap C^2([R, R+\delta])$ *satisfies*

$$\begin{cases} u''(r) + \dfrac{N-1}{\tanh r}u'(r) + \lambda u + (\sinh r)^{\alpha}|u(r)|^{p-1}u(r) = 0 \quad \textit{in} \quad (0,\delta), \\ u(R) = \beta_1, \quad u'(R) = \beta_2. \end{cases} \tag{2.16}$$

(ii) $u \in C([R, R+\delta])$ *satisfies* (2.8).

Here, we can verify Lemma 2.4 by direct calculations. Therefore, we shall prove only Lemma 2.3.

Proof of Lemma 2.3. To begin with, we shall prove that the assertion (ii) implies the assertion (i). By Lemma 2.1, we have $u(0) = \beta$. Moreover, differentiating (2.1) with respect to r, we obtain

$$u'(r) = -\frac{1}{(\sinh r)^{N-1}}\int_0^r (\sinh t)^{N-1}\{\lambda u(t) + (\sinh t)^{\alpha}|u(t)|^{p-1}u(t)\}dt.$$

Then multiplying the equality by $(\sinh r)^{N-1}$ and differentiating with respect to r again, we derive

$$((\sinh r)^{N-1}u'(r))' = -(\sinh r)^{N-1}\{\lambda u(r) + (\sinh r)^{\alpha}|u(r)|^{p-1}u(r)\}. \tag{2.17}$$

This implies that the equation in (2.14) holds. Hence, if the assertion (ii) holds then the assertion (i) is followed. Now, we shall show that the assertion (i) implies that the assertion (ii). It follows from l'Hospital's rule that

$$\beta = \lim_{r\to 0} u(r) = \lim_{r\to 0}\frac{(\sinh r)u(r)}{\sinh r} = \lim_{r\to 0}\frac{(\cosh r)u(r) + (\sinh r)u'(r)}{\cosh r}.$$

Hence, using $\lim_{r\to 0}\cosh r = 1$, we see that the asymptotic behavior (2.15) in Lemma 2.3 holds. On the other hand, the equation in (2.14) is equivalent to (2.17). Integrating the equation (2.17) over $(0,r)$, we obtain from (2.15),

$$-(\sinh r)^{N-1}u'(r) = \int_0^r (\sinh s)^{N-1}\{\lambda u(s) + (\sinh s)^{\alpha}|u(s)|^{p-1}u(s)\}ds.$$

Moreover multiplying the equality by $(\sinh r)^{-(N-1)}$ and integrating over $(0,r)$ again, we see that u satisfies (2.1). □

Lemma 2.1 and Lemma 2.3 imply that there exists a unique solution of the initial value problem (2.14) in the local interval. Similarly, Lemma 2.2 and Lemma 2.4 imply that there exists a unique solution of the Cauchy problem (2.16) in the local interval. Now we shall verify that the local solution can be extended globally.

Lemma 2.5. *Let $N \geq 2$, $p > 1$, $\alpha > -2$, and $\lambda \geq 0$. Then, for $\beta > 0$, there exists a unique solution $u_\beta \in C([0,\infty]) \cap C^2((0,\infty])$ satisfies* (1.2). *Moreover, u_β satisfies*

$$(2.18) \quad u(r) = \beta - \int_0^r \frac{1}{(\sinh s)^{N-1}} \int_0^s (\sinh t)^{N-1}\{\lambda u(t) + (\sinh t)^\alpha |u(t)|^{p-1}u(t)\}dtds$$

in $[0,\infty)$.

Proof. Lemma 2.1 and Lemma 2.3 imply that there exists $\delta > 0$ such that (1.2) has a unique solution $u_\beta \in C([0,\delta]) \cap C^2((0,\delta])$ with $u(0) = \beta$. Now we shall claim that u_β is bounded in C^1 in $[\delta,\infty)$. For the case of $-2 < \alpha \leq 0$, multiplying the equation in (2.14) by u'_β, we obtain the following identity:

$$F'_\alpha(r) = \frac{\alpha}{p+1}(\sinh r)^\alpha \frac{|u_\beta(r)|^{p+1}}{\tanh r} - \frac{N-1}{\tanh r}u'^2_\beta(r),$$

where

$$F_\alpha(r) = \frac{u'^2_\beta(r)}{2} + \lambda\frac{u^2_\beta(r)}{2} + (\sinh r)^\alpha \frac{|u^{p+1}_\beta(r)|}{p+1}.$$

Hence we observe from $\alpha \leq 0$ that $F'_\alpha < 0$ on $r > 0$, i.e., F_α is strictly monotone decreasing on $r > 0$. Then it follows from the definition of F_α that u_β is locally bounded in $C^1(0,\infty)$. For the case of $\alpha \geq 0$, multiplying the equation in (2.14) by $u'_\beta/(\sinh r)^\alpha$, we observe that the following identity holds:

$$F'_\alpha(r) = -\left(N - 1 + \frac{\alpha}{2}\right)\frac{u'^2_\beta(r)}{(\sinh r)^\alpha(\tanh r)} - \frac{\alpha\lambda u^2_\beta(r)}{2(\sinh r)^\alpha(\tanh r)},$$

where

$$F_\alpha(r) = \frac{u'^2_\beta(r)}{2(\sinh r)^\alpha} + \frac{\lambda u^2_\beta(r)}{2(\sinh r)^\alpha} + \frac{|u^{p+1}_\beta(r)|}{p+1}.$$

It follows from $\alpha \geq 0$ that $F'_\alpha < 0$ on $r > 0$, i.e., F_α is strictly monotone decreasing on $r > 0$. Then we see that u_β is locally bounded in $C^1(0,\infty)$. Therefore, $u_\beta \in C^1([\delta,\infty))$. Using Lemma 2.2 and Lemma 2.4, we observe that u_β can be extended in $(0,\infty)$. Indeed, suppose not, there exists $R > 0$ such that $u_\beta(r) \to \infty$ or $u'_\beta(r) \to \infty$ as $r \to R$. However this is a contradiction to $u_\beta \in C^2((0,\delta]) \cap C^1([\delta,\infty))$. We complete the proof. □

From Lemma 2.5, we showed the uniquely existence of the solution to the initial value problem (1.2). Now, in order to complete the proof of Remark 1.1, we shall evaluate the value of u'_β and u''_β at the origin.

Lemma 2.6. *Let $N \geq 2$, $p > 1$, $\alpha > -2$, $\beta > 0$, and $\lambda \geq 0$. Suppose that u_β is the unique solution of* (1.2). *Then the following hold:*

(i) *If $\alpha > -1$, then $u'_\beta(0) = 0$;*

(ii) *If $\alpha = -1$, then $u'_\beta(0) = -\dfrac{\beta^p}{N-1}$;*

(iii) *If $-2 < \alpha < -1$, then $\lim_{r\to 0} u'_\beta(r) = -\infty$.*

Proof. Differentiating the equation (2.18) with respect to r, we have

$$u'_\beta(r) = -\frac{1}{(\sinh r)^{N-1}}\int_0^r (\sinh s)^{N-1}\{\lambda u_\beta(s) + (\sinh s)^\alpha |u_\beta(s)|^{p-1}u_\beta(s)\}ds. \tag{2.19}$$

Since $u_\beta(0) = \beta > 0$, we choose sufficiently small $\delta > 0$ such that $u_\beta(r) > 0$ in $[0, \delta)$. Then it holds from (2.19) that $u'_\beta(r) < 0$ in $(0, \delta)$, i.e., u_β is strictly monotone decreasing in $[0, \delta)$. Therefore, using (2.19), we obtain the following two inequalities for $r \in (0, \delta)$:

$$u'_\beta(r) < -\frac{1}{(\sinh r)^{N-1}}\left\{\lambda u_\beta(r)\int_0^r (\sinh s)^{N-1}ds + u_\beta^p(r)\int_0^r (\sinh s)^{N-1+\alpha}ds\right\}; \tag{2.20}$$

$$u'_\beta(r) > -\frac{1}{(\sinh r)^{N-1}}\left\{\lambda\beta\int_0^r (\sinh s)^{N-1}ds + \beta^p\int_0^r (\sinh s)^{N-1+\alpha}ds\right\}. \tag{2.21}$$

Here, from l'Hospital's rule, it holds that

$$\lim_{r\to 0}\frac{\int_0^r (\sinh s)^{N-1}ds}{(\sinh r)^{N-1}} = \lim_{r\to 0}\frac{\tanh r}{N-1} = 0. \tag{2.22}$$

Now, for the case of $\alpha > -1$, it follows from $N - 1 + \alpha > 0$ that

$$\lim_{r\to 0}\frac{\int_0^r (\sinh s)^{N-1+\alpha}ds}{(\sinh r)^{N-1}} = \lim_{r\to 0}\frac{(\sinh r)^{\alpha+1}}{(N-1)\cosh r} = 0. \tag{2.23}$$

Thus using $u_\beta(0) = \beta$ and (2.22)-(2.23), we see that the right-hand side of the estimate (2.20) converges to 0 as $r \to 0$. Similarly, from (2.22)-(2.23), it follows that the right-hand side of the estimate (2.21) converges to 0 as $r \to 0$. Therefore, the assertion (i) holds. For the case of $\alpha = -1$, using l'Hospital's rule, we observe that

$$\lim_{r\to 0}\frac{\int_0^r (\sinh s)^{N-2}ds}{(\sinh r)^{N-1}} = \lim_{r\to 0}\frac{1}{(N-1)(\cosh r)} = \frac{1}{N-1} \quad \text{if} \quad N > 2, \tag{2.24}$$

$$\lim_{r\to 0}\frac{\int_0^r (\sinh s)^{N-2}ds}{(\sinh r)^{N-1}} = \lim_{r\to 0}\frac{r}{\sinh r} = 1 \quad \text{if} \quad N = 2. \tag{2.25}$$

Hence by $u_\beta(0) = \beta$, (2.22), and (2.24)-(2.25), the right-hand sides of (2.20)-(2.21) converges to $-\beta^p/(N-1)$ as $r \to 0$. Then, the assertion (ii) is followed. Finally, for the case of $-2 < \alpha < -1$, it holds from (2.24)-(2.25) that

$$\frac{\int_0^r (\sinh s)^{N-1+\alpha}ds}{(\sinh r)^{N-1}} > (\sinh r)^{\alpha+1}\frac{\int_0^r (\sinh s)^{N-2}ds}{(\sinh r)^{N-1}} \to \infty \quad \text{as} \quad r \to 0. \tag{2.26}$$

Therefore, from $u_\beta(0) = \beta$, (2.22), and (2.26), it holds that the right-hand side of (2.20) diverges to $-\infty$ as $r \to 0$. This implies that the assertion (iii) holds. We complete the proof. □

Lemma 2.7. *Let $N \geq 2$, $p > 1$, $\alpha \geq 0$, $\beta > 0$, and $\lambda \geq 0$. Suppose that u_β is the unique solution of* (1.2). *Then the following hold:*

(i) *If $\alpha > 0$, then* $u_\beta''(0) = -\dfrac{\lambda\beta}{N}$;

(ii) *If $\alpha = 0$, then* $u_\beta''(0) = -\dfrac{\lambda\beta}{N} - \dfrac{\beta^p}{N}$.

Proof. From the equation in (1.2), we have

$$u_\beta''(r) = -\frac{N-1}{\tanh r}u_\beta'(r) - \lambda u_\beta(r) - (\sinh r)^\alpha |u_\beta(r)|^{p-1}u_\beta(r). \tag{2.27}$$

Then by $u_\beta(0) = \beta$, the following hold:

$$\lim_{r\to 0}(\sinh r)^\alpha |u_\beta(r)|^{p-1}u_\beta(r) = \begin{cases} 0 & \text{if} \quad \alpha > 0, \\ \beta^p & \text{if} \quad \alpha = 0. \end{cases} \tag{2.28}$$

We shall study the asymptotic behavior of $u_\beta'(r)/\tanh r$ as $r \to 0$. From $u_\beta(0) = \beta > 0$, there exists sufficiently small $\delta > 0$ such that $u_\beta(r) > 0$ in $[0,\delta)$. Then from (2.19), $u_\beta'(r) < 0$ in $(0,\delta)$, i.e., u_β is strictly monotone decreasing in $[0,\delta)$. Using (2.20)-(2.21), we see that the following two estimates hold for $r \in (0,\delta)$:

$$\frac{u_\beta'(r)}{\tanh r} < -\frac{\cosh r}{(\sinh r)^N}\left\{\lambda u_\beta(r)\int_0^r (\sinh s)^{N-1}ds + u_\beta^p(r)\int_0^r (\sinh s)^{N-1+\alpha}ds\right\}; \tag{2.29}$$

$$\frac{u_\beta'(r)}{\tanh r} > -\frac{\cosh r}{(\sinh r)^N}\left\{\lambda\beta\int_0^r (\sinh s)^{N-1}ds + \beta^p\int_0^r (\sinh s)^{N-1+\alpha}ds\right\}. \tag{2.30}$$

L'Hospital's rule yields the following asymptotic behaviors:

$$\lim_{r\to 0}\frac{\int_0^r (\sinh s)^{N-1}ds}{(\sinh r)^N} = \lim_{r\to 0}\frac{\tanh r}{N(\sinh r)} = \lim_{r\to 0}\frac{1}{N(\cosh r)} = \frac{1}{N}; \tag{2.31}$$

$$\begin{aligned}\lim_{r\to 0}\frac{\int_0^r (\sinh s)^{N-1+\alpha}ds}{(\sinh r)^N} &= \lim_{r\to 0}\frac{(\sinh r)^{-1+\alpha}\tanh r}{N} = \lim_{r\to 0}\frac{(\sinh r)^\alpha}{N\cosh r} \\ &= \begin{cases} 0 & \text{if} \quad \alpha > 0, \\ \dfrac{1}{N} & \text{if} \quad \alpha = 0. \end{cases}\end{aligned} \tag{2.32}$$

Hence, for the case of $\alpha > 0$, from $\cosh 0 = 1$, $u_\beta(0) = \beta$ and (2.31)-(2.32), it follows that the right-hand side of the inequality (2.29) converges to $-\lambda\beta/N$ as $r \to 0$. Moreover, by (2.31)-(2.32), the right-hand side of the inequality (2.30) converges to $-\lambda\beta/N$ as $r \to 0$. Therefore, if $\alpha > 0$, then

$$\lim_{r\to 0}\frac{u_\beta'(r)}{\tanh r} = -\frac{\lambda\beta}{N}. \tag{2.33}$$

Combining (2.27) with (2.28) and (2.33), we obtain the assertion (i). Similarly, if $\alpha = 0$, then

$$\lim_{r\to 0} \frac{u'_\beta(r)}{\tanh r} = -\frac{\lambda\beta}{N} - \frac{\beta^p}{N}. \tag{2.34}$$

Thus from (2.27)-(2.28) and (2.34), the assertion (ii) is followed. □

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